Numerical Simulation of Sawtooth Solutions of Burgers Equation on a Finite Interval

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The problem

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Initial value (IV) $u(x,0) = u_0$

Boundary problem (BP) $u(0,t) = u_0 + a\sin(\omega t), \quad u_x(L,t) = 0$

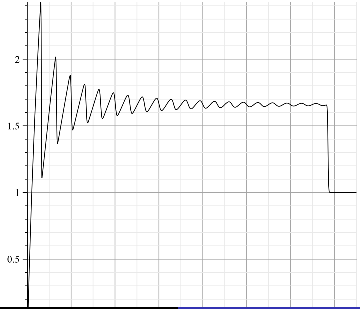
For a half-line interval $x \in [0; \infty)$ and a periodic perturbation at $x = 0$ of the form $u(0,t) = u_0 + a\sin(\omega t)$ the asymptotic of the corresponding solution is (Fay [1])

$$u = \frac{a}{R}\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{\sinh(n(1+X)/2\cdot R};$$

here R is the Reynolds number, $\theta = \omega(t - x/u_0)$.
The graph of such a sawtooth-shaped wave typically has the form presented on the next slide.

Sawtooth-shaped Waves

There can be seen two effects of the viscosity. First, the amplitude of fluctuations decreases exponentially and $\lim_{x\to\infty} = u_0$. Second, breaks of the u_x-derivative (sharp teeth) are concentrated near the source (at $x = 0$) and quickly disappear for greater x.



Development of sawtooth-shaped Waves

Consider a problem

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} - 2uu_x \\ u(x,0) &= 1, \quad u(0,t) = 1 + 2\sin(2\pi t), \quad u_x(L,t) = 0, \end{aligned}$$

where $\varepsilon = 0.5$ and $L = 75$. Four next figures show the steps of the passage from the constant initial condition to a sawtooth profile The viscosity corresponding to $\varepsilon = 0.5$ is relatively strong and sawteeth show only in vicinity of the perturbation source at the left end; as a spatial decay proceeds a single shock wave moving to the right develops. This shock profile is a well known Galilean-invariant travelling wave solution

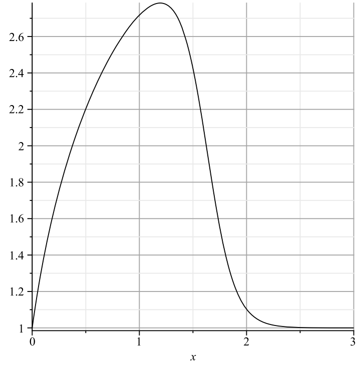


Figure: $t = 0.5$

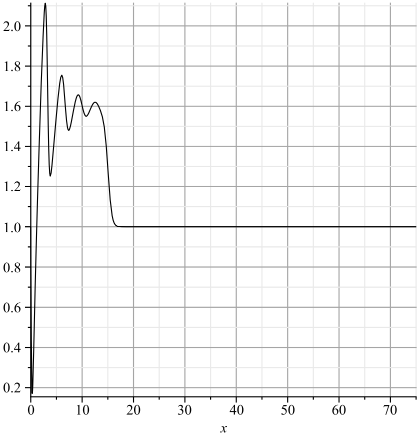


Figure: $t = 5$

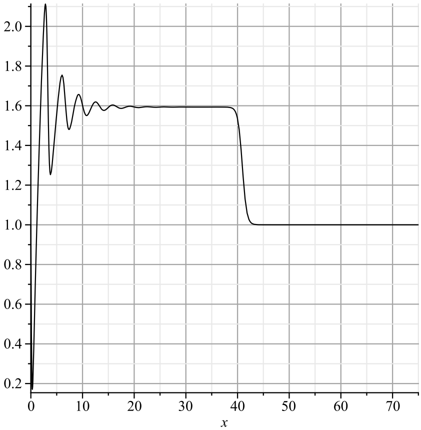


Figure: $t = 15$

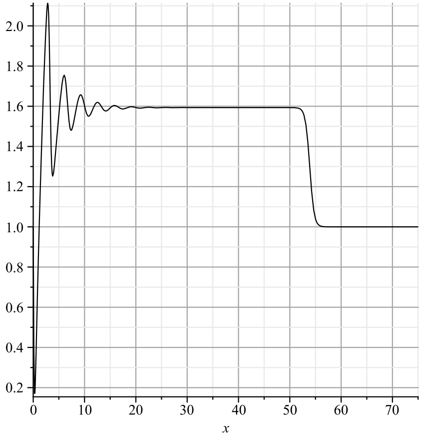
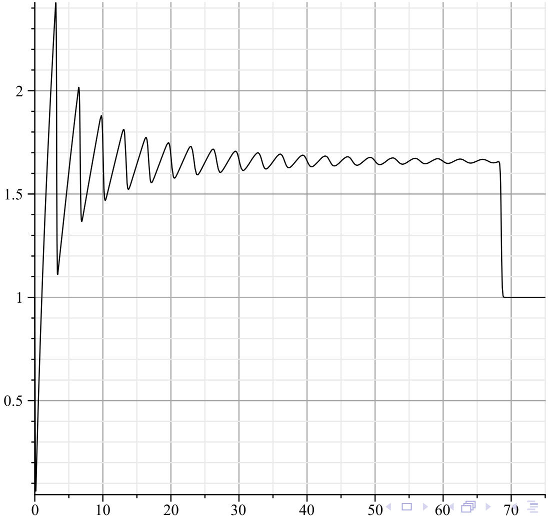


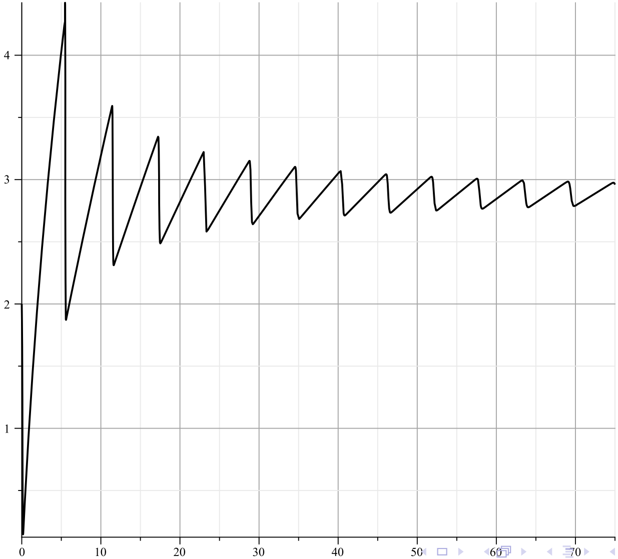
Figure: $t = 25$

For a lesser viscosity, $\varepsilon = 0.2$, the sawtooth effect stays for longer as it can be seen below. The IVBP are the same as in previous examples. The figure illustrates visually the process of gradual transformation of sharp teeth with almost vertical front edges into smooth and sloping oscillations.



For still lesser $\varepsilon = 0.15$ teeth are sharp throughout the interval; here

$$u(x,0) = 2, \quad u(0,t) = 2 + 3\sin(2\pi t), \quad u_x(75,t) = 0.$$



Amplitudes

Since the Burgers equation may be written as a conservation law, $(u)_t = (\varepsilon^2 u_x - u^2)_x$, we have

$$\oint_{\partial\mathcal{D}} [u\,dx + (\varepsilon^2 u_x - u^2)\,dt] = 0,$$

where $\mathcal{D}$ is a rectangle

$$\{0 \le x \le L,\ 0 \le t \le T\}$$

Bearing in mind the initial value/boundary conditions the integral reads

$$\int_0^L u(x,0)\,dx + \int_0^T [\varepsilon^2 u_x(L,t) - u^2(L,t)]\,dt +$$

$$\int_L^0 u(x,T)\,dx + \int_T^0 (\varepsilon^2 u_x(0,t) - u^2(0,t))\,dt = 0$$

Let A be a mean of the solution over $[0, L]$; because of dampening effect of the viscosity for large enough L and T the solution becomes constant $u(L, T) = A$. Using $u_x(L, t) = 0$ we deduce

$$u_0 L - A^2 T - AL - \int_0^T (\varepsilon^2 u_x(0, t) - u^2(0, t))\, dt = 0$$

Dividing by T we obtain the quadratic equation on A:

$$-A^2 - A\frac{L}{T} - \frac{1}{T}\int_0^T (\varepsilon^2 u_x(0, t) - u^2(0, t))\, dt + u_0\frac{L}{T} = 0$$

Since integrals of sine and cosine over their period are zero, for $T \gg 0$ we have $\frac{1}{T}\int_0^T (\varepsilon^2 u_x(0, t) - u^2(0, t))\, dt + u_0\frac{L}{T} \approx -u_0^2 - \frac{a^2}{2}$, so

$$-A^2 - Ak + u_0^2 + \frac{a^2}{2} + u_0 k = 0 \qquad (2)$$

where $k = \frac{L}{T}$. It follows $A = (-k + \sqrt{k^2 + 4u_0^2 + 2a^2 + 4u_0 k})/2$.

Compare to numeric results

Thus

$$\lim_{T\to\infty} A = \lim_{k\to 0} A = \sqrt{u_0^2 + a^2/2}$$

and

$$\lim_{L\to\infty} A = \lim_{k\to\infty} A = u_0$$

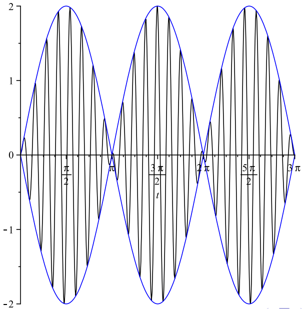
— as it should for a half-line problem.
As it was illustrated by previous figures, in the region where the initial harmonic oscillations decay, the perturbation transforms to a travelling wave with constant speed and profile, that is like an invariant solution of the form $u = -b/2f - \varepsilon^2 f \tanh(c + fx + bt)$. The velocity of this travelling wave solution is $V = -b/f$. On the other hand, as it can be seen on the graphs, the *tanh* function's shift upwards is $(A + u_0)/2$. So $V = A + u_0$ is an estimate for the velocity of the signal propagation.
For figures 2–3 we have $u_0 = 1$, $a = 2$, $A \approx \sqrt{3}$ and $V \approx 1 + \sqrt{3}$ in good conformity with numerical results.

A packet

One more example deals with interaction between periodic perturbations with different frequencies. Consider the following problem ($\varepsilon = 0,2$):

$$u(0,t) = 1 + 2\sin(t)\sin(5\pi t), \quad u(x,0) = 1, \quad u_x(75,t) = 0 \quad (3)$$

Note that $u(0,t)$ is a superposition since $2\sin(t)\sin(5\pi t) = \cos(t - 5\pi t) - \cos(t + 5\pi t)$, a wave packet; it is presented on the graph below, followed by the resulting solution.



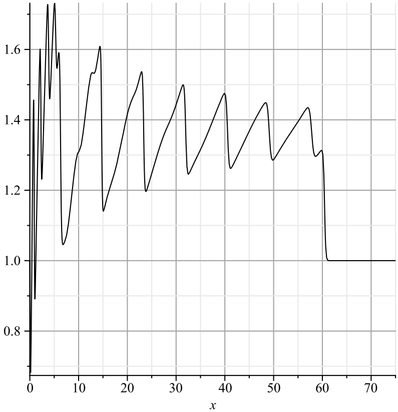


Figure: $\varepsilon = 0.2,\ t = 25$

Note that the frequency of the resulting sawtooth solution presented on this graph 9 is twice that of the envelope of the packet. It can be readily seen since $t = 25$ roughly corresponds to 7 periods.

There is a probable explanation:
only greatest peaks survive in viscous medium as they move faster and decay slower. They define the profile of the solution far from origin of perturbation; yet such peaks occur in every single packet of the length π, while the envelope $1 + 2\sin t$ period is 2π.

The estimate for the average at $t \to \infty$ in this case is

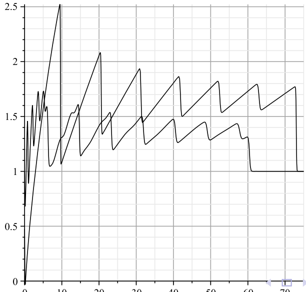
$$\sqrt{\frac{-1}{2\pi}\int_0^{2\pi}(\varepsilon^2((1+2\sin(t)\sin(5\pi t)) - (1+2\sin(t)\sin(5\pi t))^2))\,dt}$$

$$= \sqrt{2}$$

It is not, however, that the solution on the last graph tends to the solution with the twice-frequency envelope as a boundary perturbation. Indeed, the solution of the problem

$$u(0,t) = 1 + 2\sin(2t), \quad u(x,0) = 1, \quad u_x(75,t) = 0 \qquad (4)$$

is presented on the following graph together with the solution for the problem (3). The former solution has a higher initial peak and therefore is faster: at $t = 25$ it runs up (approximately) to 73 while the former only to 60. Also, an estimate for the average at $t \to \infty$ of the problem (4) is $\sqrt{3}$ which differs significantly from 1.41. More detailed analysis of this type of interactions will appear elsewhere.



Boundary - initial conditions

$$u(0,t) = 1 + 2\sin(t) + 3\cos(5\pi t), \quad u(x,0) = 1, \quad u_x(75,t) = 0$$

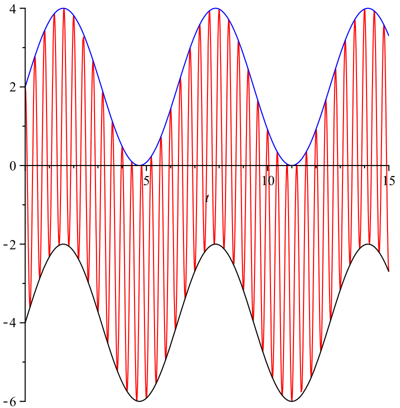
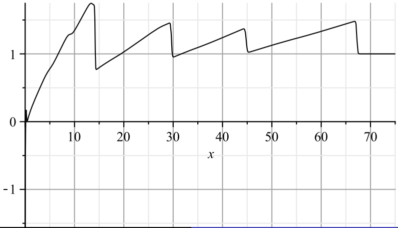


Figure: $\varepsilon = 0.2, \ t = 25$

The resulting period here is 2π coinciding with that of the envelope. Asymptotics for amplitude A is calculated using the full quadratic equation since neither $k \to 0$ nor $k \to \infty$ holds. Here $k = \frac{L}{T} = \frac{75}{25} = 3$

$$-A^2 - 3A - \frac{1}{25}\int_0^{25} (\varepsilon^2 u_x(0,t) - u^2(0,t))\, dt + 3u_0 = 0 \Rightarrow A \approx 1.1$$



Conclusion and numeric considerations

The figures in this paper were generated numerically using Maple PDETools package. The method used is Crank-Nicholson. The centered time centered space method is an implicit single stage method that can be used to find solutions to PDEs that are first order in time, and arbitrary order in space, with mixed partial derivatives. The method is $O(h^2, k^2)$ accurate. This implicit scheme is unconditionally stable for many problems (though this may need to be checked).

It is worth to notice that the presence of points of derivative's discontinuity is intrinsic for the model considered. Yet the standard procedures used with the default parameters may easily loose stability at these points. This instability leads to multi-oscillations and a general loss of precision. This problem was dealt with mainly by adapting (that is, shortening) the *spacestep* and/or *timestep* parameters. The spacing must be small enough that a sufficient number of points are in the spatial domain for the given method, boundary conditions, and spatial interpolation.

References

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