Invariant divisors and equivariant line bundles

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Introduction

Let $\mathfrak{g}$ be a Lie algebra of (holomorphic) vector fields on the (complex) manifold M.

- What are the $\mathfrak{g}$-invariant submanifolds?
  - **What are the $\mathfrak{g}$-invariant hypersurfaces?**

Contents:

- Relative invariants and their weights
- Divisors and line bundles
- Invariant divisors and equivariant line bundles
- Application to jet spaces

Introduction

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- What are the $\mathfrak{g}$-invariant submanifolds?
  - **What are the $\mathfrak{g}$-invariant hypersurfaces?**

Contents:

- Relative invariants and their weights (Chevalley-Eilenberg cohomology)
- Divisors and line bundles (Čech cohomology)
- Invariant divisors and equivariant line bundles
- Application to jet spaces

Example

Let $M = \mathbb{R}^2(x, y)$ and take the Lie algebra $\mathfrak{g} = \langle X = x\partial_x - y\partial_y \rangle$ corresponding to the group action $(x, y) \mapsto (tx, t^{-1}y)$, $t \in \mathbb{R} \setminus \{0\}$.

Absolute invariant: $I = xy$.

- Invariant hypersurfaces: $\{I = C\} \subset M$, $C \in \mathbb{R}$.
- Solution to linear PDE system: $X(I) = 0$.
- Well understood in several settings:
  - local smooth: Frobenius' theorem
  - global algebraic: Rosenlicht's theorem

Relative invariants: $R_1 = x, R_2 = y$.

- Invariant hypersurfaces: $\{R_1 = 0\}$ and $\{R_2 = 0\}$.
- Solution to $X(R) = \lambda(X)R$, with $\lambda(X) = \pm 1$.
- $\exists$ some results in the local smooth setting.

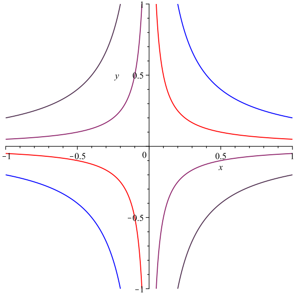

Figure: Orbits of $\mathfrak{g}$.

Relative invariants

Definition

A relative invariant wrt. $\mathfrak{g} \subset \mathcal{D}(M)$ is a function $f \in \mathcal{O}(M)$ satisfying

$$X(f) = \lambda(X)f, \qquad \forall X \in \mathfrak{g},$$

for some $\lambda \in \mathfrak{g}^* \otimes \mathcal{O}(M)$. We call λ the weight of f.

$$\lambda([X,Y])f = [X,Y](f) = X(Y(f)) - Y(X(f)) = (X(\lambda(Y)) - Y(\lambda(X)))f$$

for each pair $X, Y \in \mathfrak{g}$. Thus, the weight of a relative invariant satisfies

$$(d^1\lambda)(X,Y) := X(\lambda(Y)) - Y(\lambda(X)) - \lambda([X,Y]) = 0.$$

Furthermore, for $\mu \in \mathcal{O}(M)$,

$$X(e^\mu f) = X(\mu)e^\mu f + e^\mu \lambda(X)f = (X(\mu) + \lambda(X))e^\mu f,$$

meaning that the difference between weights of the two equivalent relative invariants is

$$(d^0\mu)(X) := X(\mu).$$

Chevalley-Eilenberg cohomology

The Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in $\mathcal{O}(M)$:

$$\mathcal{O}(M) \xrightarrow{d^0} \mathfrak{g}^* \otimes \mathcal{O}(M) \xrightarrow{d^1} \Lambda^2\mathfrak{g}^* \otimes \mathcal{O}(M) \xrightarrow{d^2} \cdots$$

$$(d^0\mu)(X) := X(\mu)$$

$$(d^1\lambda)(X,Y) := X(\lambda(Y)) - Y(\lambda(X)) - \lambda([X,Y])$$

Weights of relative invariants are elements in

$$\mathrm{H}^1(\mathfrak{g}, \mathcal{O}(M)) = \frac{\ker(d^1)}{\mathrm{im}(d^0)}.$$

Small modification: If $\mu \in \mathcal{O}(M)^\times$ is a nonvanishing function, then

$$X(\mu f) = X(\mu)f + \mu X(f) = (X(\mu) + \mu\lambda(X))f = \left(\frac{X(\mu)}{\mu} + \lambda(X)\right)\mu f.$$

$$\mathcal{O}^\times(M) \xrightarrow{d^0 \log} g^* \otimes \mathcal{O}(M) \xrightarrow{d^1} \Lambda^2\mathfrak{g}^* \otimes \mathcal{O}(M) \xrightarrow{d^2} \cdots$$

Example

$$M = \mathbb{C}^1, \qquad \mathfrak{g} = \langle X = x\partial_x, Y = x^2\partial_x \rangle.$$

Use notation $\lambda(X) = a(x), \lambda(Y) = b(x)$ for a representative of $[\lambda] \in \mathrm{H}^1(\mathfrak{g}, \mathcal{O}(M))$.

$$(\lambda - d^0\mu)(X) = a(x) - x\mu'(x)$$

By subtracting a coboundary, we set $a(x) = A_1 \in \mathbb{C}$.

$$X(b) - Y(a) = \lambda([X, Y]) = \lambda(Y) = b \quad \Leftrightarrow \quad xb'(x) = b(x)$$

Has solution $b(x) = A_2 x,\ A_2 \in \mathbb{C}$. $\mathrm{H}^1(\mathfrak{g}, \mathcal{O}(M)) = \mathbb{C}^2$.

The general relative invariant is given by $R = x^{A_1}$. It has weight

$$\lambda(X) = A_1, \quad \lambda(Y) = A_1 x. \qquad (A_1 = A_2)$$

Not all elements of $\mathrm{H}^1(\mathfrak{g}, \mathcal{O}(M))$ are realized as weights of relative invariants.

Lift of $\mathfrak{g}$ to $M \times \mathbb{C}$

Construct the lift

$$\hat{\mathfrak{g}}^\lambda = \{\hat{X} = X + \lambda(X) u \partial_u \mid X \in \mathfrak{g}\} \subset \mathcal{D}(M \times \mathbb{C}).$$

- $\widehat{[X, Y]} = [\hat{X}, \hat{Y}] \quad \Leftrightarrow \quad \lambda \in \ker(d^1).$
- Changing fiber coordinate, $\tilde{u} = \mu u$, gives equivalence relation:

$$\lambda \sim \tilde{\lambda} = \lambda + d^0 \log \mu.$$

Conclusion: $\mathrm{H}^1(\mathfrak{g}, \mathcal{O}(M))$ can be identified with the space of lifts of $\mathfrak{g}$ to $M \times \mathbb{C}$.

If f is a relative invariant with weight λ, and $\hat{\mathfrak{g}}^\lambda$ the corresponding lift to $M \times \mathbb{C}$, then $u = Cf$ is a $\hat{\mathfrak{g}}^\lambda$-invariant section of $M \times \mathbb{C}$.

Realizability of weight

Given $[\lambda] \in H^1(\mathfrak{g}, \mathcal{O}(M))$, when does there exist a relative invariant f with weight λ? The answer is given (in local smooth setting) by Fels and Olver (1997).

Construct the lift

$$\hat{\mathfrak{g}}^\lambda = \{\hat{X} = X + \lambda(X)u\partial_u \mid X \in \mathfrak{g}\} \subset \mathcal{D}(M \times \mathbb{C}).$$

If f is a relative invariant $\big(X(f) = \lambda(X)f\big)$ then

$$\hat{X}(u/f) = \frac{\hat{X}(u)f - u\hat{X}(f)}{f^2} = \frac{u(\lambda(X)f - X(f))}{f^2} = 0,$$

implying that u/f is an absolute invariant.

Theorem

If there exists a relative invariant f with weight λ, then the dimension of generic $\mathfrak{g}$-orbits on M is equal to the dimension of generic $\hat{\mathfrak{g}}^\lambda$-orbits on $M \times \mathbb{C}$.

Line bundles over M

Any holomorphic line bundle $\pi\colon L\to M$ is locally trivial: There exists cover $\mathcal{U}=\{U_\alpha\}$ such that $\pi^{-1}(U_\alpha)\simeq U_\alpha\times\mathbb{C}$. The line bundle is uniquely defined through its (holomorphic) transition functions:

$$g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\mathbb{C}^\times.$$

$$\prod_\alpha \mathcal{O}^\times(U_\alpha)\xrightarrow{\delta^0}\prod_{\alpha\neq\beta}\mathcal{O}^\times(U_\alpha\cap U_\beta)\xrightarrow{\delta^1}\prod_{\alpha\neq\beta\neq\gamma\neq\alpha}\mathcal{O}^\times(U_\alpha\cap U_\beta\cap U_\gamma)\to\cdots$$

$$(\delta^0\mu)_{\alpha\beta}=\mu_\alpha/\mu_\beta,\qquad \mu=\{\mu_\alpha\}\in\prod_\alpha\mathcal{O}^\times(U_\alpha),$$

$$(\delta^1\nu)_{\alpha\beta\gamma}=\nu_{\alpha\gamma}/(\nu_{\alpha\beta}\nu_{\beta\gamma}),\qquad \nu=\{\nu_{\alpha\beta}\}\in\prod_{\alpha\neq\beta}\mathcal{O}^\times(U_\alpha\cap U_\beta).$$

The collection $g=\{g_{\alpha\beta}\}$ represents an element of $\check{\mathrm{H}}^1(\mathcal{U},\mathcal{O}^\times)=\ker(\delta^1)/\mathrm{im}(\delta^0)$. There is a group isomorphism $\mathrm{Pic}(M)\simeq\check{\mathrm{H}}^1(M,\mathcal{O}^\times):=\varinjlim\check{\mathrm{H}}^1(\mathcal{U},\mathcal{O}^\times)$, with the Picard group $\mathrm{Pic}(M)$ being the group of line bundles.

Divisors on M

Any holomorphic hypersurface in M can be defined locally as the zeros of a holomorphic function.

A divisor D on M is a global section of the quotient sheaf $\mathcal{M}^\times/\mathcal{O}^\times$. It can be given on some open cover $\mathcal{U} = \{U_\alpha\}$ as a collection $\{f_\alpha \in \mathcal{M}^\times(U_\alpha)\}$ of meromorphic functions, such that $f_\beta/f_\alpha \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$, where f_α is defined only up to a factor of $\mathcal{O}^\times(U_\alpha)$.

$$\mathrm{Div}(M) \to \mathrm{Pic}(M)$$
$$D = \{f_\alpha\} \mapsto [D] \text{ defined by transition functions } g_{\alpha\beta} = f_\alpha/f_\beta$$

The divisor $D = \{f_\alpha\}$ defines a meromorphic section of $[D]$: $x \mapsto (x, f_\alpha(x))$ on $U_\alpha \times \mathbb{C}$.

Idea: $\mathfrak{g}$-invariant hypersurfaces are given by $\mathfrak{g}$-invariant divisors, or sections of $\mathfrak{g}$-equivariant line bundles.

$\mathfrak{g}$-equivariant line bundles

Definition

Let $\mathfrak{g}$ be a Lie algebra of vector fields on M. A $\mathfrak{g}$-equivariant line bundle over M is a pair $(\pi, \hat{\mathfrak{g}})$ where $\pi\colon L \to M$ is a line bundle and $\hat{\mathfrak{g}}$ is a lift of $\mathfrak{g}$ to L.

Locally, line bundles are trivial: $\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}$. Lifts in these local charts are of the form $\hat{\mathfrak{g}}|_{U_\alpha} = \{X|_{U_\alpha} + \lambda_\alpha(X) u \partial_u \mid X \in \mathfrak{g}\}$.

Local lifts: $\lambda = \{\lambda_\alpha\} \in \prod_\alpha \mathfrak{g}^* \otimes \mathcal{O}(U_\alpha), \quad d^1\lambda = 0.$

Trans. functions: $g = \{g_{\alpha\beta}\} \in \prod_{\alpha,\beta} \mathcal{O}^\times(U_{\alpha\beta}), \quad \delta^1 g = 0.$

On $U_{\alpha\beta} := U_\alpha \cap U_\beta$:

$$X + \lambda_\alpha(X) u_\alpha \partial_{u_\alpha} = X + \lambda_\beta(X) u_\beta \partial_{u_\beta}$$

$$u_\alpha = g_{\alpha\beta} u_\beta \quad \Rightarrow \quad X + \lambda_\alpha(X) u_\alpha \partial_{u_\alpha} = X + (\lambda_\alpha(X) - X(g_{\alpha\beta})/g_{\alpha\beta}) u_\beta \partial_{u_\beta}$$

Compatibility: $\lambda_\alpha(X) - \lambda_\beta(X) = X(g_{\alpha\beta})/g_{\alpha\beta}, \qquad \forall X \in \mathfrak{g},$

Double complex

A $\mathfrak{g}$-equivariant line bundle is given by a pair (g, λ) satisfying

$$d^1\lambda = 0, \quad \delta^1 g = 0, \quad \delta^0\lambda = d^0 \log g.$$

$$\begin{array}{ccccccc}
\prod_\alpha \mathcal{O}^\times(U_\alpha) & \xrightarrow{d^0 \log} & \prod_\alpha(\mathfrak{g}^* \otimes \mathcal{O}(U_\alpha)) & \xrightarrow{d^1} & \prod_\alpha(\Lambda^2\mathfrak{g}^* \otimes \mathcal{O}(U_\alpha)) & \xrightarrow{d^2} & \\
\delta^0 \downarrow & & & & & & \\
\prod_{\alpha,\beta} \mathcal{O}^\times(U_{\alpha\beta}) & & & & & & \\
\delta^1 \downarrow & & & & & & \\
\prod_{\alpha,\beta,\gamma} \mathcal{O}^\times(U_{\alpha\beta\gamma}) & & & & & & \\
\delta^2 \downarrow & & & & & &
\end{array}$$

Double complex

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\delta^0\downarrow & & \delta^0\downarrow & & \delta^0\downarrow & \\
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\delta^1\downarrow & & \delta^1\downarrow & & \delta^1\downarrow & \\
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\delta^1\downarrow & & \delta^1\downarrow & & \delta^1\downarrow & \\
\prod_{\alpha,\beta,\gamma} \mathcal{O}^\times(U_{\alpha\beta\gamma}) & \xrightarrow{d^0 \log} & \prod_{\alpha,\beta,\gamma}(\mathfrak{g}^* \otimes \mathcal{O}(U_{\alpha\beta\gamma})) & \xrightarrow{d^1} & & \\
\delta^2\downarrow & & \delta^2\downarrow & & &
\end{array}$$

The corresponding total complex

A $\mathfrak{g}$-equivariant line bundle is given by a pair (g, λ) satisfying

$$d^{1,0}\lambda = 0, \quad \delta^{0,1}g = 0, \quad \delta^{1,0}\lambda = d^{0,1}g.$$

$$\begin{array}{ccccccc}
C^{0,0} & \xrightarrow{d^{0,0}} & C^{1,0} & \xrightarrow{d^{1,0}} & C^{2,0} & \xrightarrow{d^{2,0}} & \\
\delta^{0,0}\downarrow & & \delta^{1,0}\downarrow & & \delta^{2,0}\downarrow & & \\
C^{0,1} & \xrightarrow{d^{0,1}} & C^{1,1} & \xrightarrow{d^{1,1}} & C^{2,1} & \xrightarrow{d^{2,1}} & \\
\delta^{0,1}\downarrow & & \delta^{1,1}\downarrow & & \delta^{2,1}\downarrow & & \\
C^{0,2} & \xrightarrow{d^{0,2}} & C^{1,2} & \xrightarrow{d^{1,2}} & & & \\
\delta^{0,2}\downarrow & & \delta^{1,2}\downarrow & & & &
\end{array}$$

$$\mathrm{Tot}^r(C) = \prod_{r=p+q} C^{p,q}, \qquad \partial^r = \sum_{p+q=r} \left(d^{p,q} + (-1)^p \delta^{p,q}\right) \colon \mathrm{Tot}^r(C) \to \mathrm{Tot}^{r+1}(C).$$

The corresponding total complex

$$\mathrm{Tot}^r(C) = \prod_{r=p+q} C^{p,q}, \qquad \partial^r = \sum_{p+q=r} (d^{p,q} + (-1)^p \delta^{p,q}) \colon \mathrm{Tot}^r(C) \to \mathrm{Tot}^{r+1}(C).$$

- A line bundle is given by a ∂^1-cocycle $(g, \lambda) \in \mathrm{Tot}^1(C)$.
- The equivalence relation $(g, \lambda) \sim (g, \lambda) + \partial^0(\mu) = (g \cdot \delta^{0,0}\mu, \lambda + d^{0,0}\mu)$ corresponds exactly to the freedom in the fiber coordinate $\tilde{u}_\alpha = \mu_\alpha u_\alpha$:

$$\tilde{u}_\alpha = \mu_\alpha u_\alpha = \mu_\alpha g_{\alpha\beta} u_\beta = \frac{\mu_\alpha}{\mu_\beta} g_{\alpha\beta} \tilde{u}_\beta$$

$\mathfrak{g}$-equivariant line bundles

Denote the group of $\mathfrak{g}$-equivariant line bundles by $\mathrm{Pic}_{\mathfrak{g}}(M)$. If the cover $\{U_\alpha\}$ is sufficiently nice, then $\mathrm{Pic}_{\mathfrak{g}}(M) \simeq \mathrm{H}^1(\mathrm{Tot}^\bullet(C)) = \ker(\partial^1)/\mathrm{im}(\partial_0)$.

More generally:

Theorem

$$\mathrm{Pic}_{\mathfrak{g}}(M) \simeq \mathbb{H}^1(\mathfrak{g}, \mathcal{O}) := \varinjlim \mathrm{H}^1(\mathrm{Tot}^\bullet(C)).$$

$\mathbb{H}^1(\mathfrak{g}, \mathcal{O})$ is called the first hypercohomology of the Chevalley-Eilenberg sheaf complex

$$\mathcal{O}^\times \to \mathfrak{g}^* \otimes \mathcal{O} \to \Lambda^2 \mathfrak{g}^* \otimes \mathcal{O} \to \cdots .$$

Example

$$M = \mathbb{C}P^1, \qquad \mathfrak{g} = \langle X, Y \rangle, \qquad U_0 \simeq \mathbb{C}^1(x), \quad U_\infty \simeq \mathbb{C}^1(y), \quad y = 1/x.$$

The vector fields are given by:

$$X|_{U_0} = x\partial_x, \quad Y|_{U_0} = x^2\partial_x, \qquad X|_{U_\infty} = -y\partial_y, \quad Y|_{U_\infty} = -\partial_y.$$

Representatives of elements $[\lambda_0] \in H^1(\mathfrak{g}, \mathcal{O}(U_0))$ and $[\lambda_\infty] \in H^1(\mathfrak{g}, \mathcal{O}(U_\infty))$:

$$\lambda_0(X) = A_1, \quad \lambda_0(Y) = A_2 x, \qquad \lambda_\infty(X) = B_1, \quad \lambda_\infty(Y) = 0.$$

Compatibility:

$$\lambda_0(X) - \lambda_\infty(X) = X(g_{0\infty})/g_{0\infty}, \qquad \lambda_0(Y) - \lambda_\infty(Y) = Y(g_{0\infty})/g_{0\infty},$$
$$(A_1 - B_1)g_{0\infty} = x\partial_x(g_{0\infty}), \qquad A_2 x g_{0\infty} = x^2\partial_x(g_{0\infty}).$$

We get $B_1 = A_1 - A_2$ and $g_{0\infty} = x^{A_2}$, $A_2 \in \mathbb{Z}$. In other words, $\mathrm{Pic}_{\mathfrak{g}}(M) \simeq \mathbb{C} \times \mathbb{Z}$.

$\mathfrak{g}$-invariant divisors

Definition

A divisor $D = \{f_\alpha\}$ is $\mathfrak{g}$-invariant if there exists an element $\lambda = \{\lambda_\alpha\} \in C^{1,0}$ such that for each α

$$X(f_\alpha) = \lambda_\alpha(X) f_\alpha, \qquad \forall X \in \mathfrak{g}.$$

Theorem

Let $\mathfrak{g}$ be a Lie algebra of vector fields on M, and let $D = \{f_\alpha\}$ be a $\mathfrak{g}$-invariant divisor with weight $\lambda = \{\lambda_\alpha\}$. The pair (g, λ) defines a $\mathfrak{g}$-equivariant line bundle, where $g_{\alpha\beta} = f_\alpha / f_\beta$. In other words, there is a map:

$$\mathrm{Div}_{\mathfrak{g}}(M) \to \mathrm{Pic}_{\mathfrak{g}}(M).$$

Example

$$M = \mathbb{C}P^1, \qquad \mathfrak{g} = \langle X, Y \rangle, \qquad U_0 \simeq \mathbb{C}^1(x), \quad U_\infty \simeq \mathbb{C}^1(y), \quad y = 1/x.$$

$$X|_{U_0} = x\partial_x, \quad Y|_{U_0} = x^2\partial_x, \qquad X|_{U_\infty} = -y\partial_y, \quad Y|_{U_\infty} = -\partial_y.$$

Representatives of elements $[\lambda_0] \in H^1(\mathfrak{g}, \mathcal{O}(U_0))$ and $[\lambda_\infty] \in H^1(\mathfrak{g}, \mathcal{O}(U_\infty))$:

$$\lambda_0(X) = A_1, \quad \lambda_0(Y) = A_2 x, \qquad \lambda_\infty(X) = B_1, \quad \lambda_\infty(Y) = 0.$$

From the compatibility condition, we get $B_1 = A_1 - A_2$ and $g_{0\infty} = x^{A_2}$, $A_2 \in \mathbb{Z}$.

$$\hat{\mathfrak{g}}|_{U_\infty} = \langle -y\partial_y + (A_1 - A_2)u\partial_u,\ -\partial_y \rangle$$

A necessary condition for invariant divisors to exist is that the generic orbit dimension of $\hat{\mathfrak{g}}$ is the same as that of $\mathfrak{g}$: in this case $A_1 = A_2$. The general invariant divisor is

$$f_0 = x^{A_2}, \qquad f_\infty = 1, \qquad A_2 = A_1 \in \mathbb{Z}.$$

$$\mathrm{Div}_{\mathfrak{g}}(M) \simeq \mathbb{Z}, \qquad \mathrm{Pic}_{\mathfrak{g}}(M) \simeq \mathbb{C} \times \mathbb{Z}.$$

Differential invariants of curves in $\mathbb{C}P^2$

Choose open chart $U_1 = \{[u : v : w] \in \mathbb{C}P^2 \mid w \neq 0\}$, and coordinates $x = u/w, y = v/w$. Then choose y as "dependent variable". This gives coordinate chart $U_{1y} \simeq \mathbb{C}^3(x, y, y_1)$ of $J^1(\mathbb{C}P^2, 1)$. The basis of $\mathfrak{sl}(3, \mathbb{C})$:

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = y\partial_x - y_1^2\partial_{y_1}, \quad X_4 = x\partial_y + \partial_{y_1}, \quad X_5 = x\partial_x - y\partial_y - 2y_1\partial_{y_1},$$
$$X_6 = x\partial_x + y\partial_y, \quad X_7 = x^2\partial_x + xy\partial_y + (y - xy_1)\partial_{y_1}, \quad X_8 = xy\partial_x + y^2\partial_y + (y - xy_1)y_1\partial_{y_1}.$$

The general element $[\lambda] \in H^1(\mathfrak{sl}(3, \mathbb{C}), \mathcal{O}(U_{1y}))$ is given by

$$\lambda(X_1) = 0, \quad \lambda(X_2) = 0, \quad \lambda(X_3) = A_1 y_1, \quad \lambda(X_4) = 0, \quad \lambda(X_5) = A_1,$$
$$\lambda(X_6) = A_2, \quad \lambda(X_7) = \frac{3A_2 + A_1}{2}x, \quad \lambda(X_8) = A_1 x y_1 + \frac{3A_2 - A_1}{2}y.$$

$$H^1(\mathfrak{sl}(3, \mathbb{C}), \mathcal{O}(U_{1y})) = \mathbb{C}^2$$

Differential invariants of curves in $\mathbb{C}P^2$

Keep the chart $U_1 \simeq \mathbb{C}^2(x,y)$, but choose x as "dependent variable". This gives coordinate chart $U_{1x} \simeq \mathbb{C}^3(x,y,x_1)$ of $J^1(\mathbb{C}P^2,1)$. The basis of $\mathfrak{sl}(3,\mathbb{C})$:

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = y\partial_x + \partial_{x_1}, \quad X_4 = x\partial_y - x_2^2\partial_{x_1}, \quad X_5 = x\partial_x - y\partial_y + 2x_1\partial_{x_1},$$
$$X_6 = x\partial_x + y\partial_y, \quad X_7 = x^2\partial_x + xy\partial_y + (x - yx_1)x_1\partial_{x_1}, \quad X_8 = xy\partial_x + y^2\partial_y + (x - yx_1)\partial_{x_1}.$$

The general element $[\lambda] \in H^1(\mathfrak{sl}(3,\mathbb{C}), \mathcal{O}(U_{1x}))$ is given by

$$\lambda(X_1) = 0, \quad \lambda(X_2) = 0, \quad \lambda(X_3) = 0, \quad \lambda(X_4) = B_1 x_1, \quad \lambda(X_5) = -B_1,$$
$$\lambda(X_6) = B_2, \quad \lambda(X_7) = B_1 y x_1 + \frac{3B_2 - B_1}{2}x, \quad \lambda(X_8) = \frac{3B_2 + B_1}{2}y.$$

$$H^1(\mathfrak{sl}(3,\mathbb{C}), \mathcal{O}(U_{1x})) = \mathbb{C}^2$$

Compatibility $\delta^{1,0}\lambda = -d^{0,1}g$:

$$A_1 = B_1 \in \mathbb{Z}, \quad A_2 = B_2 \in \mathbb{C}, \quad g_{xy} = y_1^{A_1}.$$

Differential invariants of curves in $\mathbb{C}P^2$

To cover $\mathbb{C}P^2$, we need two additional charts:

$$U_2 = \{[u : v : w] \in \mathbb{C}P^2 \mid v \neq 0\}, \quad U_3 = \{[u : v : w] \in \mathbb{C}P^2 \mid u \neq 0\}.$$

Doing the same computations for these charts, requiring analyticity, for example on $U_{1y} \cap U_{2y} \subset J^1(\mathbb{C}P^2, 1)$, gives the additional constraint $(A_1 + 3A_2)/2 \in \mathbb{Z}$.

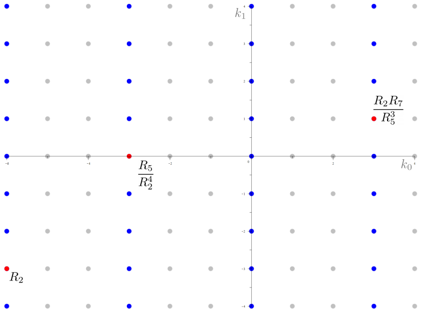
Let ω denote the contact form on $J^1(\mathbb{C}P^1, 1)$, and $\mathcal{O}_{\mathbb{C}P^2}(-1)$ denote the tautological line bundle over $\mathbb{C}P^2$.

Theorem

$$\mathrm{Pic}_{\mathfrak{sl}(3,\mathbb{C})^{(1)}}(J^1(\mathbb{C}P^2, 1)) = \left\{ \langle\omega\rangle^{\otimes k_1} \otimes \pi_{1,0}^* \mathcal{O}_{\mathbb{C}P^2}(k_0) \mid k_0, k_1 \in \mathbb{Z} \right\} \simeq \mathbb{Z}^2.$$

Relation between parameters: $A_1 = -k_1 + \frac{2}{3}k_0, A_2 = k_1$.

Differential invariants of curves in $\mathbb{C}P^2$



Relative invariants on $J^7(\mathbb{C}P^2, 1)$:

$$
\begin{aligned}
R_2 &= y_2,\\
R_5 &= 9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3,\\
R_7 &= 18y_2^4(9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3)y_7 - 189y_2^4y_5^2(4y_3^2\\
&\quad + 15y_2y_4) - 189y_2^6y_6^2 + 126y_2^4(9y_2y_3y_5 + 15y_2y_4^2 - 25y_3^2y_4)y_6\\
&\quad + 210y_2^2y_3(63y_2^2y_4^2 - 60y_2y_3^2y_4 + 32y_3^4)y_5 - 4725y_2^4y_4^4\\
&\quad - 7875y_2^3y_3^2y_4^3 + 31500y_2^2y_3^4y_4^2 - 33600y_2y_3^6y_4 + 11200y_3^8.
\end{aligned}
$$

- Only $(k_0, k_1) \in (3\mathbb{Z}) \times \mathbb{Z}$ are realized by rational relative invariants.
- The weights of R_2, R_5, R_7 depend only on $J^1(\mathbb{C}P^2, 1)$.

Polynomial invariant divisors on jet spaces

We call a divisor on $J^k(E,m)$ polynomial if its restriction to fibers of $J^k(E,m) \to J^1(E,m)$ is polynomial. This notion is independent of the choice of coordinates on E.

Theorem

Let $\mathfrak{g}$ be a Lie algebra of point vector fields on $J^0(E,m)$ and D a $\mathfrak{g}^{(k)}$-invariant polynomial divisor on $J^k(E,m)$. Then $[D] = \pi_{k,1}^* L$ for some $\mathfrak{g}^{(1)}$-equivariant line bundle $L \to J^1(E,m)$.

Proof.

Let $D = \{f_\alpha\}$ be a polynomial divisor on $J^k(E,m)$ defined on the open cover $\{\pi_{k,1}^{-1}(U_\alpha)\}$. The transition functions $g_{\alpha\beta} = f_\alpha / f_\beta \in \mathcal{O}^\times(\pi_{k,1}^{-1}(U_\alpha \cap U_\beta))$ are independent of jet variables of order ≥ 2.

$$(y\partial_x)^{(4)} = y\partial_x - y_1^2 \partial_{y_1} - \underbrace{3y_1y_2}_{\text{w. deg. 2}} \partial_{y_2} - \underbrace{(4y_1y_3 + 3y_2^2)}_{\text{w. deg. 4}} \partial_{y_3} - \underbrace{(5y_1y_4 + 10y_2y_3)}_{\text{w. deg. 5}} \partial_{y_4}$$

If f_α has weighted degree d, then $X^{(k)}(f_\alpha)$ has weighted degree $\leq d+1$ for any $X \in \mathfrak{g}$. Since $X^{(k)}(f_\alpha) = \lambda_\alpha(X) f_\alpha$, it follows that $\lambda_\alpha(X)$ has weighted degree ≤ 1, and is therefore a function on $J^1(E,m)$. □

Thank you for your attention!