Variational, Symplectic and Hamiltonian Operators

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This is a summary of the paper with Emrullah Yasar:

Variational Operators, Symplectic Operators, and the Cohomology of Scalar Evolution Equations

1. Variational Operators and Symplectic Operators and the Variational Bicomplex
2. Coverings and reduction for Hamiltonian Evolution equations.

The Multiplier Problem in Calculus of Variations

Given a differential equation/system

$$\Delta(\mathbf{x}, u, \partial u) = 0$$

does there exists a function $A(\mathbf{x}, u, \partial u)$ and a Lagrangian $L(\mathbf{x}, u, \partial u)$ such that

$$A\ \Delta = \mathbf{E}(L)$$

here $\mathbf{E}$ is the Euler-Lagrange operator, and the function $A(\mathbf{x}, u, \partial u)$ is called the variational multiplier

Long history, going back to Helmholtz, and maybe even longer....

Example

For a 4th order ODE

$$\frac{d^4 u}{dx^4} = F(x, u, u_x, u_{xx}, u_{xxx})$$

admits a variational multiplier if and only if

$$0 = \frac{\partial^3 F}{\partial u_{xxx}^3}$$

$$0 = \frac{\partial F}{\partial u_x} + \frac{1}{2}\frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xxx}} - \frac{d}{dx}\frac{\partial f}{\partial u_{xx}} - \frac{3}{4}\frac{\partial f}{\partial u_{xxx}}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{2}\frac{\partial f}{\partial u_{xx}}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{8}\left(\frac{\partial f}{\partial u_{xxx}}\right)^3$$

This is about the best we can hope for.
Shown using the cohomology of the *variational bicomplex*.

The bicomplex approach also produces the multiplier A and Lagrangian L in a geometric way.

Variational Operator

The multiplier problem can be generalized : Given a differential equation

$$\Delta(\mathbf{x}, u, \partial u) = 0$$

does exists a differential operator $\mathcal{E}$ and a Lagrangian $L(\mathbf{x}, u, \partial u)$ such that

$$\mathcal{E}(\Delta) = \mathbf{E}(L).$$

If $\mathcal{E}$ is a function, then this is the variational multiplier problem as before. In general call $\mathcal{E}$ a variational operator.

The focus here will be scalar evolution equations $\Delta = u_t - K(t, x, u, u_x, \ldots, u_n)$. This is related to the Symplectic/Hamiltonian formulation of evolution equations.

Examples:

$$D_x(u_t - u_{xxx}) = u_{tx} - u_{xxxx} = \mathbf{E}\left(-\frac{1}{2}(u_t u_x + u_{xx}^2)\right)$$

$$tD_x\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) = \mathbf{E}\left(-\frac{1}{2}tu_x u_t + \frac{1}{2}tu_x u_{xxx} + \frac{1}{6}tu_x^3\right) \quad \text{PCKdV.}$$

Low Order Case : $u_t = K(t, x, u, u_x, u_{xx}, u_{xxx})$

We find using the bicomplex

Theorem

$u_t = K(t, x, u, u_x, u_{xx}, u_{xxx})$ admits a first order variational operator

$$\mathcal{E} = R(t, x, u, u_x, \ldots)D_x + \frac{1}{2}D_x R$$

if and only if the following is a trivial conservation law,

$$\begin{aligned}\kappa =& \hat{K}_2 dx + \\ & \left(-K_0 + K_1\hat{K}_2 - \frac{1}{2}(X(K_3)\hat{K}_2^2 + K_3\hat{K}_2^3) + X(K_3)X(\hat{K}_2) + K_3X^2(\hat{K}_2)\right) dt\end{aligned}$$

where $K_{,i} = \partial_i K$, $\hat{K}_2 = \frac{2}{3K_{,3}}(K_{,2} - X(K_{,3}))$, and X is the total x derivative

$$X = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \ldots,$$

Furthermore, when $\kappa = d_H(\log R)$ then $u_t = K$ admits the first order variational operator $\mathcal{E} = RD_x + \frac{1}{2}D_x R$.

Examples:

PCKdV: $u_t = u_{xxx} + \frac{1}{2}u_x^2 - \frac{u}{2t}$

$$\kappa = \frac{1}{t}dt = d_H \log t, \qquad \mathcal{E} = tD_x$$

KN/SCH-KdV: $u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}$

$$\kappa = -2\frac{u_{xx}}{u_x}dx + \frac{6u_x u_{xx} u_{xxx} - 2u_{xxxx}u_x^2 - 3u_{xx}^2}{u_x^3}dx = d_H(\log u_x^{-2})$$

$$\mathcal{E} = \frac{1}{u_x^2}D_x - \frac{u_{xx}}{u_x^3}$$

KdV : $u_t = u_{xxx} + uu_x$

$$\kappa = -2u_x dt$$

and κ is not a conservation law. No first order operator.

Part 1:

The Variational Bicomplex.

The Unconstrained Jet Space

Reference: Anderson , Kamran, *The Variational Bicomplex for Hyperbolic Second-order Scalar Partial Differential Equations in the Plane*.

On $J^\infty(\mathbb{R}^2, \mathbb{R}) = (t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots)$ the t and x total vector fields are

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} \ldots,$$
$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} \ldots.$$

With $u_i = D_x^i(u) = u_{xxxx\ldots}$, $u_{t,i} = D_x^i(u_t)$, the contact forms are

$$\vartheta^0 = du - u_t dt - u_x dx,$$
$$\vartheta^i = D_x^i(\vartheta^0) = du_i - u_{t,i}dt - u_{i+1}dx^{i+1}, \quad i \geq 1,$$
$$\vartheta^{a,i} = (D_t)^a D_x^i(\vartheta^0), \quad a \geq 1, i \geq 0.$$

$D_x^i(\vartheta^0)$ is the repeated Lie derivative.

The Unconstrained Bicomplex

The contact forms together with dt, dx give a coframe for $J^\infty(\mathbb{R}^2, \mathbb{R})$.

This gives rise to a bi-grading of forms

$$\Omega^{r,s}(J^\infty(\mathbb{R}^2, \mathbb{R})) \subset \Omega^{r+s}(J^\infty(\mathbb{R}^2, \mathbb{R}))$$

$r = 0, 1, 2$ - the degree of dt, dx or horizontal forms

$s \geq 0$ - the degree of contact forms or vertical forms.

Example: $\omega \in \Omega^{1,2}(J^\infty(\mathbb{R}^2, \mathbb{R}))$,

$$\omega = (tu_x + x^2 u)dt \wedge \vartheta^{1,1} \wedge \vartheta^2 + u_{tx} \sin(xt)dx \wedge \vartheta^{2,3} \wedge \vartheta^3$$

$$\vartheta^{1,1} = D_t D_x(\theta^0) = du_{tx} - u_{ttx}dt - u_{txx}dx,$$
$$\vartheta^2 = D_x^2(\theta^0) = du_{xx} - u_{txx}dt - u_{xxx}dx, \ldots$$

The Unconstrained Differentials d_H and d_V

The horizontal differential is an anti-derivation,

$$d_H : \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R})) \to \Omega^{r+1,s}(J^\infty(\mathbb{R}^2,\mathbb{R})),$$

computed using the Lie derivative $D_t(\omega)$ and $D_x(\omega)$, $\omega \in \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ by,

$$d_H\omega = dt \wedge D_t(\omega) + dx \wedge D_x(\omega).$$

The vertical differential is an anti-derivation

$$d_V : \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R})) \to \Omega^{r,s+1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$$

which satisfies,

$$d_V f(t,x,u,u_t,u_x,\ldots) = \frac{\partial f}{\partial u}\vartheta^0 + \frac{\partial f}{\partial u_t} D_t(\vartheta^0) + \frac{\partial f}{\partial u_x} D_x(\vartheta^0) + \ldots,$$
$$d_V \vartheta^{a,i} = 0, \qquad d_V dt = 0, \quad d_V dx = 0.$$

The important properties of d_H, d_V are

$$d_H^2 = 0, \qquad d_V^2 = 0, \qquad d_H d_V + d_V d_H = 0$$

The Unconstrained Variational Bicomplex.

The rows and columns of the unconstrained variational bicomplex are exact.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \Omega^{0,2}(J) & \xrightarrow{d_H} & \Omega^{1,2}(J) & \xrightarrow{d_H} & \Omega^{2,2}(J) \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \Omega^{0,1}(J) & \xrightarrow{d_H} & \Omega^{1,1}(J) & \xrightarrow{d_H} & \Omega^{2,1}(J) \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
\mathbb{R} & \longrightarrow & \Omega^{0,0}(J) & \xrightarrow{d_H} & \Omega^{1,0}(J) & \xrightarrow{d_H} & \Omega^{2,0}(J) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \Omega^{0}(\mathbb{R}^2) & \xrightarrow{d} & \Omega^{1}(\mathbb{R}^2) & \xrightarrow{d} & \Omega^{2}(\mathbb{R}^2)
\end{array}
$$

The Equation Manifold

Start with the function Δ whose zero set defines a scalar evolution equation,

$$\Delta = u_t - K(t, x, u, u_x, \dots, u_n), \qquad K \in C^\infty(J^\infty(\mathbb{R}^2, \mathbb{R})).$$

Let $\mathcal{R}^\infty = (t, x, u, u_x, u_{xx}, \dots)$ and $\iota : \mathcal{R}^\infty \to J^\infty(\mathbb{R}^2, \mathbb{R})$,

$$\iota = (t, x, u, u_x = u_x, u_t = K, u_{tt} = T(K), u_{tx} = X(K), u_{xx} = u_{xx}, \dots).$$

the inclusion of the infinite prolongation of $\Delta = 0$ where T and X are

$$\begin{aligned} T &= \partial_t + K\partial_u + X(K)\partial_{u_x} + X^2(K)\partial_{u_{xx}} + \dots\ , \\ X &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \dots \end{aligned} \tag{1.1}$$

and T, X are the restriction of D_t and D_x to $\mathcal{R}^\infty$,

The Pfaffian system $\mathcal{I}$ on $\mathcal{R}^\infty$ is the pullback of the contact system on $J^\infty(\mathbb{R}^2, \mathbb{R})$,

$$\begin{aligned} \theta^0 &= \iota^*(du - u_t dt - u_x dx) = du - K dt - u_x dx, \\ \theta^i &= \iota^* \vartheta^i = du_i - X^i(K) dt - u_{i+1} dx \end{aligned} \tag{1.2}$$

generate $\mathcal{I}$.

Solutions to $\Delta = 0$ are integral manifolds of $\mathcal{I} = \{\theta^i\}_{i \geq 0}$.

The Constrained Variational Bicomplex

The bicomplex $\Omega^{r,s}(\mathcal{R}^\infty) = \iota^* \Omega^{r,s}(J^\infty(\mathbb{R}^2, \mathbb{R}))$, $r = 0, 1, 2$ and $s = 0, 1, \ldots$.
If $\omega \in \Omega^{1,2}(\mathcal{R}^\infty)$ then

$$\omega = dx \wedge (a_{ij}\theta^i \wedge \theta^j) + dt \wedge (b_{ij}\theta^i \wedge \theta^j)$$

where $a_{ij}(t, x, u, u_x, u_{xx}, \ldots), b_{ij}(t, x, u, u_x, u_{xx}, \ldots) \in C^\infty(\mathcal{R}^\infty)$.

The induced anti-derivation $d_H : \Omega^{r,s}(\mathcal{R}^\infty) \to \Omega^{r+1,s}(\mathcal{R}^\infty)$ is

$$d_H\omega = dt \wedge T(\omega) + dx \wedge X(\omega), \quad T = D_t|_{\mathcal{R}^\infty}, X = D_x|_{\mathcal{R}^\infty}$$

The induced vertical differential $d_V : \Omega^{r,s}(\mathcal{R}^\infty) \to \Omega^{r,s+1}(\mathcal{R}^\infty)$ is $d_V = d - d_H$.
The operations d_H and d_V satisfy as in the unconstrained case,

$$d_H^2 = 0 \qquad d_V^2 = 0, \qquad d_H d_V = -d_V d_H. \tag{1.3}$$

Except : The horizontal d_H complex may not be exact (vertical d_V is), and

$$H^{r,s}(\mathcal{R}^\infty) = \frac{\mathrm{Ker}\,\{d_H : \Omega^{r,s}(\mathcal{R}^\infty) \to \Omega^{r+1,s}(\mathcal{R}^\infty)\}}{\mathrm{Im}\{d_H : \Omega^{r-1,s}(\mathcal{R}^\infty) \to \Omega^{r,s}(\mathcal{R}^\infty)\}}.$$

$(1, s)$- Conservation Laws

The kernel of $d_H : \Omega^{1,s}(\mathcal{R}^\infty) \to \Omega^{2,s}(\mathcal{R}^\infty)$ are $(1, s)$ conservation laws.

Example: $u_t = \sqrt{\frac{1}{u_{xxx}}}$, The form $\kappa \in \Omega^{1,0}(\mathcal{R}^\infty)$

$$\kappa = \sqrt{u_{xxx}}dx - \frac{4u_{xxx}u_{xxxxx} - 5u_{xxxx}^2}{16u_{xxx}^3}dt$$

satisfies

$$d_H\kappa = (T(u_{xxx}) + X\left(\frac{4u_{xxx}u_{xxxxx} - 5u_{xxxx}^2}{16u_{xxx}^3}\right))dt \wedge dx = 0$$

is a conservation law and $[\kappa] \in H^{1,0}(\mathcal{R}^\infty)$. The form $\eta \in \Omega^{1,1}(\mathcal{R}^\infty)$,

$$\eta = dx \wedge \theta^0 \cdot \left(-\frac{2}{3}u_x u_{xxx} - \frac{1}{3}uu_{xxxx}\right) - dt \wedge \sum_{a=1}^{3}(-X)^{a-1}\left(\frac{uu_{xxxx} + 2u_x u_{xxx}}{6u_{xxx}^{\frac{3}{2}}}\theta^{3-a}\right)$$

satisfies $d_H\eta = 0$ so is a $(1, 1)$-conservation law and $[\eta] \in H^{1,1}(\mathcal{R}^\infty)$.

Here $[\eta] \neq d_V[\xi]$, $[\xi] \in H^{1,0}(\mathcal{R}^\infty)$.

Part 2:

The Cohomology $H^{1,2}(\mathcal{R}^\infty)$ and Variational Operators.

Normal Form for $H^{1,s}(\mathcal{R}^\infty)$

Reference: Anderson , Kamran, *The Variational Bicomplex for Hyperbolic Second-order Scalar Partial Differential Equations in the Plane*.

Theorem

For any $[\omega] \in H^{1,s}(\mathcal{R}^\infty), s \geq 1$ *there exists a representative,*

$$\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \beta, \tag{2.1}$$

where $\rho \in \Omega^{0,s-1}(\mathcal{R}^\infty),\ \beta \in \Omega^{0,s}(\mathcal{R}^\infty)$ *and*

$$\mathbf{L}^*_\Delta(\rho) = -T(\rho) - (-X)^i(K_i\rho) = 0.$$

If $s = 1$, ρ is a function and $\mathbf{L}^*_\Delta(\rho) = 0$ is the "equation for the characteristic".

Corollary

1. *For* $s \geq 3$ *there are no non-zero solutions to* $\mathbf{L}^*_\Delta(\rho) = 0$, $\rho \in \Omega^{0,s-1}(\mathcal{R}^\infty)$, *and so* $H^{1,s}(\mathcal{R}^\infty) = 0$, $s \geq 3$.
2. *For all* $[\omega] \in H^{1,2}(\mathcal{R}^\infty)$ *there exists a* d_V *closed representative.*
3. *For* Δ *even order there are no non-zero solutions to* $\mathbf{L}^*_\Delta(\rho) = 0$, $\rho \in \Omega^{0,1}(\mathcal{R}^\infty)$, *and so* $H^{1,2}(\mathcal{R}^\infty) = 0$, $s \geq 2$.

Variational Operators and $H^{1,2}(\mathcal{R}^\infty)$

A Lagrangian λ is a differential form,

$$\lambda = L(t,x,u,\partial u)dt \wedge dx \in \Omega^{2,0}(J^\infty(\mathbb{R}^2,\mathbb{R}))$$

The fundamental computation in Calculus of Variations is:

$$d_V\lambda = d_V(L(t,x,u,\partial u)dt \wedge dx) = dt \wedge dx \wedge \theta^0 \cdot \mathbf{E}(L) + d_H\eta \tag{2.2}$$

$\mathbf{E}(L)$ is the Euler-Lagrange expression for L
$\eta \in \Omega^{1,1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ is the boundary term.

If Δ admits a variational operator

$$\mathcal{E}(\Delta) = r^i(t,x,u,u_x,\ldots)D_x^i(\Delta) = \mathbf{E}(L)$$

Equation 2.2 is then

$$d_V(Ldt \wedge dx) = dt \wedge dx \wedge \theta^0 \cdot \mathcal{E}(\Delta) + d_H\eta \tag{2.3}$$

Restrict 2.3 to $\Delta = 0$ (pullback by $\iota : \mathcal{R}^\infty \to J^\infty(\mathbb{R}^2,\mathbb{R})$) so $\mathcal{E}(\Delta) = 0$,

$$d_V\iota^*(Ldt \wedge dx) = d_H\iota^*\eta.$$

Continuing from

$$d_V \iota^*(L dt \wedge dx) = d_H \iota^* \eta, \qquad \eta \in \Omega^{1,1}((J^\infty(\mathbb{R}^2, \mathbb{R})) \tag{2.4}$$

Lemma

The form $\omega = d_V(\iota^*\eta) \in \Omega^{1,2}(\mathcal{R}^\infty)$ is d_H-closed so that $[d_V \iota^* \eta] \in H^{1,2}(\mathcal{R}^\infty)$. Futhermore ω is a d_V closed representative.

Proof.

Take d_H of $\omega = d_V(\iota^*\eta)$ and use $d_H d_V = -d_V d_H$, and $d_V^2 = 0$ in equation 2.4. ∎

This gives an onto linear mapping $\Phi : \mathcal{V}_{op}(\Delta) \to H^{1,2}(\mathcal{R}^\infty)$,

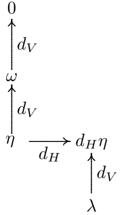
$$\Phi(\mathcal{E}) = [d_V \iota^* \eta].$$

a) Find for $[\omega]$ a d_V closed representative.

b) By vertical exactness $\omega = d_V \eta$.

c) $d_V d_H \eta = -d_H d_V \eta = -d_H \omega = 0$

d) Use vertical exactness $d_H \eta = d_V \lambda$ and lift off $\mathcal{R}^\infty$ (**key argument produces** $\mathcal{E}, \ \lambda$)



The normal form for $[\omega] \in H^{1,2}(\mathcal{R}^\infty)$ representative given previously-

$$\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \beta, \tag{2.5}$$

can be modified to a full canonical form leading to the following correspondence.

Theorem

Let $\mathcal{E} = r_i(t, x, u, u_x, \ldots)D_x^i$ $i = 0, \ldots, k$ be a k^{th} order differential operator and $\Delta = u_t - K(t, x, u, u_x, \ldots, u_{2m+1}), m \geq 1$ an odd order evolution equation.

1. *$\mathcal{E}$ is a variational operator for Δ if and only if $\mathcal{E}$ is skew-adjoint and*

$$\omega = dx \wedge \theta^0 \wedge \epsilon - dt \wedge \sum_{j=1}^{2m+1} \left(\sum_{a=1}^{j} (-X)^{a-1} \left(\frac{\partial K}{\partial u_j} \epsilon \right) \wedge \theta^{j-a} \right) \tag{2.6}$$

   is d_H closed on $\mathcal{R}^\infty$, where $\epsilon = -\frac{1}{2}\iota^*\mathcal{E}(\vartheta^0) = -\frac{1}{2} r_i X^i(\theta^0)$.

2. *Let $\mathcal{V}_{op}(\Delta)$ be the vector space of variational operators for Δ. The function $\Phi : \mathcal{V}_{op}(\Delta) \to H^{1,2}(\mathcal{R}^\infty)$ defined from equation 2.6 by*

$$\Phi(\mathcal{E}) = [\,\omega\,], \tag{2.7}$$

   is an isomorphism.

KN/SCH-KdV: $u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}$

$$\mathcal{E} = \frac{1}{u_x} D_x \frac{1}{u_x}$$

$$\omega = -\frac{1}{2u_x^2} dx \wedge \theta^0 \wedge \theta^1 + dt \wedge \left[\theta^0 \wedge \left(\frac{4u_{xxx}u_x - 3u_{xx}^2}{4u_x^4}\theta^1 + \frac{u_{xx}}{2u_x^3}\theta^2 - \frac{1}{2u_x^2}\theta^3\right) + \frac{1}{u_x^2}\theta^1 \wedge \theta^2\right].$$

Longer for

$$\mathcal{E}_0 = \frac{1}{u_x^2} D_x^3 - 3\frac{u_{xx}}{u_x^3} D_x^2 + \left(3\frac{u_{xx}^2}{u_x^4} - \frac{u_{xxx}}{u_x^3}\right) D_x.$$

PCKdV: $u_t = u_{xxx} + \frac{1}{2}u_x^2 - \frac{u}{2t}$

$$\begin{aligned}\mathcal{E} = &\quad tD_x \\ \omega = &\quad -tdx \wedge \theta^0 \wedge \theta^1 + dt \wedge \left(tu_x\theta^0 \wedge \theta^1 + t\theta^0 \wedge \theta^3 - 2t\theta^1 \wedge \theta^2\right)\end{aligned}$$

Longer for

$$\mathcal{E}_0 = t^2 D_x^3 + \frac{1}{3}(2t^2 u_x + tx)D_x + \frac{1}{6}(2t^2 u_{xx} + t).$$

Summary

The existence of Variational operators is equivalent to $H^{1,2}(\mathcal{R}^\infty) \neq 0$.

All variational operators and Lagrangians can be found using the 3 steps

1. Find $[\omega] \in H^{(1,2)}(\mathcal{R}^\infty)$
2. Go to canonical form representative ω to find $\mathcal{E}$
3. Use the snake lemma to find L.

Part 3:

Bicomplex formulation of Symplectic Hamiltonian Evolution Equations

Symplectic Hamiltonian Evolution Equations

A time-independent evolution equation defined through

$$\Delta = u_t - K(x, u, u_x, \ldots, u_n)$$

is a **Symplectic Hamiltonian Evolution Equation** if

1) there exists a symplectic differential operator $\mathcal{S} = s_i(x, u, u_x, \ldots)D_x^i$ and

2) a function $H(x, u, u_x, \ldots)$ such that,

$$\mathcal{S}(K) = \mathbf{E}(H). \tag{3.1}$$

A time dependent formulation is tougher to track down....

The time-independent formulation in terms of the variational bicomplex leads easily to the time dependent one, which I'll give.

The $\Omega^{r,s}_{\mathrm{t_{sb}}}(E)$ Bicomplex

Let $E = \mathbb{R} \times J^\infty(\mathbb{R}, \mathbb{R})$ with coordinates $(t, x, u, u_x, u_{xx}, \ldots)$.
The total x derivative vector field is

$$D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + \ldots .$$

The contact forms on E are,

$$\theta^i_E = du_i - u_{i+1} dx. \tag{3.2}$$

Let $\Omega^{r,s}_{\mathrm{t_{sb}}}(E)$ be the bicomplex of t semi-basic forms on E,

$$\Omega^{r,s}_{\mathrm{t_{sb}}}(E) = \{ \ \omega \in \Omega^{r,s}(E) \mid \partial_t \lrcorner\, \omega = 0 \quad , r = 0, 1; s = 0 \ldots \ \}.$$

A generic form $\omega \in \Omega^{1,2}_{\mathrm{t_{sb}}}(E)$ is given by

$$\omega = dx \wedge \theta^i_E \wedge \theta^j_E \cdot \xi_{ij}, \qquad \xi_{ij}(t, x, u, u_x, u_{xx}, \ldots) \in C^\infty(E).$$

The anti-derivations $d^E_H : \Omega^{r,s}_{\mathrm{t_{sb}}}(E) \to \Omega^{r+1,s}_{\mathrm{t_{sb}}}(E)$ and $d^E_V : \Omega^{r,s}_{\mathrm{t_{sb}}}(E) \to \Omega^{r,s+1}_{\mathrm{t_{sb}}}(E)$ are

$$d^E_H(\omega) = dx \wedge D_x(\omega), \quad d^E_V(f) = f_i \theta^i_E, \quad d^E_V \theta^i_E = 0, \tag{3.3}$$

and satisfy $(d^E_H)^2 = 0, (d^E_V)^2 = 0, d^E_H d^E_V + d^E_V d^E_H = 0$. However $d \neq d^E_H + d^E_V$.

Integration by parts operator

The integration by parts operator $I_E : \Omega^{1,s}_{\mathrm{t_{sb}}}(E) \to \Omega^{1,s}_{\mathrm{t_{sb}}}(E)$ $(s \geq 1)$ is

$$I_E(\Sigma) = \frac{1}{s}\theta^0_E \wedge \sum_{i=0}^{\infty}(-1)^i (D_x)^i(\partial_{u_i} \lrcorner \Sigma), \quad \Sigma \in \Omega^{1,s}_{\mathrm{t_{sb}}}(E) \tag{3.4}$$

We let the **space of functional s-forms** be the image,

$$\mathcal{F}^s_{\mathrm{t_{sb}}}(E) = I_E\left(\Omega^{1,s}_{\mathrm{t_{sb}}}(E)\right). \tag{3.5}$$

Equations 3.4 and 3.5 shows if $\Sigma \in \mathcal{F}^2_{\mathrm{t_{sb}}}(E)$ then there exists $\rho \in \Omega^{0,1}_{\mathrm{t_{sb}}}(E)$ such that

$$\Sigma = dx \wedge \theta^0_E \wedge \rho\,, \qquad \rho = s_i\theta^i_E. \tag{3.6}$$

The operator I_E has the properties,

$$\Sigma = I_E(\Sigma) + d^E_H\eta, \qquad I^2_E = I_E, \qquad \mathrm{Ker}\, I_E = \mathrm{Image}\ d^E_H. \tag{3.7}$$

The property $\mathrm{Ker}\, I = \mathrm{Image}\ d^E_H$ leads to the *Augmented Variational Bicomplex*

The Augmented Variational Bicomplex on $\mathbb{R} \times J^\infty(\mathbb{R}, \mathbb{R})$

$$\operatorname{Im} d_H^E = \ker I_E, \quad \operatorname{Im}(\delta_V^E)^i = \ker(\delta_V^E)^{i+1}.$$

Exact rows, columns, and δ_V^E (+lower row is Euler Complex)

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d_V^E} & & \uparrow{\scriptstyle d_V^E} & & \uparrow{\scriptstyle \delta_V^E} & & \\
0 & \longrightarrow & \Omega^{0,2}_{\mathrm{t_{sb}}}(E) & \xrightarrow{d_H^E} & \Omega^{1,2}_{\mathrm{t_{sb}}}(E) & \xrightarrow{I_E} & \mathcal{F}^2_{\mathrm{t_{sb}}}(E) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_V^E} & & \uparrow{\scriptstyle d_V^E} & & \uparrow{\scriptstyle \delta_V^E} & & \\
0 & \longrightarrow & \Omega^{0,1}_{\mathrm{t_{sb}}}(E) & \xrightarrow{d_H^E} & \Omega^{1,1}_{\mathrm{t_{sb}}}(E) & \xrightarrow{I_E} & \mathcal{F}^1_{\mathrm{t_{sb}}}(E) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_V^E} & & \uparrow{\scriptstyle d_V^E} & \nearrow{\scriptstyle \delta_V^E} & & & \\
R & \longrightarrow & \Omega^{0,0}_{\mathrm{t_{sb}}}(E) & \xrightarrow{d_H^E} & \Omega^{1,0}_{\mathrm{t_{sb}}}(E) & & & &
\end{array}$$

Time Dependent Symplectic Forms and Hamiltonian Vector Fields

Definition

A form $\Sigma \in \mathcal{F}^2_{\mathrm{t_{sb}}}(E)$ is symplectic on Γ if Σ is non-vanishing and $\delta^E_V \Sigma = 0$. A differential operator $\mathcal{S} = s_i D^i_x$ is symplectic if $dx \wedge \theta^0_E \wedge \mathcal{S}(\theta^0_E)$ is a symplectic form

Since δ^E_V complex is exact, then for Σ symplectic

$$\Sigma = dx \wedge \theta^0_E \wedge \mathcal{S}(\theta^0_E) = dx \wedge \theta^0_E \wedge (s_i \theta^i_E) = \delta^E_V(dx \wedge \theta^0_E \cdot P).$$

$\phi = dx \wedge \theta^0_E \cdot P \in \mathcal{F}^1_{\mathrm{t_{sb}}}(E)$ is a **symplectic potential**.

The **suspension** of the evolutionary vector field $Y = \mathrm{pr}(K\partial_u)$ is

$$T = \partial_t + Y = \partial_t + K\partial_u + D_x(K)\partial_{u_x} + \dots \ , .$$

Definition

$u_t = K$ is a symplectic Hamiltonian evolution equation and $Y = pr(K\partial_u)$ is Hamiltonian vector field with respect to the symplectic form $\Sigma \in \mathcal{F}^2_{\mathrm{t_{sb}}}(E)$ if

$$\mathcal{L}^\natural_T \Sigma = I_E \circ \pi^{1,2} \circ \mathcal{L}_T \Sigma = I_E \circ \pi^{1,2} \circ T(\Sigma) = 0. \tag{3.8}$$

Here $\mathcal{L}^\natural_T = I_E \circ \pi^{1,2} \circ \mathcal{L}_T$ is the Lie derivative on functional 2-forms.

Lemma

The vector field $Y = pr(K\partial_u)$ is Hamiltonian for $\Sigma = dx \wedge \theta^0_E \wedge \mathcal{S}(\theta^0_E)$ if and only if there exists $H(t, x, u, u_x, \dots)$ such that

$$\frac{1}{2}P_t + \mathcal{S}(K) = \mathbf{E}(H)$$

where $dx \wedge \theta^0_E \cdot P$ is a symplectic potential.

For time independent Σ this gives the standard condition

$$\mathcal{S}(K) = \mathbf{E}(H)$$

Symplectic if and only if Variational

We find-

Theorem

The form $\Sigma = dx \wedge \theta_E^0 \wedge (s_i \theta_E^i)$ is symplectic, and $Y = pr(K\partial_u)$ is a Hamiltonian vector field for Σ if and only if

$$\omega = dx \wedge \theta^0 \wedge \epsilon - dt \wedge \sum_{j=1}^{2m+1} \left(\sum_{a=1}^{j} (-X)^{a-1} \left(\frac{\partial K}{\partial u_j} \epsilon \right) \wedge \theta^{j-a} \right) \tag{3.9}$$

satisfies $d_H \omega = 0$, where $\epsilon = \mathcal{S}(\theta^0) = s_i \theta^i$

Corollary

The induced map $\Pi : H^{1,2}(\mathcal{R}^\infty) \to \mathcal{F}^2_{\mathrm{tsb}}(E)$ given by

$$\Pi(dx \wedge \theta^0 \wedge (r_i \theta^i) - dt \wedge \sum_{j=1}^{2m+1} \left(\sum_{a=1}^{j} (-X)^{a-1} \left(\frac{\partial K}{\partial u_j} \epsilon \right) \wedge \theta^{j-a} \right)) = dx \wedge \theta_E^0 \wedge (r_i \theta_E^i)$$

is isomorphism to symplectic forms for which $u_t = K$ is a symplectic Hamiltonian equation.

Part 4:

Hamiltonian and Variational/Symplectic Operator Reduction

First Order Hamiltonians

Suppose we have a first order Hamiltonian evolution in canonical form

$$z_t = D_x \left(\frac{\delta H}{\delta z} \right) \tag{4.1}$$

Going to potential form $z = u_x$ gives

$$u_{tx} = \left[D_x \left(\frac{\delta H}{\delta z} \right) \right] \Big|_{z=u_x} = D_x \left[\left(\frac{\delta H}{\delta z} \right) \Big|_{z=u_x} \right] \tag{4.2}$$

integrating gives a potential form,

$$u_t = \frac{\delta H}{\delta z} \Big|_{z=u_x} \tag{4.3}$$

The translation in u invariant u_x satisfies 4.2, and hence $z = u_x$ satisfies 4.1 and 4.1 is the quotient of 4.3 by translation in u.

Applying D_x to the potential form gives

$$D_x\left(u_t - \frac{\delta H}{\delta z}\bigg|_{z=u_x}\right) = u_{tx} - D_x\left(\frac{\delta H}{\delta z}\bigg|_{z=u_x}\right). \tag{4.4}$$

On the other hand the change of variables formula in CV gives

$$\frac{\delta}{\delta u}\left(H|_{z=u_x}\right) = -D_x\left(\frac{\delta H}{\delta z}\bigg|_{z=u_x}\right)$$

and equation 4.4 is

$$D_x\left(u_t - \left(\frac{\delta H}{\delta z}\right)\bigg|_{z=u_x}\right) = E\left(-\frac{1}{2}u_t u_x + H|_{z=u_x}\right). \tag{4.5}$$

Therefore D_x is a variational/symplectic operator for the potential form, with the Lagrangian being the pullback of the Hamiltonian.

Theorem

Every Hamiltonian evolution equation $z_t = \mathcal{D}(\delta H)$ *with first order Hamiltonian* $\mathcal{D}$ *is the symmetry reduction of an equation* $u_t = K$*, of the same order, which admits a first order variational operator* $\mathcal{E}$ *and* $\pi_* \mathcal{E} = \mathcal{D}$*.*

(Ie. The symmetry reduction of an integrable extension which admits a Variational operator).

Compatible Bi-Hamiltonian Scalar Evolution Equations

Theorem

Let

$$z_t = K(x, z, z_x, \ldots, z_{2m+1}) = D_x\left(\frac{\delta H_1}{\delta z}\right) \tag{4.6}$$

with potential form

$$u_t = \left.\frac{\delta H_1}{\delta z}\right|_{z=u_x}. \tag{4.7}$$

Let $\mathcal{D}$ be a Hamiltonian operator satisfying the compatibility condition

$$\mathcal{D}\left(\frac{\delta H_1}{\delta z}\right) = D_x\left(\frac{\delta H_2}{\delta z}\right) \tag{4.8}$$

Then the potential form satisfies $\mathcal{E}(u_t) = -\frac{\delta}{\delta u}(H_2|_{z=u_x})$ where $\pi_*\mathcal{E} = \mathcal{D}$.

Proof.

We apply $\mathcal{E}$ to RHS of equation 4.7, and use condition 4.8

$$\begin{aligned} \mathcal{E}\left(\left.\frac{\delta H_1}{\delta z}\right|_{z=u_x}\right) &= \left[\mathcal{D}\left(\frac{\delta H_1}{\delta z}\right)\right]\Big|_{z=u_x} \\ &= D_x\left(\frac{\delta H_2}{\delta z}\right)\Big|_{z=u_x} \\ &= -\frac{\delta}{\delta u}(H_2|_{z=u_x}). \end{aligned} \tag{4.9}$$

Where the last line follows as before from the change of variables formula for variations. ∎

REMARK : For third order $\mathcal{D}$, the operator $\mathcal{E}$ is symplectic and hence a variational operator for the potential form and the pullback H_2 in equation 4.8 is part of the Lagrangian for the second variational operator.

REMARK : Conversely invariant Variational operators quotient to Hamiltonian ones.

The Potential Cylindrical KdV: $\Delta = u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}$

The third order variational operator for the potential cylindrical KdV is

$$\mathcal{E}_0 = t^2 D_x^3 + \frac{1}{3}(2t^2 u_x + tx)D_x + \frac{1}{6}(2t^2 u_{xx} + t).$$

$$\mathcal{E}_0\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) = \mathbf{E}\left(Q_0(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}) - \frac{1}{72}\left(t^2 u_x^4 + 2txu_x^3\right)\right)$$

where

$$Q_0 = -\frac{1}{6}\left(t^2 u_x^2 + txu_x + 3u_{xxx}t^2\right)$$

The reduction of the potential cylindrical KdV by ∂_u is the cylindrical KdV. Substitute $w = \sqrt{t}\,u_x$ into the x-derivative of potential cylindrical KdV gives

$$w_t = w_{xxx} + \frac{1}{\sqrt{t}}ww_x = \mathcal{D}_1(\mathbf{E}(H_1)) = \mathcal{D}_0(\mathbf{E}(H_0)) \qquad (4.10)$$

where

$$\mathcal{D}_1 = D_x,\ H_1 = \frac{1}{2}w_x^2 + \frac{1}{6\sqrt{t}}w^3, \quad \mathcal{D}_0 = D_x^3 + \frac{2w}{3\sqrt{t}}D_x + \frac{w_x}{3\sqrt{t}}, \quad H_0 = \frac{1}{2}w^2.$$

Equation 4.10 is obtained from the standard form of the cylindrical KdV equation by $w = \sqrt{t}\,v$. Some references say no Hamiltonians for the cylindrical KdV.

Harry-Dym/KN

The quotient of the KN/Schwartzian KdV

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x}.$$

by translation in x gives Harry-Dym

$$z_t = z^3 z_{xxx}$$

and the KN equation is the potential form.
The Hamiltonian operators for the HD equation are reduction of the symplectic for KN.