"Quantisation of Poisson structures on Painleve monodromy varieties."

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(joint work in progress with
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Talk at **tcmWINE2015 "Workshop on Integrable Nonlinear Equations".**

Mikulov, Czech Republic, October, 18-23, 2015

Plan:

- Painlevé equations;
- Isomonodromy and Riemann-Hilbert;
- Affine cubics;
- Singularities and cluster transformations;
- Quantisation an relations to Sklyanin algebras
- Perspectives and output;

Painlevé equations

The Painlevé equations are non linear second order ODE of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \qquad z \in \mathbb{C},$$

where $F(z, w, y)$ is a rational function of z, w, y and the solutions $w(z; c_1, c_2)$ satisfy

1. Painlevé–Kowalevski property: $w(z; c_1, c_2)$ have no *critical points* that depend on c_1, c_2.
2. Otherwise, they are the only second order ODE without movable singularities (branching points).
3. For generic c_1, c_2, $w(z; c_1, c_2)$ are new functions, Painlevé Transcendents.

Painlevé property:

- **Example for 1-st ordre ODE**:

$$
\begin{array}{lcll}
w' = w & \Longrightarrow & w = e^{z-z_0}, & \checkmark \\
w' = w^2 & \Longrightarrow & w = \dfrac{1}{z_0 - z}, & \checkmark \\
w' = w^3 & \Longrightarrow & w \sim \dfrac{1}{\sqrt{z - z_0}}. & \text{X}
\end{array}
$$

Painlevé I,II,II,IV

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

Painlevé V and VI

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 -$$

$$-\frac{1}{z}\frac{dw}{dz} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right)\frac{\delta w(w+1)}{w-1},$$

$$\frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1}\right)w_z +$$

$$+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

Painlevé parameters

Denote $z = t$ and

$$\alpha := (\theta_\infty - 1/2)^2; \qquad \beta := -\theta_0^2;$$

$$\gamma := \theta_1^2; \qquad \delta := (1/4 - \theta_t)^2.$$

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- All Painlevés (except for P_I) admit one-parameter family of solutions (in terms of special functions) and for some special values of parameteres they have particular rational solutions;
- Recently: P_{II} - has a genuine fully NC analogue (V. Retakh-V.R.)

- All Painlevé equations are isomonodromic deformation equations (Miwa-Jimbo 1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}z} = [A, B]\,, \tag{1}$$

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- The monodromy data play the role of initial conditions.

PVI as isomonodromic deformation

Painlevé sixth equation

- The Painlevé VI equation describes the isomonodromic deformations of the rank 2 meromorphic connections on $\mathbb{P}^1$ with simple poles.

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = \left(\frac{A_1(z)}{\lambda} + \frac{A_2(z)}{\lambda - t} + \frac{A_3(z)}{\lambda - 1}, \right) Y, \; \lambda \in \mathbb{C} \setminus \{0, t, 1\} \quad (3)$$

where $A_1, A_2, A_3 \in \mathfrak{sl}_2(\mathbb{C})$, $A_1 + A_2 + A_3 = -A_\infty$, diagonal.

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- Fundamental matrix: $Y_\infty(\lambda) = (1 + O(\frac{1}{\lambda}))\lambda^{A_\infty}$.
- Monodromy matrices $\gamma_j(Y_\infty) = Y_\infty M_j$
- Describes by generators of the fundamental group under the anti-isomorphism

$$\rho : \pi_1 \left(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}, \lambda_1\right) \to \mathrm{SL}_2(\mathbb{C}).$$

- $\text{eigen}(M_j) = \text{eigen}(\exp(2\pi i A_j))$
- We fix the base point λ_1 at infinity and the generators of the fundamental group to be $\gamma_1, \gamma_2, \gamma_3$ such that γ_j encircles only the pole i once and are oriented in such a way that

$$M_1 M_2 M_3 M_\infty = \mathbb{I}, \qquad M_\infty = \exp(2\pi i A_\infty). \tag{4}$$

- Eigenvalues of A_j are $(\theta_j, -\theta_j)$, $j = 0, t, 1, \infty$.

Let:

$$G_j := \mathrm{Tr}(M_j) = 2\cos(\pi\theta_j), \quad j = 0, t, 1, \infty,$$

The Riemann-Hilbert correspondence

$$\mathcal{F}(\theta_0, \theta_t, \theta_1, \theta_\infty)/\mathcal{G} \to \mathcal{M}(G_1, G_2, G_3, G_\infty)/\mathrm{SL}_2(\mathbb{C}),$$

where $\mathcal{G}$ is the gauge group, is defined by associating to each Fuchsian system its monodromy representation class. The representation space $\mathcal{M}(G_1, G_2, G_3, G_\infty)$ is realised as an affine cubic surface (Jimbo)

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4 = 0, \qquad (5)$$

where:

$$x_1 = \mathrm{Tr}\left(M_2 M_3\right), \quad x_2 = \mathrm{Tr}\left(M_1 M_3\right), \quad x_3 = \mathrm{Tr}\left(M_1 M_2\right).$$

and

$$-\omega_i := G_k G_j + G_i G_\infty, i \neq k, j,$$

$$\omega_\infty = G_1^2 + G_2^2 + G_3^2 + G_\infty^2 + G_1 G_2 G_3 G_\infty - 4.$$

Iwasaki proved that the triple (x_1, x_2, x_3) satisfying the cubic relation (5) provides a set of coordinates on a large open subset

$$S \subset \mathcal{M}(G_1, G_2, G_3, G_\infty).$$

In what follows, we restrict to such open set.

Following Sakai, there are eight Painlevé equations corresponding to the eight extended Dynkin diagrams:

$$\widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8,$$

corresponding respectively to PVI, PV, three different cases of PIII, PIV, PII and PI.
Their monodromy manifolds were studied by several authors, but were recently presented in a unified way:

$$\widetilde{D}_4 \quad x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \tag{6}$$

$$\widetilde{D}_5 \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0,$$

$$\widetilde{D}_6 \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1 = 0,$$

$$\widetilde{D}_7 \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 = 0,$$

$$\widetilde{D}_8 \quad x_1x_2x_3 + x_1^2 + x_2^2 + 1 = 0,$$

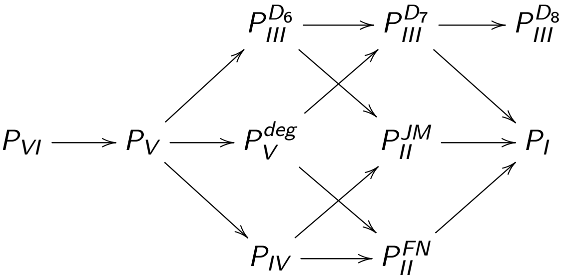
$$\widetilde{E}_6 \quad x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2(x_2 + x_3) + 1 + \omega_4 = 0,$$

$$\widetilde{E}_7^* \quad x_1x_2x_3 + x_1 + x_2 + x_3 + \omega_4 = 0,$$

$$\widetilde{E}_7^{**} \quad x_1x_2x_3 + x_1 + \omega_2 x_2 + x_3 - \omega_2 + 1 = 0,$$

$$\widetilde{E}_8 \quad x_1x_2x_3 + x_1 + x_2 + 1 = 0,$$

Confluence of Painlevé equations (Sakai)



General Affine Cubic

The main object studied in this talk is the affine irreducible cubic surface $M_\phi := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3]/_{\langle \phi=0\rangle})$ where

$$\phi = x_1x_2x_3+\epsilon_1^{(d)}x_1^2+\epsilon_2^{(d)}x_2^2+\epsilon_3^{(d)}x_3^2+\omega_1^{(d)}x_1+\omega_2^{(d)}x_2+\omega_3^{(d)}x_3+\omega_4^{(d)} = 0, \tag{7}$$

According to Saito and Van der Put, the monodromy manifolds $\mathcal{M}^{(d)}$ have all the form of M_ϕ

Here d is an index running on the list of the extended Dynkin diagrams $\widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8$ and the parameters $\epsilon_i^{(d)}$, $\omega_i^{(d)}$, $i = 1, 2, 3$ are given by:

$$\epsilon_1^{(d)} = \begin{cases} 1 & \text{for } d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \\ 0 & \text{for } d = \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8, \end{cases}$$
$$\epsilon_2^{(d)} = \begin{cases} 1 & \text{for } d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8 \\ 0 & \text{for } d = \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8, \end{cases} \tag{8}$$
$$\epsilon_3^{(d)} = \begin{cases} 1 & \text{for } d = \widetilde{D}_4, \\ 0 & \text{for } d = \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8. \end{cases}$$

The coefficients $\omega^{(d)}$ are defined by:

$$\begin{aligned}
\omega_1^{(d)} &= -G_1^{(d)} G_\infty^{(d)} - \epsilon_1^{(d)} G_2^{(d)} G_3^{(d)}, \\
\omega_2^{(d)} &= -G_2^{(d)} G_\infty^{(d)} - \epsilon_2^{(d)} G_1^{(d)} G_3^{(d)}, \\
\omega_3^{(d)} &= -G_3^{(d)} G_\infty^{(d)} - \epsilon_3^{(d)} G_1^{(d)} G_2^{(d)}, \\
\omega_4^{(d)} &= \epsilon_2^{(d)} \epsilon_3^{(d)} \left(G_1^{(d)} \right)^2 + \epsilon_1^{(d)} \epsilon_3^{(d)} \left(G_2^{(d)} \right)^2 + \epsilon_1^{(d)} \epsilon_2^{(d)} \left(G_3^{(d)} \right)^2 + \\
&\left(G_\infty^{(d)} \right)^2 + G_1^{(d)} G_2^{(d)} G_3^{(d)} G_\infty^{(d)} - 4\epsilon_1^{(d)} \epsilon_2^{(d)} \epsilon_3^{(d)},
\end{aligned} \tag{9}$$

Here $G_1^{(d)}, G_2^{(d)}, G_3^{(d)}, G_\infty^{(d)}$ are some constants related to the parameters appearing in the Painlevé equations as follows:

$$G_1^{(d)} = \begin{cases} 2\cos\pi\theta_0 & d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{E}_6 \\ e^{-\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^{*} \\ e^{-i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 1 & d = \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_8 \\ e^{\frac{i\pi(\theta_0+\theta_\infty)}{2}} + e^{\frac{-i\pi(\theta_0+\theta_\infty)}{2}} & d = \widetilde{D}_6, \end{cases}$$

$$G_2^{(d)} = \begin{cases} 2\cos\pi\theta_1 & d = \widetilde{D}_4, \widetilde{D}_5, \\ 2\cos\pi\theta_\infty & d = \widetilde{E}_6 \\ e^{-\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^{*} \\ e^{i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 1 & d = \widetilde{D}_8, \widetilde{E}_8 \\ e^{\frac{i\pi(\theta_0-\theta_\infty)}{2}} + e^{\frac{i\pi(-\theta_0+\theta_\infty)}{2}} & d = \widetilde{D}_6 \end{cases}$$

$$G_3^{(d)} = \begin{cases} 2\cos\pi\theta_t & d = \widetilde{D}_4, \\ 1 & d = \widetilde{D}_5, \widetilde{D}_7 \\ 2\cos\pi\theta_\infty & d = \widetilde{E}_6 \\ e^{-\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^* \\ e^{-i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 0 & d = \widetilde{D}_6, \widetilde{D}_8, \widetilde{E}_8 \end{cases}$$

$$G_\infty^{(d)} = \begin{cases} 2\cos\pi\theta_\infty & d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{E}_6 \\ e^{\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^* \\ e^{i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 1 & d = \widetilde{D}_8, \widetilde{E}_8 \\ e^{\frac{i\pi(\theta_0+\theta_\infty)}{2}} & d = \widetilde{D}_6 \\ 0 & d = \widetilde{D}_7 \end{cases}$$

This family of cubics is a variety
$M_\phi = \{(\bar{x}, \bar{\omega}) \in \mathbb{C}^3 \times \Omega) : \phi(\bar{x}, \bar{\omega}) = 0\}$ where
$\bar{x} = (x_1, x_2, x_3), \quad \bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and the
"$\bar{x}-$forgetful"projection $\pi : M_\phi \to \Omega : \pi(\bar{x}, \bar{\omega}) = \bar{\omega}$. This projection defines a family of affine cubics with generically non–singular fibres $\pi^{-1}(\bar{\omega})$
The cubic surface $M_{\phi_{\bar{\omega}}}$ has a volume form $\vartheta_{\bar{\omega}}$ given by the Poincaré residue formulae:

$$\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial \phi_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial \phi_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial \phi_{\bar{\omega}})/(\partial x_2)}. \qquad (10)$$

The volume form is a holomorphic 2-form on the non-singular part of $M_{\phi_{\bar{\omega}}}$ and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

$$\{x_1, x_2\}_{\bar{\omega}} = \frac{\partial \phi_{\bar{\omega}}}{\partial x_3} \tag{11}$$

The other brackets are defined by circular transposition of x_1, x_2, x_3. For $(i, j, k) = (1, 2, 3)$:

$$\{x_i, x_j\}_{\bar{\omega}} = \frac{\partial \phi_{\bar{\omega}}}{\partial x_k} = x_i x_j + 2\epsilon_i^{(d)} x_k + \omega_i^{(d)} \tag{12}$$

and the volume form (10) reads as

$$\vartheta_{\bar{\omega}} = \frac{dx_i \wedge dx_j}{(\partial \phi_{\bar{\omega}} / \partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2\epsilon_i^{(d)} x_k + \omega_i^{(d)})}. \tag{13}$$

Observe that for any $\phi \in \mathbb{C}[x_1, x_2, x_3]$ the following formulae define a Poisson bracket on $\mathbb{C}[x_1, x_2, x_3]$:

$$\{x_i, x_{i+1}\} = \frac{\partial \phi}{\partial x_{i+2}}, \qquad x_{i+3} = x_i, \tag{14}$$

and ϕ itself is a central element for this bracket, so that the variety

$$M_\phi := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3]/_{\langle \phi = 0 \rangle})$$

inherits the Poisson variety structure $[\mathrm{Nambu} \sim 70]$.
Today I am going to quantize it.

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- In algebraic geometry - projective completion:

$$\overline{M}_{\tilde{\phi}} := \{(u,v,w,t) \in \mathbb{P}^3 \,| x_1^2 t + x_2^2 t + x_3^2 t - x_1 x_2 x_3 +$$

$$+\omega_3 x_1 t^2 + \omega_2 x_2 t^2 + \omega_3 x_3 t^2 + \omega_4 t^3 = 0\}$$

is a del Pezzo surface of degree three and differs from it by three smooth lines at infinity forming a triangle [Oblomkov] $t = 0, \qquad x_1 x_2 x_3 = 0.$

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- PVI ($\tilde{D}_4$) cubic with only $\omega_4 \neq 0$ admits the log-canonical symplectic structure $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ under isomorphism $\mathbb{C}^* \times \mathbb{C}^*/\imath \to M_\phi$ by

$$(u, v) \to (x_1 = -(u + \frac{1}{u}), x_2 = -(v + \frac{1}{v}), x_3 = -(uv + \frac{1}{uv})$$

and $\imath : \mathbb{C}^* \to \mathbb{C}^*$ is the involution $\imath(u) = \frac{1}{u}$, $\imath(v) = \frac{1}{v}$.

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and $\imath : \mathbb{C}^* \to \mathbb{C}^*$ is the involution $\imath(u) = \frac{1}{u}$, $\imath(v) = \frac{1}{v}$.
- The family (7) can be "uniformize"by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces (M. Gross, P. Hacking and S.Keel (see Example 5.12 of "Mirror symmetry for log-Calabi-Yau varieties I, arXiv:1106.4977).

Basic ideas

- The character variety of a Riemann sphere with 4 holes $\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, \mathrm{t}, 1, \infty\}); \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is the monodromy cubic of the Painlevé VI $(\mathrm{Goldman\text{-}Toledo})$.

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- The confluent Painlevé monodromy manifolds are "decorated character varieties
  (Chekhov-Mazzocco -R.2015).
- The real slice of the $\mathrm{SL}_2(\mathbb{C})$ character variety is the Teichmüller space.

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- The confluent Painlevé monodromy manifolds are "decorated character varieties
  (Chekhov-Mazzocco -R.2015).
- The real slice of the $\mathrm{SL}_2(\mathbb{C})$ character variety is the Teichmüller space.
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- Closed paths in the fat graph $\Gamma_{0,4} \iff$ conjug. classes in $\Delta_{0,4}$

Shear coordinates on 4-holed sphere

In the D_4 case the parameterisation of the cubic in Thurston shear coordinates on the fat-graph of a 4–holed sphere was found by Chekhov-Mazzocco:

$$x_1 = -e^{\tilde{s}_2+\tilde{s}_3} - e^{-\tilde{s}_2-\tilde{s}_3} - e^{-\tilde{s}_2+\tilde{s}_3} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2}$$

$$x_2 = -e^{\tilde{s}_3+\tilde{s}_1} - e^{-\tilde{s}_3-\tilde{s}_1} - e^{-\tilde{s}_3+\tilde{s}_1} - G_3 e^{\tilde{s}_1} - G_1 e^{-\tilde{s}_3},$$

$$x_3 = -e^{\tilde{s}_1+\tilde{s}_2} - e^{-\tilde{s}_1-\tilde{s}_2} - e^{-\tilde{s}_1+\tilde{s}_2} -$$

$$-G_1 e^{\tilde{s}_2} - G_2 e^{-\tilde{s}_1}$$

where

$$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \qquad i = 1, 2, 3, \, G_\infty = e^{\tilde{s}_1+\tilde{s}_2+\tilde{s}_3} + e^{-\tilde{s}_1-\tilde{s}_2-\tilde{s}_3},$$

and $\tilde{s}_i$ are actually the shifted shear coordinates $\tilde{s}_i = s_i + \frac{p_i}{2}$, $i = 1, 2, 3$.

The geodesic length functions, which are traces of hyperbolic elements in the Fuchsian group $\Delta_{0,4}$ are obtained by decomposing each hyperbolic matrix $\gamma \in \Delta_{0,4}$ into a product of the so–called *right, left and edge matrices:*

$$R := \left(\begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right), \, L := \left(\begin{array}{cc} 0 & 1 \\ -1 & {\scriptstyle 1} \end{array} \right), \, X_{s_i} := \left(\begin{array}{cc} 0 & -\exp\left(\frac{s_i}{2}\right) \\ \exp\left(-\frac{s_i}{2}\right) & 0 \end{array} \right) \tag{15}$$

In this setting our x_1, x_2, x_3 are the geodesic lengths of three geodesics which go around two holes without self–intersections, for example x_3 corresponds to the dashed geodesic in Fig.1.

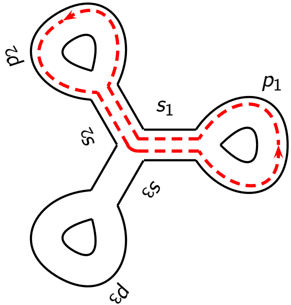

Figure: The fat graph of the 4 holed Riemann sphere. The dashed geodesic corresponds to x_3. The corresponding hyperbolic element $\gamma_{1;2} = \mathrm{Tr}(\mathrm{X}_{\mathrm{s}_1}\mathrm{L}\mathrm{X}_{\mathrm{p}_1}\mathrm{L}\mathrm{X}_{\mathrm{s}_1}\mathrm{R}\mathrm{X}_{\mathrm{s}_2}\mathrm{L}\mathrm{X}_{\mathrm{p}_2}\mathrm{L}\mathrm{X}_{\mathrm{s}_2}\mathrm{L})$

V. Fock: The fat graph associated to a Riemann surface $\Sigma_{g,n}$ of genus g and with n holes is a connected three–valent graph drawn without self-intersections on $\Sigma_{g,n}$ with a prescribed cyclic ordering of labelled edges entering each vertex; it must be a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole).

$$G_{\gamma_{12}} = \mathrm{Tr}(\gamma_{12}) = 2\cosh(\mathrm{l}_{\gamma_{12}}/2)$$

where $l_{\gamma_{12}}$ is actual length of the closed geodesic on $\Sigma_{0,4}$

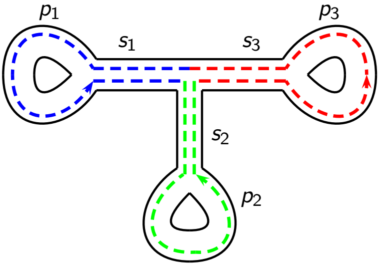

Figure: The fat graph of the 4 holed Riemann sphere. The geodetic corresponding to x_1 is obtained by going along first the green loop then the red one.

The confluence from the cubic associated to PVI to the one associated to PV is realised by

$$p_3 \to p_3 - 2\log[\epsilon],$$

in the limit $\epsilon \to 0$. We obtain the following shear coordinate description for the *PV* cubic:

$$\begin{aligned}
x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}},\\
x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}},\\
x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} -
\end{aligned}$$

where

$$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \quad i = 1, 2, \quad G_3 = e^{\frac{p_3}{2}}, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1x_2x_3 + x_1^2 + x_2^2 - (G_1G_\infty + G_2G_3)x_1 - (G_2G_\infty + G_1G_3)x_2 - \\ -G_3G_\infty x_3 + G_\infty^2 + G_3^2 + G_1G_2G_3G_\infty = 0. \tag{17}$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV- cubic, we need to pick $p_3 = -p_1 - p_2 - 2s_1 - 2s_2 - 2s_3$ so that $G_\infty = 1$.

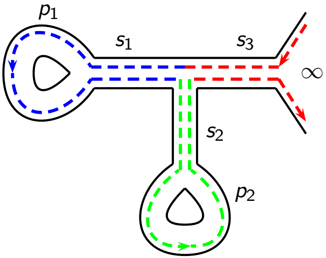

Figure: The fat graph corresponding to PV.

Geometrically speaking, sending the perimeter p_3 to infinity means that we are performing a chewing-gum move:
two holes, one of perimeter p_3 and the other of perimeter $s_1 + s_2 + s_3 + \frac{p_1}{2} + \frac{p_2}{2} + \frac{p_3}{2}$, become infinite, but the area between them remains finite.
This is leading to a Riemann sphere with three holes and two cusps on one of them. In terms of the fat-graph, this is represented by Figure 2.
The geodesic x_3 corresponds to the closed loop obtained going around p_1 and p_2 (green and red loops), while x_1 and x_2 are "asymptotic geodesics"starting at one cusp, going arond p_1 and p_2 respectively, and coming back to the other cusp.

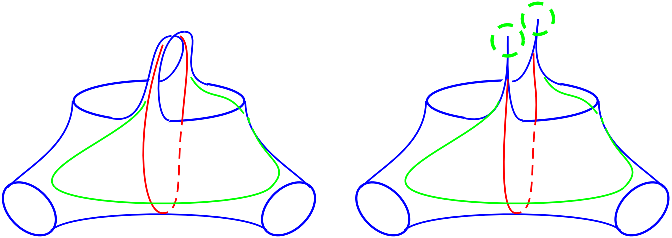

Figure: The process of confluence of two holes on the Riemann sphere with four holes. Chewing-gum move: hook two holes together and stretch to infinity by keeping the area between them finite (see Fig.). As a result we obtain a Riemann sphere with one less hole, but with two new cusps on the boundary of this hole. The red geodesic line which was initially closed becomes infinite, therefore two horocycles (the green dashed circles) must be introduced in order to measure its length.

Theorem

(Chekhov-Mazzocco-R.) The decorated character variety of a Riemann sphere with 3 holes, one of which with two cusps, is given by the monodromy manifold of the Painlevé V equation:

$$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4.$$

The character variety of a Riemann sphere with three holes and two cusps on one boundary is 7-dimensional (rather than 2-dimensional like in PVI case). The fat-graph admits a complete cusped lamination as displayed in the figure below. A full set of coordinates on the character variety is given by the five elements in the lamination and the two parameters G_1 and G_2 which determine the perimeter of the two non-cusped holes.

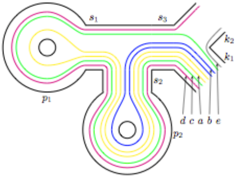

FIGURE 6. The system of arcs for PV.

Notice that there are two shear coordinates associated to the two cusps, they are denoted by k_1 and k_2, their sum corresponds to what we call p_3 above.
These shear coordinates do not commute with the other ones, they satisfy the following relations:

$$\{s_3, k_1\} = \{k_1, k_2\} = \{k_2, s_3\} = 1.$$

As a consequence in the character variety, the elements G_3 and G_∞ are not Casimirs.
In terms of shear coordinates, the elements in the lamination are expressed as follows:

$$a = e^{k_1+s_1+2s_2+s_3+\frac{p_1}{2}+p_2}, \qquad b = e^{k_1+s_2+s_3+\frac{p_2}{2}}, \qquad e = e^{\frac{k_1}{2}+\frac{k_2}{2}},$$
$$c = e^{k_1+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}, \qquad d = e^{\frac{k_1}{2}+\frac{k_2}{2}+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}. \qquad (18)$$

They satisfy the following Poisson relations:

$$\{a,b\} = ab, \quad \{a,c\} = 0, \quad \{a,d\} = -\frac{1}{2}ad, \quad \{a,e\} = \frac{1}{2}ae, \tag{19}$$

$$\{b,c\} = 0, \quad \{b,d\} = -\frac{1}{2}bd, \quad \{b,e\} = \frac{1}{2}be, \tag{20}$$

$$\{c,d\} = -\frac{1}{2}cd, \quad \{c,e\} = \frac{1}{2}ce, \quad \{d,e\} = 0, \tag{21}$$

so that the element $G_3 G_\infty = de$ is a Casimir.

The symplectic leaves are determined by the values of the three Casimirs G_1, G_2 and $G_3 G_\infty$.
On each symplectic leaf, the PV monodromy manifold (17) is the subspace defined by those functions of a, b, c (and of the Casimir values $G_1,\ G_2,\ G_3 G_\infty$) which commute with $G_3 = e$. To see this, we can use relations (18) to determine the exponentiated shear coordinates in terms of a, b, c, d, e and then deduce he expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1 = -e\frac{a}{c} - d\frac{b}{c}, \qquad x_2 = -e\frac{b}{c} - G_1 d\frac{b}{a} - d\frac{b^2}{ac} - d\frac{c}{a}, \tag{22}$$

$$x_3 = -G_2\frac{c}{b} - G_1\frac{c}{a} - \frac{b}{a} - \frac{c^2}{ab} - \frac{a}{b}, \tag{23}$$

which automatically satisfy (17).

Due to the Poisson relations (19) the functions that commute with e are exactly the functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Such functions may depend on the Casimir values G_1, G_2 and $G_3 G_\infty$ and e itself, so that $d = G_\infty$ becomes a parameter now. With this in mind, it is easy to prove that x_1, x_2, x_3 are algebraically independent functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ so that x_1, x_2, x_3 form a basis in the space of functions which commute with e.

Remark

It is worth reminding that the exponentials of the shear coordinates satisfy the log-canonical Poisson bracket.

Cusps removal:

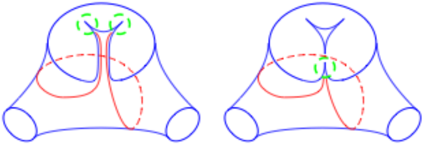

FIGURE 2. The process of breaking up a Riemann surface with boundary cusps: by grabbing together two cusps and pulling we tear apart an ideal triangle.

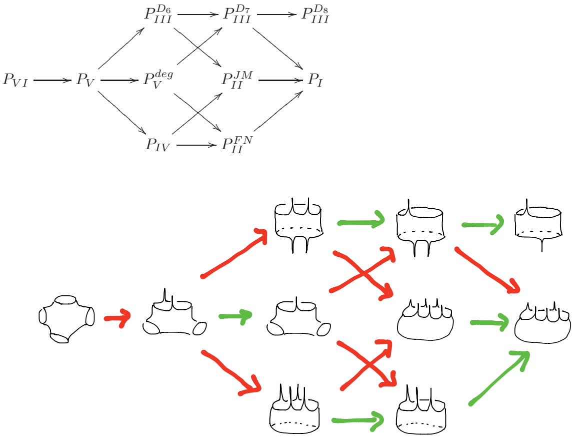

Рис.: Sakai confluence and decoration

Quantization

To produce the quantum Painlevé cubics, we introduce the Hermitian operators S_1, S_2, S_3 subject to the commutation inherited from the Poisson bracket of $\tilde{s}_i$:

$$[S_i, S_{i+1}] = i\pi\hbar\{\tilde{s}_i, \tilde{s}_{i+1}\} = i\pi\hbar, \quad i = 1,2,3, \; i+3 \equiv i.$$

Observe that thanks to this fact, the commutators $[S_i, S_j]$ are always numbers and therefore we have

$$\exp(aS_i)\exp(bS_j) = \exp\left(aS_i + bS_i + \frac{ab}{2}[S_i, S_j]\right),$$

for any two constants a, b. Therefore we have the Weyl ordering:

$$e^{S_1+S_2} = q^{\frac{1}{2}} e^{S_1} e^{S_2} = q^{-\frac{1}{2}} e^{S_2} e^{S_1}, \quad q \equiv e^{-i\pi\hbar}.$$

Theorem

*(*L. Chekhov-M. Mazzocco-V.R*)*

Denote by X_1, X_2, X_3 the quantum Hermitian operators corresponding to x_1, x_2, x_3 as above. The quantum commutation relations are:

$$q^{-\frac{1}{2}} X_i X_{i+1} - q^{\frac{1}{2}} X_{i+1} X_i = \left(\frac{1}{q} - q\right) \epsilon_k^{(d)} X_k - (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \omega_k^{(d)} \quad (24)$$

where $\epsilon_i^{(d)}$ and $\omega_i^{(d)}$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^{\frac{1}{2}} X_3 X_1 X_2 - q \epsilon_3^{(d)} X_3^2 - q^{-1} \epsilon_1^{(d)} X_1^2 - q \epsilon_2^{(d)} X_2^2 +$$

$$q^{\frac{1}{2}} \epsilon_3^{(d)} \omega_3 X_3 + q^{-\frac{1}{2}} \omega_1^{(d)} X_1 + q^{\frac{1}{2}} \omega_2^{(d)} X_2 - \omega_4^{(d)} = 0.$$

Remark

The Hermitian operators X_1, X_2, X_3 corresponding to x_1, x_2, x_3 are introduced as follows: consider the classical expressions for x_1, x_2, x_3 in terms of s_1, s_2, s_3 and p_1, p_2, p_3. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version. For example (the case $\tilde{D}_5$): the classical x_1 is

$$x_1 = -e^{s_2+s_3} - e^{-(\tilde{s}_2+\tilde{s}_3)} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2},$$

and its quantum version is defined as

$$X_1 = -e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - e^{S_3-S_2} - e^{S_3+S_2} =$$

$$-e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - q^{-1/2}e^{-S_2}e^{S_3} - q^{1/2}e^{S_2}e^{S_3}.$$

Remark

- Our theorem and close results of Marta Mazzocco show that we can interpret the Cherednik algebra and their close "relatives"as a quantisation of the (group algebra of the) monodromy group of the Painlevé equations. Here $q := e^{-i\pi\hbar}$ and $q^n \neq 1$.

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- The Askey-Wilson $AW(3)$ (or Zhedanov algebra) can be obtained from (24) for a special constant choice after a proper "rescaling".

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- $D-$brane world: live on $D3-$brane $\perp$ $6D-$affine variety $\mathcal{M}$. $1+3D-$world-volume with SUSY YM and product gauge group.

$D-$brane algebras and superpotentials. Basic principles:

- One can associate an algebra to the category of $D-$branes at a singular point p. In every known example, the collection of possible $D-$branes at p can be described as a collection of QFT with the same Lagrangian for each of the theories.

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- More precisely, one does specify the "matter representation"(as a collection of adjoint and bifundamental fields for the gauge groups G_i) and one specifies a superpotential $W-$ the trace of a polynomial in the matter fields.
- To such data one can assign a quiver whose vertices label the groups G_i and whose directed edges specify the bifundamental and adjoint fields in the matter representation.

Quiver Theory

- Action

$$\int d^4x[\int d^4\theta \Psi_i^\dagger e^V \Psi_i + (\frac{1}{4g^2}\int d^2\theta \mathrm{Tr}\mathcal{W}_\alpha \mathcal{W}^\alpha + \int \mathrm{d}^2\theta \mathrm{W}(\overline{\psi}) + \mathrm{h.c.})]$$

W = superpotential;
$V(\phi_i;\bar{\phi}_i) = \sum_i \mid \frac{\partial W}{\partial \phi_i} \mid^2 + \frac{g^2}{4}(\sum_i q_i \mid \phi_i \mid^2)^2$

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- Encode in a Quiver:
k nodes $\Longleftrightarrow \mathrm{V}^{n_1}, \ldots \mathcal{V}^{n_k} \Longleftrightarrow \prod_{j=1}^k U(n_j)$ gauge group;
Each arrow $i \to j \Longleftrightarrow$ bifundamental fields X_{ij} of $U(n_i) \times U(n_j)$;
Each loop $i \to i \Longleftrightarrow$ adjoint fields ϕ_i of $U(n_i)$;
Superpotential $W \Longleftrightarrow$ linear combination of cycles: $\sum_i c_i$ gauge invariant operators;
Relations $\Longleftrightarrow$ jacobian of $W(\phi_i, X_{ij})$.
Vacuum:$\rightsquigarrow V(\phi_i;\bar{\phi}_i) = 0 \Rightarrow \frac{\partial W}{\partial \phi_i} = 0; \sum_i q_i \mid \phi_i \mid^2 = 0.$

Superpotential algebra

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- This is called a superpotential algebra, which is a Calabi - Yau algebra.

Elementary example

- First example, we consider the case in which P is a smooth point. In physics language, the conformal fields theory is the $N = 4$ SUSY Yang-Mills theory, written in $N = 1$ language. The $N = 4$ gauge multiplet decomposes as an $N = 1$ gauge multiplet plus three complex scalar fields X, Y, Z each transforming in the adjoint representation of the group.

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- Thus, we find

$$\mathcal{A} = \mathbb{C}[X, Y, Z],$$

the (commutative) polynomial algebra in three variables.

Example 2. Sklyanin algebra-1

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- This algebra denotes by $Q_3(\mathcal{E}, a, b, c)$ where $(a, b, c) \in \mathbb{C}^3$ such that $Q_3(\mathcal{E}, a, b, c) = \mathbb{C} < X, Y, Z > / J_W$ with $J_W =< aYZ + bZY + cX^2, aZX + bXZ + cY^2, aXY + bYX + cZ^2 >$

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- The ideal J_W can be written as a non-commutative jacobian ideal $J_W =< \partial_X, \partial_Y, \partial_Z > \in \mathbb{C} < X, Y, Z >$ for superpotential

$$W = aXYZ + bYXZ + c(X^3 + Y^3 + Z^3)$$

Example 2. Sklyanin algebra-2

- Here we consider W as a cyclic word of three variables X, Y, Z, i.e. like an element of the quotient $A_\natural := \mathbb{C} < X, Y, Z > / [\mathbb{C} < X, Y, Z >, \mathbb{C} < X, Y, Z >]$ with

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- cyclic derivatives $\partial_X, \quad \partial_Y, \quad \partial_Z$ where

$$\partial_j : A_\natural \to \mathbb{C} < X, Y, Z >, j = X, Y, Z$$

defines for any cyclic word $\varphi \in A_\natural$ by

$$\partial_j \varphi := \sum_{k | i_k = j} X_{i_k+1} X_{i_k+2} \ldots X_{i_N} \ldots X_{i_1} X_{i_2} .. X_{i_k-1} \in \mathbb{C} < X, Y, Z >$$

Example 2. Sklyanin algebra-3

Etingof-Ginzburg:

- One can identify the Sklyanin algebra $Q_3(\mathcal{E}, 1, -q, \frac{c}{3})$ with the flat deformation of the Poisson algebra $(\mathbb{C}[x, y, z], \{-, -\}_\phi)$ as above with $\phi = \frac{1}{3}(x^3 + y^3 + z^3) + \tau xyz$ and $W = XYZ - qYXZ + \frac{c}{3}(X^3 + Y^3 + Z^3)$.

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- There is a degree 3 central element $\Phi \in Z(Q_3(\mathcal{E}, 1, -q, \frac{c}{3}))$ and the quotient of the Sklyanin 3-Calabi-Yau algebra by two-sided ideal $< \Phi >$ is a flat deformation of the Poisson algebra B_ϕ.

Superpotentials of marginal and relevant deformations-1

- There is a "physical interpretation"of the Sklyanin superpotential (Berenstein-Leigh) as a marginal deformation of the superpotential from the Example 1:

$$W + W_{marg} =$$

$$= g\,\mathrm{tr}(\mathrm{X}[\mathrm{Y},\mathrm{Z}]) + \mathrm{tr}(\mathrm{aXYZ} + \mathrm{bYXZ} + \frac{\mathrm{c}}{3}(\mathrm{X}^3 + \mathrm{Y}^3 + \mathrm{Z}^3)) \in \mathrm{A}_\natural.$$

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- The structure of the vacua of D-brane gauge theories relates to the Non-Commutative Geometry also via another superpotentials (relevant deformations) having the form

$$W_{rel} = \mathrm{tr}(\frac{\mathrm{m}_1}{2}\mathrm{X}^2 + \frac{\mathrm{m}_2}{2}(\mathrm{Y}^2 + \mathrm{Z}^2) + \mathrm{e}_1\mathrm{X} + \mathrm{e}_2\mathrm{Y} + \mathrm{e}_3\mathrm{Z})$$

Superpotentials of marginal and relevant deformations-2

- The "vacua"of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

$$\partial_i W_{tot} = 0, i = X, Y, Z.$$

Superpotentials of marginal and relevant deformations-2

- The "vacua"of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

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- The defining equations (for $a = 1, b = -q$):

$$\left\{ \begin{array}{rl} X_1X_2 - qX_2X_1 = & -cX_3^2 - m_2X_3 - e_3 \\ X_2X_3 - qX_3X_2 = & -cX_1^2 - m_1X_1 - e_1 \\ X_3X_1 - qX_1X_3 = & -cX_2^2 - m_2X_2 - e_2 \end{array} \right. \tag{25}$$

This relations contain our (24) (again, after a special constant choice and a "rescaling").

Etingof-Ginzburg ideology-1:

- Let $M = \mathbb{C}^3$ considering as the simplest Calabi-Yau manifold and $\phi \in \mathcal{A} = \mathbb{C}[x_1, x_2, x_3]$ defines the Poisson bracket of jacobian type as above.

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- $M_\phi : \phi(x_1, x_2, x_3) = 0$ is an affine surface in M and the coordinate ring $\mathcal{B}_\phi := \mathbb{C}[M_\phi] = \mathcal{A}/(\phi)$ is a commutative Poisson algebra with the structure induced by ϕ

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- Let $\phi^{\tau,\nu} = \tau x_1 x_2 x_3 + \frac{\nu}{3}(x_1^3 + x_2^3 + x_3^3) + P(x_1) + Q(x_2) + R(x_3) = 0$ be the family of affine surfaces containing the E_6 del Pezzo. Here $\mathrm{degP}, \mathrm{degQ}$ and $\mathrm{degQ} < 3$.

Etingof-Ginzburg ideology-2:

- Let $A = \mathbb{C} < X_1, X_2, X_3 >$ and $A_\natural$ be defined as above and

$$\Phi^{q,\nu}_{P,Q,R} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) + P(X_1) + Q(X_2) + R(X \tag{26}$$

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- $\mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})$ is a filtered algebra defined by three inhomogeneous "jacobian"relations:

$$X_iX_j - qX_jX_i = \nu X_k^2 + \frac{\mathrm{d}P(Q,R)}{\mathrm{d}X_k}, (i,j,k) = (1,2,3) \tag{27}$$

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- The superpotential $\Phi^{q,\nu}_{P,Q,R} = \Phi^{q,\nu} + \Phi_{P,Q,R}$ where $\Phi^{q,\nu} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) \in A_\natural^{(3)}$ and $\Phi_{P,Q,R} \in A_\natural^{(\leq 2)}$ is a 3-CY-superpotential (for generic parameters)

Etingof-Ginzburg ideology-3:

$$
\begin{array}{ccc}
\mathcal{A}_\phi & \overset{\text{fl. def.}}{\rightsquigarrow} & \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R}) \\
\downarrow & & \downarrow \\
\mathcal{B}_\phi & \rightsquigarrow & B(\Phi^{q,\nu}_{P,Q,R}, \Psi) = \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})/(\Psi).
\end{array}
$$

In our case $\Phi^{q,0}_{P,Q,R} := X_1X_2X_3 - qX_2X_1X_3$

$$\Psi^{q,\epsilon,\omega} = X_1X_2X_3 - q^2X_2X_1X_3 + \epsilon_1^{(d)}\frac{q-1}{\sqrt{q}}X_1^2 + \epsilon_2^{(d)}q^{3/2}(q-1)X_2^2 +$$

(28)

$$\epsilon_3^{(d)}\frac{q^3-1}{\sqrt{q}}X_3^2 - -\omega_1^{(d)}(q-1)X_1 - \omega_2^{(d)}q(q-1)X_2 - \omega_3^{(d)}(q^2-1)X_3,$$

Links and open problems

- There are various links to Sklyanin algebras and their degenerations;
- Toric character varieties, their "uniformization"("toric theta-functions");
- Deformations of cubic divisors.
- Interesting and intriguing problems are related to a construction of NC cubic surfaces and their relations to NC cluster algebras.

Thank you