Conservation laws in action: an approach and implementations

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DIFFIETIES, COHOMOLOGICAL PHYSICS AND OTHER ANIMALS

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Equations

Korteweg-de Vries equation:

$$u_t = 2uu_x + u_{xxx}.$$

Korteweg-de Vries- Burgers equation:

$$u_t = u_{xx} + 2uu_x + u_{xxx}.$$

Generalised Korteweg-de Vries- Burgers equation (nonhomogeneous media):

$$u_t = g(x)u_{xx} + 2uu_x + f(x)u_{xxx}.$$

Here $g(x)$ and $f(x)$ are viscosity/dissipation and dispersion coefficients respectively.

KdV, $u_t = u_{xx} + 2uu_x + u_{xxx}$.

KdV possess

- Compact travelling wave solutions (solitons) $u(x,t) = 6a^2 \operatorname{sech}^2(4a^3 t + a(x+s))$;
- infinitely many conservation laws.
  The first four conserved quantities for KdV are

$$\begin{aligned}
I_1(u)) &= \int_{-\infty}^{+\infty} u(x,t)\,dx \text{ — mass},\\
I_2(u) &= \int_{-\infty}^{+\infty} u^2(x,t)\,dx \text{ — momentum},\\
I_3(u) &= \int_{-\infty}^{+\infty} \left(2u^3(x,t) - 3(u_x(x,t))^2\right)\,dx \text{ — energy},\\
I_4(u) &= \int_{-\infty}^{+\infty} \left(5u^4 - 30u(u_x)^2 + 9(u_{xx})^2\right)\,dx,
\end{aligned}$$

If $u(x,t)$ is a KdV solution then $\dfrac{\partial}{\partial t} I_k(u) = 0$, that is $u(x,0) = f(x) \Rightarrow I_k(u) = I_k(f)$, and $I_k(f)$ is constant in time. In particular, for solitons $I_k(6a^2 \operatorname{sech}^2(a(x+s) + 4a^3 t)) = I_k(a)$,

$$
\begin{aligned}
I_1(a) &= \int_{-\infty}^{+\infty} 6a^2 \operatorname{sech}^2(ax)\, dx = 12a, \qquad (1)\\
I_2(a) &= \int_{-\infty}^{+\infty} (6a^2 \operatorname{sech}^2(ax))^2\, dx = 48a^3,\\
I_3(a) &= \frac{1728}{5} a^5,\\
I_4(a) &= \frac{20736}{7} a^7,\\
\dots \quad & \dots \quad \dots\\
I_l(a) &= K_l a^{2l-1}.
\end{aligned}
$$

There is a simple recurrent procedure to generate $I_k(u) \to I_{k+1}(u)$ using the bi-hamiltonian structure of KdV.

For the KdV of the form $u_t = u_{xxx} + 2uu_x$ the hamiltonian operators are D and $(D^3 + uD + u_x)$, where D is a total derivative with respect to x.

We assume that the initial data $u(x,0) = f(x)$ is bounded and has a compact support.
Then the asymptotic form (at $t \to \infty$) of the N-soliton solution is

$$\sum_{i=1}^{N} 6a_i^2 \operatorname{sech}^2(a_i x + p_i + 4a_i^3 t) + R(x,t),$$

where $R(x,t)$ is a tail, and phase shifts are given by the formula

$$p_i = \frac{1}{2}\log\left(\frac{\gamma_i}{2a_i} \cdot \prod_{j=i+1}^{N} \left(\frac{a_j - a_i}{a_j + a_i}\right)^2\right).$$

Here $\{-a_i^2\}$ is the the discrete specter of the differential operator $-\frac{d^2}{dx^2} + f(x)$ and γ_i are the norming constants from the inverse scattering procedure. For an arbitrary $f(x)$ this data is hard to get, so estimations based solely on conserved quantities may be useful.

Initial data producing 4 and 5-soliton splitting

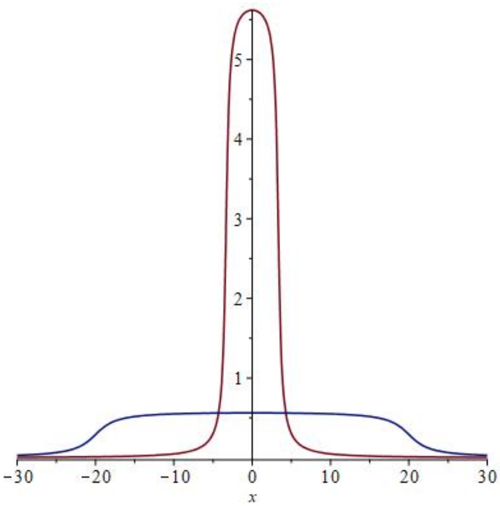


Figure: The upper red line splits into 4 solitons and the lower one into 5

Splitting of $g(x) = 3(\tanh(3(x+4)) - \tanh(3(x-4)))$

Splitting of $f(x) = 0.4(\tanh(x+15) - \tanh(x-15))$, movie

Reflectionless splitting

Since after some deliberation $q(x)$ splits (at least numerically) into a disconnected sum of N different-speed solitons S_a, we get

$$
\begin{aligned}
I_1(q) &= \int_{-\infty}^{+\infty} q(x)\,dx = \sum_{i=1}^{N} \int_{-\infty}^{+\infty} S_a\,dx = 12\sum_{i=1}^{N} a_i \\
I_2(q) &= \int_{-\infty}^{+\infty} q^2(x)\,dx = 48\sum_{i=1}^{N} a_i^3 \\
I_3(q) &= \int_{-\infty}^{+\infty} \left(2q^3(x) - 3(q_x(x))^2\right)\,dx = \frac{1728}{5}\sum_{i=1}^{N} a_i^5 \\
I_4(q) &= \int_{-\infty}^{+\infty} \left(5q^4 - 30q(q_x)^2 + 9q_{xx}^2\right)\,dx \\
&= \frac{20736}{7}\sum_{i=1}^{N} a_i^7 \qquad (2) \\
\ldots & \quad \ldots
\end{aligned}
$$

Thus we obtain the system on $a_i,\ i = 1 \dots N$:

$$K_j \sum_{i=1}^{N} a_i^{2j+1} = I_j(q),\ j = 1 \dots N,$$

where K_j is the constant specific to the j-th conserved quantity and $a_1 > a_2 > \dots a_N > 0$ is assumed.

The above equations hold for all $j = 1 \dots \infty$, but to find N solitons it suffice to consider only first N equations.

General case

$$
\begin{aligned}
I_1(q)) &= 12\sum_{i=1}^{N} a_i + \int_{-\infty}^{+\infty} R(x,t)\,dx \\
I_2(q) &= 48\sum_{i=1}^{N} a_i^3 + \int_{-\infty}^{+\infty} R^2(x,t)\,dx \\
I_3(q) &= \frac{1728}{5}\sum_{i=1}^{N} a_i^5 + \int_{-\infty}^{+\infty} \left(2R^3(x,t) - 3(R_x(x,t))^2\right)\,dx \\
\dots & \quad \dots
\end{aligned}
$$

It follows that the discrepancies $I_j(R(x,t)) = I_j(q) - K_j\sum_{i=1}^{N} a_i^{2j+1}$ are also constant.

Signs of disrepancies

The first four of $I_j(R)$ are alternating in sign.

Indeed, at least the initial perturbation mass is carried away by solitons, so $I_1(R) \leqslant 0$.

Since momentum of any part of solution is non-negative, it follows that $I_2(R) \geqslant 0$.

The reflected tail is oscillating around zero value, therefore

$$\int_{-\infty}^{+\infty} \left(2R^3(x,t)\right)\, dx \text{ is small while } \int_{-\infty}^{+\infty} \left(-3(R_x(x,t))^2\right)\, dx$$

is negative and comparatively large; so $I_3(R) \leqslant 0$.

Signs of disrepancies

Similarly plausible argument can be applied to $I_4(R)$ if the conservation law is rewritten to equivalent quadratic form $5u^4 - 30uu_x^2 + 9u_{xx}^2 \sim$

$$5u^4 + 15u^2 u_{xx} + 9u_{xx}^2 = 9(u^2 + \frac{5+\sqrt{5}}{6}u_{xx})(u^2 + \frac{5-\sqrt{5}}{6}u_{xx});$$

Hypothesis The whole series of conserved quantities for a radiation tail is alternating in signs.

Admissible domain

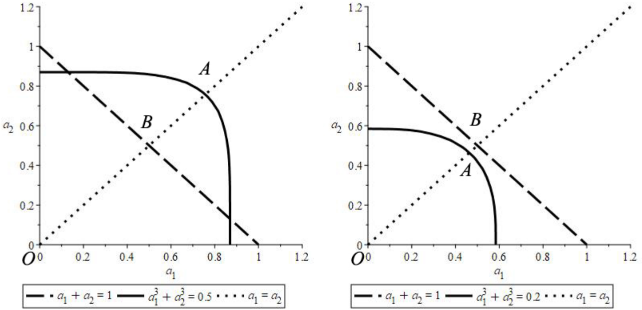
Necessary conditions

$$
\begin{aligned}
I_1(q) &\leqslant 12 \sum\nolimits_{i=1}^{N} a_i \\
I_2(q) &\geqslant 48 \sum\nolimits_{i=1}^{N} a_i^3 \\
I_3(q) &\leqslant \frac{1728}{5} \sum\nolimits_{i=1}^{N} a_i^5 \\
I_4(q) &\geqslant \frac{20736}{7} \sum\nolimits_{i=1}^{N} a_i^7 \\
\dots & \quad \dots; \\
a_i &\geqslant 0.
\end{aligned}
$$

The system (3) defines the admissible domain in $\{a_1, a_2, \dots, a_n\}$, the solitons'parameters space.

Number of solitons after splitting

In order for this domain not to be empty a number of inequalities must hold, for instance for $n = 2$, we have
$a_1 + a_2 \geqslant p_1$, $a_1^3 + a_2^3 \leqslant p_3$ (here $p_k = K_j^{-1} I_j(q)$, $j = 2k - 1$).



The admissible domain would be nonempty if $OA > OB$ as on the graph (above left), where $p_1 = 1, p_2 = 0.5$. For both points A and B $a_1 = a_2$, so $OA = \sqrt{2\left(\frac{p_1}{2}\right)^2} = \sqrt{\frac{1}{2}}$ and $OB = \sqrt{2\left(\frac{p_2}{2}\right)^2} = \sqrt{\frac{1}{8}}$

On the right graph $p_1 = 1, p_2 = 0.2$ and admissible domain is empty. We must increase the number of solitons to n. Then $a_1 = a_2 = \dots a_n = \frac{1}{n}$ for A and $a_1^3 = a_2^3 = \dots = a_n^3 = \frac{0.2}{n}$. If require $OA^2 = n\left(\frac{1}{n^2}\right) \geqslant OB^2 = n\left(\sqrt[3]{\frac{0.2}{n}}\right)^2 \Rightarrow n^2 > 5 \Rightarrow n = 3$.
For arbitrary p_1, p_2 the smallest number of solitons is the integer n such that

$$n \geqslant \sqrt{\frac{p_1^3}{p_2}}. \tag{3}$$

For other conserved quantities similar conditions of non-emptiness of the admissible domain lead to compare $n^{k-1} \vee \frac{p_1^k}{p_2}$. However usually (eg, for all examples below) it suffice to use (3) to predict the right number of resulting solitons.

Remark: The intersection points correspond to reflectionless splitting; their coordinates may be used as a estimations of the solitons' train parameters.

Back to videos

In the case of the second video

$$I_1(q) = 24,\ \ I_2(q) = 18.56,\ \ I_3(q) = 27.904,\ \ I_4(q) = 55.637.$$

$$p_1 = 24/6,\ \ p_2(q) = 18.56/48, \ldots.$$

The number of solitons

$$n > \sqrt{\frac{p_1^3}{p_2}} = \sqrt{\frac{2^3}{0.3867}} \approx 4.5 \Rightarrow n = 5$$

Note that the system for 4 solitons $\sum_{i=1}^{4} a_i^{2j-1}$, $j = 1, \ldots, 4$ has no solutions.

Non-homogeneous media. Model

Consider equation

$$u_t(x,t) = 2uu_x + f'(x)u_{xx} + f(x)u_{xxx} = \left(u_x^2 + f(x)u_{xx}\right)_x . \quad (4)$$

Since the equation has a form $u_t = F_x(u)$, the mass $\int_{-\infty}^{+\infty} u\,dx$ is conserved.
But the momentum $\langle u^2 \rangle = \displaystyle\int_{-\infty}^{+\infty} u^2\,dx$ does not:

$$\tfrac{1}{2}\langle u^2\rangle_t = \langle uu_t\rangle = \langle u(u^2 + f(x)u_{xx})_x\rangle = \langle 2u^2 u_x\rangle + \langle u(f(x)u_{xx})_x\rangle$$

$$\tfrac{2}{3}u^3\Big|_{-\infty}^{+\infty} + f(x)uu_{xx}\Big|_{-\infty}^{+\infty} - \langle u_x f(x)u_{xx}\rangle = -\tfrac{1}{2}\, f(x)u_x^2\Big|_{-\infty}^{+\infty} + \langle f'(x)u_x^2\rangle,$$

$$\langle u^2\rangle_t = 2\langle f'(x)u_x^2\rangle. \quad (5)$$

The choice of $f(x)$

We consider examples where $f(x) > 0$ is numerically constant outside a finite neighborhood of origin.

If $f(\pm\infty) = \gamma_\pm$, the equation at $x \to \pm\infty$ comes to

$$u_t = 2uu_x + \gamma_\pm u_{xxx}$$

These are KdVs, whose solitons are given by

$$6\gamma_\pm a^2 \operatorname{sech}^2(a(x+s) + 4\gamma_\pm a^3 t);$$

and they move to the left.

If start with a KdV_+ soliton to the right of the above neighborhood, it crosses the transient region and becomes

KdV_- soliton or splits into a number of them.

The number of solitons and their parameters may be evaluated using the monotonicity of the momentum evolution (5).

Conservation laws restrictions, single soliton

Suppose the problem

$$\{u_t = (u^2 + f(x)u_{xx})_x, \quad f(+\infty) = \gamma_0, f(-\infty) = \gamma_1\}$$

has a solution $u(x,t)$, such that
$u(x,0) = 6a_0^2\gamma_0 \operatorname{sech}^2(a_0((x+s)+4a_0^2 t))|_{t=0}$
and at $t \gg 0$, $u(x,t)$ it coincides with a single soliton, possibly plus reflected wave.
Let this soliton has the amplitude $6a_1^2\gamma_1$.
If it is plausible to ignore a reflected wave then

$$\begin{aligned} 12a_1\gamma_1 &= 12a_0\gamma_0 && -\text{mass conservation;} \\ 48a_1^3\gamma_1^2 &> 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \geqslant 0; \\ 48a_1^3\gamma_1^2 &< 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \leqslant 0. \end{aligned} \tag{6}$$

Then $\frac{a_1}{a_0} = \frac{\gamma_0}{\gamma_1}$ and refraction coefficient $R = \frac{V_1}{V_0} = \frac{a_1^2\gamma_1}{a_0^2\gamma_0} = \frac{\gamma_0}{\gamma_1}$

Conservation laws restrictions, 3

Suppose the problem

$$\{u_t = (u^2 + f(x)u_{xx})_x, \quad f(+\infty) = \gamma_0, f(-\infty) = \gamma_1\}$$

has a solution $u(x,t)$, such that
$u(x,0) = 6a_0^2\gamma_0 \operatorname{sech}^2(a_0((x+s)+4a_0^2 t))|_{t=0}$
and at $t \gg 0$, $u(x,t)$ it coincides with a **bi-soliton**, possibly plus reflected wave.
Let bi-soliton consist of peaks with amplitude $6a_1^2\gamma_1$ and $6a_2^2\gamma_1$.
If it is plausible to ignore a reflected wave then

$$\begin{aligned}
&12a_1\gamma_1 + 12a_2\gamma_1 = 12a_0\gamma_0 && -\text{mass conservation;}\\
&48a_1^3\gamma_1^2 + 48a_2^3\gamma_1^2 > 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \geqslant 0;\\
&48a_1^3\gamma_1^2 + 48a_2^3\gamma_1^2 < 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \leqslant 0.
\end{aligned} \tag{7}$$

Conservation laws restrictions, 4

Denote

$$y = \frac{a_1\gamma_1}{a_0\gamma_0}, \quad z = \frac{a_2\gamma_1}{a_0\gamma_0}, \quad k = \frac{\gamma_1}{\gamma_0};$$

the (7) may be rewritten to the form

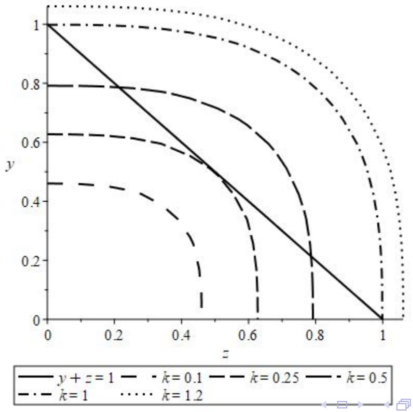
$$\begin{aligned} y + z &= 1 && -\text{mass conservation;} \\ y^3 + z^3 &> k && -\text{impulse evolution if } k < 1; \\ y^3 + z^3 &< k && -\text{impulse evolution if } k > 1. \end{aligned} \tag{8}$$

The solution of the system $\{y + z = 1,\ y^3 + z^3 = k\}$ is $\{\frac{1}{2} \pm \frac{1}{6}\sqrt{12k-3}, \frac{1}{2} \mp \frac{1}{6}\sqrt{12k-3}\}$. Since obviously $0 \leqslant y,\ z \leqslant 1$, it make sense only for $\frac{1}{4} \leqslant k \leqslant 1$, see the next figure.
In this case for the first (greater) peak it follows that

$$1 > y = \frac{a_1\gamma_1}{a_0\gamma_0} > y_+ = \frac{1}{2} + \frac{1}{6}\sqrt{12k-3}.$$

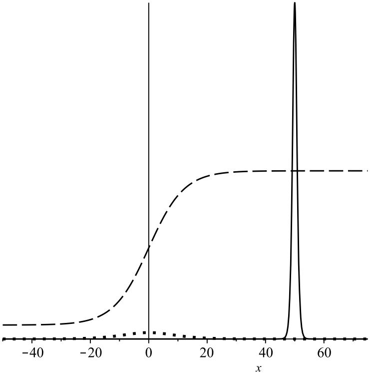
Since the refraction coefficient $R = \frac{V_1}{V_0} = \frac{4a_1^2\gamma_1}{4a_2^0\gamma_0}$ equals $\frac{y^2}{k}$ we obtain the restriction on the first peak refraction coefficient (it also coincides with the amplitudes ratio)

$$R > \frac{2k + 1 + \sqrt{12k - 3}}{6k}.$$



Example 1

The decreasing (with respect to the direction of the soliton motion) dispersion coefficient $f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right)$ in $u_t = \left(u_x^2 + f(x)u_{xx}\right)_x$. Thus $u_t = 2uu_x + u_{xxx}$ at $x = +\infty$ and $u_t = 2uu_x + \frac{1}{12}u_{xxx}$ at $x = -\infty$.



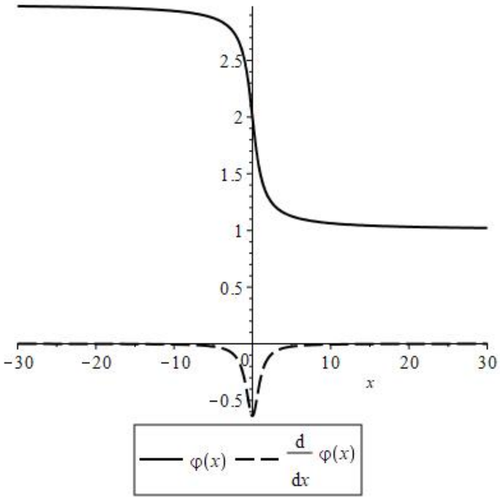
Nonhomogeneous distribution $f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right)$ and the initial soliton $6\,\mathrm{sech}^2(4t + x - 50)$.

Soliton in the case of $f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right)$

No reflected wave can be seen. The stable height of the first peak is 18.5 high. The height of the second one is about 0.37 .

Example 2. Transmitted solitary wave

Increasing (with respect to the soliton motion) dispersion coefficient $\varphi(x) = \frac{2}{3}\left(1 + \frac{1}{\pi}\arctan(x)\right)$ in $u_t = \left(u_x^2 + \varphi(x)u_{xx}\right)_x$. Thus $u_t = 2uu_x + u_{xxx}$ at $x = +\infty$ and $u_t = 2uu_x + \frac{1}{3}u_{xxx}$ at $x = -\infty$.



$\varphi(x) = \frac{2}{3}\left(1 + \frac{1}{\pi}\arctan(x)\right)$

Decay velocity

Let $\mathbf{E}(\mathbf{u}) = 0$ be a system of equations describing an ideal media state (i.e., without dissipation).

A scalar H depending on $\mathbf{u}$ and its derivatives is a conservation law if an integral of H over some fixed spatial domain, denoted by $\langle H \rangle$, is independent of time: $\left.\frac{\partial \langle H \rangle}{\partial t}\right|_{\mathbf{E}} = 0$.

Here the restriction to $\mathbf{E}$ means that $\frac{\partial \langle H \rangle}{\partial t} = 0$ on any solution $\mathbf{u}(x)$ of $\mathbf{E} = 0$.

With dissipation taken into account, the quantity H is constant no more and $\frac{\partial \langle H \rangle}{\partial t} \neq 0$ is called the decay velocity of H
A dissipative media state usually satisfies the equation $\mathbf{E}(\mathbf{u}) + \eta \mathbf{F}(\mathbf{u}) = 0$, where η is some small parameter; for $\eta = 0$ we get the ideal state equation.

The decay velocity depends on the additional summand $\eta\,\mathbf{F}(\mathbf{u})$.

A *conservation law* for the equation $\mathbf{E}$ is a differential $n-$ form $\omega = \sum_{i=0}^{n} \omega_i \hat{d}x_i$, such that $d\omega = 0$ on $\mathcal{E}^\infty$; here $\hat{d}x_i = dx_0 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$ and ω_i's are some functions on $J^\infty(\pi)$.

In established terminology ω_0 is called a conserved density, while $(-\omega_1, \omega_2, \ldots, (-1)^n \omega_n)$ is called a flux.)
The method for finding of conservation laws is as follows. Let $\ell^*_{\mathbf{E}}$ be a formal conjugate of $\ell_{\mathbf{E}}$, the linearizaton of $\mathbf{E}$.
Solutions of the equation

$$\ell^*_{\mathbf{E}}(\psi)|_{\mathcal{E}^\infty} = 0 \tag{9}$$

are called *generating functions* of conservation laws.

Generating functions

They are connected to conservation laws themselves in a following way. By the definition of a conservation law, the equation $d\omega|_{\mathcal{E}^\infty} = 0$ holds, which is equivalent to

$$d\omega = \mathcal{O}(\mathbf{E}) dx_0 \wedge \cdots \wedge dx_n,$$

where $\mathcal{O}(\mathbf{E}) \in \mathcal{J}$. $\mathcal{O}(\mathbf{E}) = \sum_{\sigma,\, r} \mathcal{O}_\sigma D_\sigma(E_r)$.

Now $\mathcal{O}^*(1)$ is the generating function of this conservation law or a solution of (9) (* stands for formal conjugation).

When the generating function is found, it remains however to find the conservation law itself and to check whether it is trivial (*trivial* by definition are are conservation laws ω which are exact, i.e., $\omega = dw$ for some $(n-1)$– differential form w).

Main theorem

Theorem

Let an equation $\mathbf{E}(\eta)$ *depends on a small parameter* η *in such a way that* $\mathbf{E}_0 = \mathbf{E}(0)$ *is a non–dissipative system. Let* $\omega = \sum_0^n \omega_i \hat{dx}_i$ *be the conservation law of* $\mathbf{E}_0$*. Then the decay velocity of the conserved quantity* $\langle \omega_0 \rangle$ *in presence of dissipation is given by*

$$\frac{d}{dt}\langle \omega_0 \rangle \bigg|_{\mathbf{E}(\eta)} = -\eta \langle \mathcal{O}^*(1) \cdot \frac{\partial \mathbf{E}}{\partial \eta}\bigg|_{\eta=0} \rangle \quad \text{up to } O(\eta^2)$$

Proof

For any domain $V \subset \mathbb{R}^{n+1}$ we have

$$\int_{\partial V} \omega = \int_{V} d\omega = 0 \quad \text{on } \mathcal{E}_0^\infty\,, \tag{10}$$

by the definition of a conservation law. If $x_0 = t$ and V is a cylinder over spatial domain S, $V = S \times [t_0,\, t_1]$,
then $Q = \int\limits_S \omega_0 dx_1 \wedge \cdots \wedge dx_n$ is a function of variable t and the former integral equals $Q(t) - Q(t_0)$ plus the flow of a vector $(-\omega_1, \ldots, (-1)^n \omega_n)$ through the $\partial S \times [t_0,\, t_1]$.
In the case this flow is trivial (such is a case when $S = \mathbb{R}^n$ and $\omega_i|_{\mathbf{E}}$ are functions rapidly decreasing at infinity), the function $Q(t)$ is constant, i.e. $Q(t)$ is a conserved quantity:

$$\frac{d}{dt} Q(t) = \frac{d}{dt} \int_S \omega_0 dx_1 \wedge \cdots \wedge dx_n \bigg|_{\mathcal{E}_0^\infty} = \frac{d}{dt} \langle \omega_0 \rangle = 0 \tag{11}$$

On the other hand

$$
\frac{d}{dt}\int\limits_S \omega_0 dx_1 \wedge \cdots \wedge dx_n = \frac{d}{dt}\int\limits_{\partial V} \omega = \frac{d}{dt}\int\limits_V d\omega =
$$

$$
\frac{d}{dt}\int\limits_V \left(\sum \mathcal{O}_\sigma D_\sigma(\mathbf{E}_0) \right) dt \wedge dx_1 \wedge \cdots \wedge dx_n =
$$

$$
\frac{d}{dt}\int\limits_{t_0}^{t}\int\limits_S \mathcal{O}^*(1)\mathbf{E}_0\, dt \wedge dx_1 \wedge \cdots \wedge dx_n = 0 \quad (12)
$$

Differentiation of the last integral by the upper limit t imply

$$
\frac{d}{dt}\int\limits_S \omega_0 dx_1 \wedge \cdots \wedge dx_n = \int\limits_S \mathcal{O}^*(1)\mathbf{E}_0\, dx_1 \wedge \cdots \wedge dx_n \qquad (13)
$$

The right-hand side of (13)) is zero on $\mathbf{E}_0 = 0$, but when restricting (13) to $\mathbf{E}(\eta) = \mathbf{E}_0 + \eta \cdot \mathbf{F} = 0$ we get

$$\frac{d}{dt} \int_S \omega_0 dx_1 \wedge \cdots \wedge dx_n = \int_S \mathcal{O}^*(1)(-\eta \mathbf{F})\, dx_1 \wedge \cdots \wedge dx_n \,, \quad (14)$$

or

$$\frac{d}{dt} \langle \omega_0 \rangle \Big|_{\mathbf{E}(\eta)} = -\eta \langle \mathcal{O}^*(1) \cdot \mathbf{F} \rangle \quad (15)$$

In a more general case of $\mathbf{E}(\eta) = \mathbf{E}_0 + \eta \cdot \frac{\partial \mathbf{E}}{\partial \eta} + O(\eta^2)$, it follows from (13) that

$$\frac{d}{dt} \langle \omega_0 \rangle \Big|_{\mathbf{E}(\eta)=0} = -\langle \eta \mathcal{O}^*(1) \cdot \frac{\partial \mathbf{E}}{\partial \eta} \Big|_{\eta=0} \rangle \quad \text{up to } O(\eta^2) \quad \square \quad (16)$$

Remark

If there is more than one small parameter, the formula (16) is readily generalized:

$$\frac{d}{dt}\langle\omega_0\rangle\Big|_{\mathbf{E}(\eta_1,\dots\eta_s)=0} = -\sum_{r=1}^{s}\langle\eta_r\mathcal{O}^*(1)\cdot\frac{\partial\mathbf{E}}{\partial\eta_r}\Big|_{\boldsymbol{\eta}=0}\rangle \quad \textit{up to } O(\eta^2) \tag{17}$$

Remark

It is noteworthy that in case of a ***system*** *it is possible for any given conserved quantity to add dissipative–like summands in such a way that this quantity still remains conserved: for any $\mathcal{O}^*(1)$ one can choose such an* **F** *that right–hand side of (15) will be zero.*

Remark

In case of evolution equation $\frac{\partial \mathbf{u}}{\partial t} = \mathbf{E}_0(\mathbf{u}) + \eta \mathbf{F}(\mathbf{u})$ *there is another explicit form of the decay velocity.*

If ω_0 *be an ideal conserved density,* $\frac{d\omega_0}{dt} \equiv \ell_{\omega_0}(\mathbf{E}_o) = 0 \quad$ *on* $\mathbf{E}_0$ *and* $\frac{d\omega_0}{dt} \equiv \ell_{\omega_0}(\mathbf{E}_0 + \eta \ell_{\omega_0}(\mathbf{F})$ *in presence of dissipation.*

Therefore

$$\frac{d\langle\omega_0\rangle}{dt} = -\eta\langle\ell_{\omega_0}(\mathbf{F})\rangle$$

is the decay velocity law for evolution equation.

Example from magnetohydrodynamics

The 3-dimensional MHD–equation, describing incompressible magnetofluids in dimensionless variables may be taken in the following form:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p^* + \mathbf{B} \cdot \nabla \mathbf{B} + \nu \nabla^2 \mathbf{v} \\ \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{B} \\ \nabla \cdot \mathbf{v} = 0 = \nabla \cdot \mathbf{B} \end{cases} \tag{18}$$

Here ν and η are reciprocal of mechanical and magnetic Reynolds numbers respectively;
$\mathbf{v}$ and $\mathbf{B}$ are velocity and magnetic fields
p^* is the total pressure.

It is assumed that mass density is constant and uniform, and that $\mathbf{v}$ and $\mathbf{B}$ are in Alfven speed units.

Equation (18) may be simplified in case of two spatial variables $(x,\ y)$ assuming $\frac{\partial}{\partial z} \equiv 0$. In this case $\mathbf{B} = (B_x, B_y, 0),\ \mathbf{v} = (v_x, v_y, 0)$.

Moreover $\mathbf{v} = \nabla \times \psi \mathbf{e}_z$ and $\mathbf{B} = \nabla \times \mathbf{a}$ for some stream function $\psi(x, y, t)$ and potential $\mathbf{a} = a(x, y, t)\mathbf{e}_z$.

Introduce dimensionless vorticity and current by $\nabla \times \mathbf{v} = \omega \mathbf{e}_z$, $\nabla \mathbf{B} = j\mathbf{e}_z$ where $j = -\nabla^2 a$ and $\omega = -\nabla^2 \psi$.

Then the last equation in the system (18) is automatically true, while the rest comes to

$$\begin{cases} \Delta u_t + u_x \Delta u_y - u_y \Delta u_x + v_y \Delta v_x - v_x \Delta v_y = \nu \Delta^2 u \\ v_t + u_x v_y - u_y v_x = \eta \Delta v \end{cases} \tag{19}$$

We restrict ourselves to low order conservation laws, that is to such a $\mathbf{f} = \begin{pmatrix} S \\ T \end{pmatrix}$ in (9) that S and T are functions on $J^0(\mathbb{R}^3, \mathbb{R}^2)$ and $J^2(\mathbb{R}^3, \mathbb{R}^2)$ respectively.

This choice may be understood by considering the structure of $\ell^*_{E_0}$ matrix: its second column is a first order operator while the first column is of third order.

Then the kernel of $\ell^*_{E_0}|_{\mathcal{E}_0^\infty}$ is linearly generated by

$$\begin{pmatrix} h(t) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x^2 + y^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p(t)x \\ 0 \end{pmatrix}, \quad \begin{pmatrix} q(t)y \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} u \\ \Delta u \end{pmatrix}, \quad \begin{pmatrix} f(v) \\ f'(v)\Delta v \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \Phi'(v) \end{pmatrix} \tag{20}$$

where h, p, q, f and Φ are arbitrary functions.

Conserved quantities

There are only three non–trivial conserved densities:

the total energy E (magnetic plus kinetic energy),

generalized 'cross helicity' H_c,

mean magnetic potential A,

$$\begin{aligned} E &= \frac{1}{2}\langle u_x^2 + u_y^2 + v_x^2 + v_y^2\rangle \\ H_c &= \langle f'(v) \cdot (u_x v_x + u_y v_y)\rangle \\ A &= \langle \Phi(v)\rangle \end{aligned} \tag{21}$$

Their generating functions are placed on the second line of (20) in respective order. Reall that f and Φ are arbitrary functions of v.

Decay rates

Once dissipation coefficients ν or η are have small but finite values, quantities (20) are conserved no more. Their decay rates are

$$\begin{aligned}
\frac{dE}{dt} &= -\int_S [\nu(\Delta u)^2 + \eta(\Delta v)^2]\, dxdy\ ; \\
\frac{dH_c}{dt} &= \frac{1}{2}\int_S [\nu f(v)\Delta^2 u + \eta f'(v)\Delta u \Delta v]\, dxdy = \\
&\quad -\frac{1}{2}(\nu+\eta)\int_S f'(v)\Delta u \Delta v\, dxdy - \frac{1}{2}\nu \int_S f''(v)\Delta u(v_x^2 + v_y^2)\, dxdy \\
\frac{dA}{dt} &= -\eta \int_S \Phi'(v)\Delta v\, dxdy = \eta \int_S \Phi''(v)(v_x^2 + v_y^2)\, dxdy
\end{aligned} \tag{22}$$

One can see that the decay of E is monotonic but those of H_c and A are not necessarily so.

Such an inequality in decay rates leads to a distinct physical phenomenon of 'self–organization' or quasi–stable states of plasma.

Depending on initial conditions competing processes called 'selective decay' and 'dynamic alignment' occur: in selective decay energy decays relatively to mean potential, and in dynamic alignment energy decays relatively to cross helicity (velocity and magnetic field being aligned.

There are also some more delicate possibilities of *self–organization*.

There exist a simple procedure for finding solutions of the described behavior. It was suggested in
J.B. Taylor. *Relaxation of toroidal plasma and generation of reverse magnetic fields.* Phys. Rev.Lett. **33**,1974 1139-1141,
and is known as 'Taylor trick'.

Let us minimize E with H_c and A as constrains. Put $\delta(E + \lambda H_c + \mu A) = 0$, A and H_c presumed constant, λ and μ being Lagrange multipliers.

The Euler–Lagrange equations are

$$\begin{cases} \Delta[u - F(v)] = 0 \\ \Delta v = f(v)\Delta u + g(v) \, , \end{cases} \tag{23}$$

where $F' = f$ and $g = \pm\Phi'$.

The system (23) generally is not compatible with (19).

But it is compatible if $\eta = \nu$ which is in particular true in the ideal case $\eta = \nu = 0$. In this case, combining (19) and (23) we get

$$\begin{cases} \Delta[u - F(v)] = 0 \\ \Delta v = \frac{ff'}{1-f^2}(v_x^2 + v_y^2) + \frac{g}{1-f^2} \\ v_t = u_y v_x - u_x v_y \\ (u_{xy} - fv_{xy})(v_x^2 - v_y^2) + [(u_{yy} - fv_{yy}) - (u_{xx} - fv_{xx})]v_x v_y = 0 \end{cases} \tag{24}$$

Solutions of (24) describe the quasistationary states with remarkable accuracy as it was demonstrated numerically for special types of f and Φ in

A.C. Ting, M.H. Matthaeus, D. Montgomery. *Turbulent relaxation processes in magnetohydrodynamics* Phys. Fluids, **29**, 1986,

Remark

The first and the last equations of (24) *form the closed system*

$$\begin{cases} \Delta w = 0 \\ z_t + w_x z_y - w_y z_x = 0 \end{cases},$$

where $w = u - F(v)$ *and* $z = v_x^2 + v_y^2$.

Remark

The second equation in (24) *may be written in a closed form* $\Delta R = \Psi(R)$ *where* $R = R(v)$, $R' = \sqrt{1 - f^2}$

Remark

The case of $u = F(v)$ in (24) is a generalization of dynamic alignment (aligned are gradients of u and v). It implies stationary solutions

$$\begin{cases} u = F(v) \\ v_t = 0 \\ \Delta R = \Psi(R)\,, \end{cases}$$

where $R'(v) = \sqrt{1 - f^2(v)}$ as in previous remark.

Thank you

for your attention