Quadratic Poisson structures and ODEs on free associative algebras.

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Reference

1. A.V. Odesskii, V. Roubtsov and V. V. Sokolov, *Non-abelian quadratic Poisson brackets*, in preparation.

Let $x_1, \ldots, x_N$ be non-commutative generators of the free associative algebra $\mathcal{A}$. We consider ODE systems of the form

$$\frac{dx_i}{dt} = F_i(\mathbf{x}), \quad \mathbf{x} = (x_1, ..., x_N), \tag{1}$$

where F_i are (non-commutative) polynomials.

The (infinitesimal) symmetry for (1) is a system

$$\frac{dx_i}{d\tau} = G_i(\mathbf{x}), \tag{2}$$

compatible with (1).

Definition. A system (1) is called integrable if it possesses infinitely many symmetries.

Example 1. Let $N = 2$, $x_1 = u$, $v = x_2$. The following system

$$u_t = v^2, \qquad v_t = u^2$$

is integrable.

Example 2. Consider the following system

$$u_t = u^2 v - v u^2, \qquad v_t = 0. \tag{3}$$

Many important *multi–component* integrable equations on associative algebras can be obtained as reductions of (3).

For instance, if u is $n \times n$ matrix such that $u^T = -u$, and v is a constant diagonal matrix, then (3) is equivalent to the n-dimensional Euler top. The integrability of this model was established by S.V. Manakov in 1976.

Consider the cyclic reduction

$$u = \begin{pmatrix} 0 & u_1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & u_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & u_{N-1} \\ u_N & 0 & 0 & 0 & \cdot & 0 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 & J_N \\ J_1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & J_2 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & J_{N-1} & 0 \end{pmatrix},$$

where u_k and J_k are matrices or even generators of a bigger free algebra. Then (3) is equivalent to the non-abelian Volterra equation

$$\frac{d}{dt}u_k = u_k u_{k+1} J_{k+1} - J_{k-1} u_{k-1} u_k, \quad k \in \mathbb{Z}_k.$$

If we assume $N = 3$, $J_1 = J_2 = J_3 = Id$ and $u_3 = -u_1 - u_2$ then the latter system yields

$$u_t = u^2 + uv + vu\,, \quad v_t = -v^2 - uv - vu\,.$$

Example 3. Let $N = 3$. Consider the system

$$x_t = [x,\ bxa+axb+xba+bax], \qquad a_t = 0, \qquad b_t = 0.$$

On the free associative algebra this system is **non**-integrable. However it is integrable on the associative algebra with identities $a^2 = b^2 = \mathbf{1}$.

The system admits the following skew-symmetric reduction

$$x^T = -x, \qquad b = a^T,$$

which leads to a non-abelian Steklov top (Odesskii-VS).

Hamiltonian structures.

Denote by $\mathcal{O}$ the associative algebra of operators on $\mathcal{A}$ generated by L_{x_i} and R_{x_i}, where

$$L_a(y) = a\,y, \qquad R_a(y) = y\,a$$

are the operators of left and right multiplication by a. We call $\mathcal{O}$ the algebra of local operators.

It follows from the associativity of algebra $\mathcal{A}$ that

$$R_a L_b = L_b R_a, \quad L_{ab} = L_a L_b, \quad R_{ab} = R_b R_a,$$

$$L_{\alpha a+\beta b} = \alpha L_a + \beta L_b, \qquad R_{\alpha a+\beta b} = \alpha R_a + \beta R_b.$$

Definition 1 *The traces for elements of $\mathcal{A}$ are defined as the corresponding elements of of quotient space $\mathcal{A}/K$, where K is the vector space spanned by all commutators in $\mathcal{A}$. If $a - b \in K$, we write $a \sim b$.*

Definition 2 *Let $a(\mathbf{x}) \in \mathcal{A}$. Then grad(tr a) is the vector uniquely defined by:*

$$\frac{d}{d\epsilon} a(\mathbf{x} + \epsilon\, \delta\mathbf{x})|_{\epsilon=0} \sim < \delta\mathbf{x}, \mathit{grad}\, a(\mathbf{x})) > ,$$

where $< (p_1, \ldots, p_N), (q_1, \ldots, q_N) >= p_1 q_1 + \cdots + p_N q_N$.

If $a \sim b$, (or, the same, $\mathrm{tr}\, a = \mathrm{tr}\, b$), then $\mathrm{grad}\, a = \mathrm{grad}\, b$.

Definition 3 *We shall call $\Theta \in \mathcal{O} \otimes gl_N$ a local Hamiltonian operator, if the Poisson bracket*

$$\{a, b\} = < \mathit{grad}\, a,\ \Theta(\mathit{grad}\, b) >, \qquad a, b \in \mathcal{A}$$

satisfies conditions

$$\{a, b\} + \{b, a\} \sim 0,$$

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} \sim 0$$

for any elements $a, b, c \in \mathcal{A}$.

We call such bracket non-abelian Poisson bracket.

Example 2 (continuation). The system (3) can be written in the Hamiltonian form

$$u_t = \Theta(\mathrm{grad}_u H),$$

where $\Theta = L_u R_v - L_v R_u$, and $H = \frac{1}{2}\mathrm{tr}\, u^2$. Indeed, $\mathrm{grad}_u H = u$ and $u_t = uuv - vuu$.

General Hamiltonian equation on $\mathcal{A}$ has the form

$$\mathbf{x}_t = \Theta\, \mathrm{grad}(\mathrm{tr}\, H(\mathbf{x})),$$

where H is a Hamiltonian of the equation, Θ is a Hamiltonian operator. We study the **local** Hamiltonian operators, i.e. assume that $\Theta \in \mathcal{O} \otimes gl_N$.

Proposition. Any linear non-abelian Poisson bracket is given by the Hamiltonian operator

$$\Theta_{i,j} = b^p_{ij} L_{x_p} - b^p_{ji} R_{x_p},$$

where b^p_{ij} are structural constant of an associative algebra.

If x_α are $m \times m$-matrices then we can extend the linear non-abelian Poisson bracket to the matrix entries in the following way. We have

$$x^j_{i,\alpha} = \mathrm{tr}(e^i_j x_\alpha), \quad x^{j'}_{i',\beta} = \mathrm{tr}(e^{i'}_{j'} x_\beta),$$

where e^i_j stand for the matrix unities. We put

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = \mathrm{tr}(e^i_j \Theta_{\alpha,\beta}(e^{i'}_{j'})).$$

Using the formula for the Hamiltonian operator, we find

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = b^\gamma_{\alpha,\beta} x^{j'}_{i,\gamma} \delta^j_{i'} - b^\gamma_{\beta,\alpha} x^j_{i',\gamma} \delta^{j'}_i.$$

Consider quadratic non-abelian Poisson brackets.

Proposition. Any quadratic non-abelian Poisson bracket is given by the Hamiltonian operator

$$\Theta_{i,j} = a_{ij}^{pq} L_{x_p} L_{x_q} - a_{ji}^{qp} R_{x_p} R_{x_q} + c_{ij}^{pq} L_{x_p} R_{x_q},$$

where $p, q, i, j = 1, ..., N$. The constants a_{ij}^{pq} and c_{ij}^{pq} satisfy identities

$$c_{ij}^{pq} = -c_{ji}^{qp},$$

$$c_{ij}^{up} c_{pk}^{vw} + c_{jk}^{vp} c_{pi}^{wu} + c_{ki}^{wp} c_{pj}^{uv} = 0,$$

$$a_{ij}^{pu} a_{kp}^{vw} = a_{ki}^{vp} a_{pj}^{wu},$$

$$a_{ij}^{pu} a_{pk}^{vw} = a_{ij}^{vp} c_{kp}^{uw} + a_{ip}^{vw} c_{jk}^{pu}.$$

and

$$a_{ij}^{up} a_{kp}^{vw} = a_{ij}^{pw} c_{pk}^{uv} + a_{pj}^{vw} c_{ki}^{pu}.$$

In the matrix case the extension to the matrix entries is given by

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = c^{\gamma,\delta}_{\alpha,\beta} x^{j'}_{i,\gamma} x^{j}_{i',\delta} + a^{\gamma,\delta}_{\alpha,\beta} x^{k}_{i,\gamma} x^{j'}_{k,\delta} \delta^{j}_{i'} - a^{\gamma,\delta}_{\beta,\alpha} x^{k}_{i',\gamma} x^{j}_{k,\delta} \delta$$

Under change of basis $x_i \to g_i^j x_j$ the constants are transformed in a standard way:

$$c^{kl}_{ij} \to g^{\alpha}_{i} g^{\beta}_{j} h^{k}_{\gamma} h^{l}_{\epsilon}\, c^{\gamma,\epsilon}_{\alpha,\beta}, \qquad a^{kl}_{ij} \to g^{\alpha}_{i} g^{\beta}_{j} h^{k}_{\gamma} h^{l}_{\epsilon}\, a^{\gamma,\epsilon}_{\alpha,\beta}. \tag{4}$$

Here $g_i^j h_j^k = \delta_i^k$. Moreover, the system of identities admits the following discrete involution:

$$c^{kl}_{ij} \to c^{kl}_{ji}, \qquad a^{kl}_{ij} \to a^{lk}_{ji}. \tag{5}$$

In the matrix case the involution corresponds to the transposition $x_i \to x_i^T$. Brackets related by (4),(5) are called *equivalent*.

Let V be a linear space with a basis v_i, $i = 1, ..., N$. Define linear operators C, A on the space $V \otimes V$ by

$$Cv_i \otimes v_j = c_{ij}^{pq} v_p \otimes v_q, \qquad Av_i \otimes v_j = a_{ij}^{pq} v_p \otimes v_q.$$

Then the identities can be rewritten in the following form:

$$C^{12} = -C^{21}, \quad C^{23}C^{12} + C^{31}C^{23} + C^{12}C^{31} = 0,$$

$$A^{12}A^{31} = A^{31}A^{12},$$

$$\sigma^{23}A^{13}A^{12} = A^{12}C^{23} - C^{23}A^{12},$$

$$A^{32}A^{12} = C^{13}A^{12} - A^{32}C^{13}.$$

Here all operators act in $V \otimes V \otimes V$, by σ^{ij} we mean transposition of i-th and j-th component of the tensor product and A^{ij}, C^{ij} mean operators A, C acting in the product of the i-th and j-th components.

A vector $\Lambda = (\lambda_1, ..., \lambda_N)$ is said to be *admissible* if for any i, j

$$(a_{ij}^{pq} - a_{ji}^{qp} + c_{ij}^{pq})\lambda_p\lambda_q = 0.$$

For any admissible vector the argument shift $x_i \to x_i + \lambda_i \mathrm{Id}$ yields a linear Poisson bracket with

$$b_{ij}^{p} = (a_{ij}^{qp} + a_{ij}^{pq} + c_{ij}^{pq})\lambda_q,$$

compatible with the quadratic one.

Classification in the case $N = 2$.

Theorem. Let $N = 2$. Then the following Cases 1-5 form a complete list of non-abelian quadratic Poisson brackets up to equivalence and proportionality.

We present non-zero components of the tensors a and c only.

Case 1. $c^{22}_{12} = 1,\ c^{22}_{21} = -1;$

Case 2. $c^{21}_{11} = 1,\ c^{12}_{11} = -1,\ a^{22}_{21} = a^{12}_{11} = -1;$

Case 3. $c^{22}_{12} = 1,\ c^{22}_{21} = -1,\ a^{12}_{11} = a^{22}_{21} = 1;$

Case 4. $a^{22}_{11} = 1.$

Case 5. $c^{21}_{11} = 1,\ c^{12}_{11} = -1.$

The matrix **Case 1** admits the following description.

Let $G = GL_n(\mathbb{R})$ and $TG \simeq \mathfrak{g} \times G = \{(X, Y) | \mathrm{det} Y \neq 0\}$. Define the following 2-form:

$$\Omega := \mathrm{tr} dX \wedge d(Y^{-1}),$$

where the matrix differential defines as $dX := ||dx_{ij}||$ and the wedge product combines with the matrix one. The matrix of Ω is written as:

$$\Delta^{-2} \begin{pmatrix} \mathbb{O} & \mathbb{S} \\ -\mathbb{S} & \mathbb{O} \end{pmatrix}, \quad \Delta := \mathrm{det} Y \neq 0,$$

where the entries of $\mathbb{S}$ are monomials $y_{pq} y_{rs}$.

The form Ω defines a Poisson structure on $\mathbb{R}^{2n^2}$ with coordinates x_{ij}, y_{kl}. The Poisson tensor Π has the matrix

$$\Delta^2 \begin{pmatrix} \mathbb{O} & -\mathbb{S}^{-1} \\ \mathbb{S}^{-1} & \mathbb{O} \end{pmatrix}.$$

Remarkably, the entries of Π are quadratic monomials in y_{ij} and the Poisson bracket is equivalent to the bracket from Case 1.

In the matrix **Case 2** the Poisson structure is also non-degenerate. We don't know any description similar to the above.

In the matrix **Case 4** the linear Casimir functions are: $\mathrm{tr}\, x_1$ and $x^i_{j,2}$ for all i, j. This Poisson structure is trivial in the following sense. If we fix $x_2 = C$, then the dynamics of $u = x_1$ has the form

$$\frac{du}{dt} = [C^2, H],$$

where H is a non-commutative polynomial in u, C.

For the matrix **Case 5** the linear Casimir functions are $x^i_{j,2}$ for all i, j. This bracket defines the Poisson structure for the non-abelian Euler-Manakov top.

Consider **Case 3**. The Casimir functions are:

$$\operatorname{tr} x_2^k, \qquad \operatorname{tr} x_1 x_2^k, \qquad k = 0, 1, ...$$

The simplest integrable non-abelian ODE with Poisson bracket **Case 3** has the form

$$u_t = uvu - u^2v, \qquad v_t = v^2u - vuv,$$

where $u = x_1, v = x_2$. The Hamiltonian is $\frac{1}{2}\operatorname{tr} u^2$.

The reduction $u = Cv$ gives rise to

$$v_t = v^2Cv - vCv^2.$$

The cyclic reduction of the latter equation yields the non-abelian modified Volterra equation.

We show that in the matrix case our quadratic Poisson bracket is equivalent to a pencil of compatible linear Poisson brackets.

Suppose that

$$x_2 = T\Lambda T^{-1}, \quad x_1 = TYT^{-1},$$

where Y is a generic matrix, $\Lambda = diag(\lambda_1, ..., \lambda_m)$, where $\lambda_i \neq \lambda_j$ and $\lambda_i \neq 0$, and T is a generic invertible matrix with $t_{1,j} = 1$.

Consider $y_{i,j}$ and $t_{i,j}$, $i > 1$ as coordinates on the corresponding $(2m^2 - m)$-dimensional Poisson submanifold. Then in this coordinates the restriction of the initial quadratic Poisson bracket $\{,\}$ has the form

$$\{,\} = \sum_{i=1}^{m} \lambda_i \{,\}_i,$$

where $\{,\}_i$ are some linear Poisson brackets.

Describe the structure of the Lie algebra $\mathcal{G}$ corresponding to the pencil. It turns out that

$$\mathcal{G} = \mathcal{Y} \oplus \mathcal{T},$$

where $[\mathcal{Y}, \mathcal{Y}] \subset \mathcal{Y}$, $[\mathcal{Y}, \mathcal{T}] \subset \mathcal{T}$, $[\mathcal{T}, \mathcal{T}] = \{0\}$. Subalgebra $\mathcal{Y}$ of dimension m^2 is generated by y_{ij} and $\mathcal{Y}$-module $\mathcal{T}$ of dimension $m(m-1)$ is generated by $t_{i,j}$, $i > 1$.

Agebra $\mathcal{Y}$ is a trivial central extension by $y_{1,1}, ..., y_{m,m}$ of algebra $\mathcal{Z}$ spanned by $z_{i,j} = y_{i,j} - y_{i,i}$, where $i \neq j$.

The radical of $\mathcal{Z}$ is spanned by $r_i = \sum_{j \neq i} \frac{1}{\lambda_j} z_{j,i}$.

The centralizer $\mathcal{S}$ of r_1 is isomorphic to $gl(m-1)$ with r_1 being the center. The isomorphism between $\mathcal{S}$ and $Mat(m-1)$ is given by

$$e^i_j \to \frac{1}{\lambda_j}(z_{j+1,1} - z_{j+1,i+1}), \qquad i, j = 1, ..., m-1,$$

where $z_{k,k} = 0$ for any k. Here e^i_j are the matrix unities.

The radical of $\mathcal{Z}$ is the direct sum of two commutative $\mathcal{S}$-modules of dimensions $m-1$ and 1. The first one is spanned by $v_i = r_i - r_1$. The second is generated by r_1. The commutator relations between the modules is given by $[r_1, v_i] = v_i$.

The module $\mathcal{T}$ is a direct sum of n-dimensional submodules $\mathcal{T}_i$ spanned by $t_{i,k}$, $i > 1$. The commutator relations are

$$[y_{i,j}, t_{k,l}] = \delta_l^i \lambda_i (t_{k,i} - t_{k,j}).$$

$\square$