"Algebraic and Geometric Structures of Painlevé monodromy varietes"

Vladimir Rubtsov, ITEP, Moscow
and LAREMA, Université d'Angers,
(joint work in progress with
M. Mazzocco and ...???)

(An attempt of) a Scrap-Talk at **Workshop "Geometry of PDEs and Integrability".**

Teplice-nad-Bečvou, October, 4, 2013

Plan:

- Painlevé equations;
- Isomonodromy and Riemann-Hilbert;
- Affine cubics;
- Singularities and cluster transformations;
- Quantisation an relations to Sklyanin algebras
- Perspectives and output;

Painlevé equations

The Painlevé equations are non linear second order ODE of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \qquad z \in \mathbb{C},$$

where $F(z, w, y)$ is a rational function of z, w, y and the solutions $w(z; c_1, c_2)$ satisfy

1. Painlevé–Kowalevski property: $w(z; c_1, c_2)$ have no *critical points* that depend on c_1, c_2.
2. Otherwise, they are the only second order ODE without movable singularities (branching points).
3. For generic c_1, c_2, $w(z; c_1, c_2)$ are new functions, Painlevé Transcendents.

Painlevé I,II,II,IV

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

Painlevé V and VI

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - $$

$$-\frac{1}{z}\frac{dw}{dz} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right)\frac{\delta w(w+1)}{w-1},$$

$$\frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1}\right)w_z +$$

$$+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

Painlevé parameters

Denote $z = t$ and

$$\alpha := (\theta_\infty - 1/2)^2; \qquad \beta := -\theta_0^2;$$

$$\gamma := \theta_1^2; \qquad \delta := (1/4 - \theta_t)^2.$$

Painlevé transcendents - paradigmatic integrable systems

- Reductions of soliton equations (KdV, KP, NLS);

Painlevé transcendents - paradigmatic integrable systems

- Reductions of soliton equations (KdV, KP, NLS);
- They admit a Hamiltonian formulation;

Painlevé transcendents - paradigmatic integrable systems

- Reductions of soliton equations (KdV, KP, NLS);
- They admit a Hamiltonian formulation;
- They can be expressed as the isomonodromic deformation of some linear differential equation with rational coefficients;

Painlevé transcendents - paradigmatic integrable systems

- Reductions of soliton equations (KdV, KP, NLS);
- They admit a Hamiltonian formulation;
- They can be expressed as the isomonodromic deformation of some linear differential equation with rational coefficients;
- All Painlevés (except for P_I) admit one-parameter family of solutions (in terms of special functions) and for some special values of parameteres they have particular rational solutions;

Painlevé transcendents - paradigmatic integrable systems

- Reductions of soliton equations (KdV, KP, NLS);
- They admit a Hamiltonian formulation;
- They can be expressed as the isomonodromic deformation of some linear differential equation with rational coefficients;
- All Painlevés (except for P_I) admit one-parameter family of solutions (in terms of special functions) and for some special values of parameteres they have particular rational solutions;
- Recently: P_{II} - has a genuine fully NC analogue (V. Retakh-V.R.)

Painlevé sixth equation The Painlevé VI equation describes the isomonodromic deformations of the following

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \left(\frac{A_1(t)}{\lambda - u_1} + \frac{A_2(t)}{\lambda - u_2} + \frac{A_3(t)}{\lambda - u_3} \right) \Phi, \tag{1}$$

where

$$\mathrm{eigen}(A_i) = \pm \frac{\theta_i}{2}, \quad \text{for } i = 1, 2, 3, \ A_\infty := -A_1 - A_2 - A_3 \tag{2}$$

$$A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & \\ & -\frac{\theta_\infty}{2} \end{pmatrix}, \tag{3}$$

In this talk: $(u_1, u_2, u_3, \infty) := (0, 1, t, \infty)$ and $(\theta_1, \theta_2, \theta_3, \theta_\infty) := (\theta_0, \theta_1, \theta_t, \theta_\infty)$.

The solution $\Phi(\lambda)$ of the system (1) is a multi-valued analytic function in the punctured Riemann sphere $\mathbb{P}^1 \setminus \{u_1, u_2, u_3, \infty\}$ and its multivaluedness is described by the so-called monodromy matrices, i.e. the images of the generators of the fundamental group under the anti-isomorphism

$$\rho : \pi_1 \left(\mathbb{P}^1 \setminus \{u_1, u_2, u_3, \infty\}, \lambda_1\right) \to SL_2(\mathbb{C}).$$

We fix the base point λ_1 at infinity and the generators of the fundamental group to be l_1, l_2, l_3 such that l_i encircles only the pole i once and are oriented in such a way that

$$M_1 M_2 M_3 M_\infty = \mathbb{I}, \qquad M_\infty = \exp(2\pi i A_\infty). \tag{4}$$

Let:

$$G_i := \mathrm{Tr}(M_i) = 2\cos(\pi\theta_i), \quad i = 1, 2, 3, \infty,$$

The Riemann-Hilbert correspondence

$$\mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty)\backslash\mathcal{G} \to \mathcal{M}(\theta_1, \theta_2, \theta_3, \theta_\infty)\backslash GL_2(\mathbb{C}),$$

where $\mathcal{G}$ is the gauge group, is defied by associating to each Fuchsian system its monodromy representation class. The representation space $\mathcal{M}(G_1, G_2, G_3, G_\infty)$ is realised as an affine cubic surface with

$$x_1 = \mathrm{Tr}\left(M_2 M_3\right), \quad x_2 = \mathrm{Tr}\left(M_1 M_3\right), \quad x_3 = \mathrm{Tr}\left(M_1 M_2\right).$$

We parameterise local solutions $w(t; c_1, c_2)$ of PVI by points on the cubic.

Analytic continuation $\rightarrow$ nonlinear action

$$\pi_1(\overline{\mathbb{C}} \setminus \{0,1,\infty\}) \ni \gamma : (c_1, c_2) \rightarrow (c_1^{[\gamma]}, c_2^{[\gamma]}).$$

Loops around $0, 1, \infty$ in $\mathbb{C}\backslash\{0,1,\infty\} \Rightarrow$ loops $(u_1, u_2, u_3) \in \mathbb{C}^3\backslash\{\Delta\}$. Pure braid goup

$$\pi_1(\mathbb{C}^3 \setminus \Delta) = P_3$$

Here Δ means the "set of diagonals" in $\mathbb{C}^3$:

$$\Delta := \{z_i = z_j\}.$$

Following Sakai, there are eight Painlevé equations corresponding to the eight extended Dynkin diagrams:

$$\widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8,$$

corresponding respectively to PVI, PV, three different cases of PIII, PIV, PII and PI.
Their monodromy manifolds were studied by several authors, but were recently presented in a unified way:

$$\widetilde{D}_4 \quad x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0,$$

$$\widetilde{D}_5 \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0,$$

$$\widetilde{D}_6 \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1 = 0,$$

$$\widetilde{D}_7 \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 = 0,$$

$$\widetilde{D}_8 \quad x_1x_2x_3 + x_1^2 + x_2^2 + 1 = 0,$$

$$\widetilde{E}_6 \quad x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 (x_2 + x_3) + 1 + \omega_4 = 0,$$

$$\widetilde{E}_7^{*} \quad x_1x_2x_3 + x_1 + x_2 + x_3 + \omega_4 = 0,$$

$$\widetilde{E}_7^{**} \quad x_1x_2x_3 + x_1 + \omega_2 x_2 + x_3 - \omega_2 + 1 = 0,$$

General Affine Cubic

The main object studied in this talk is the affine irreducible cubic surface $M_\phi := \mathbb{C}[x_1, x_2, x_3]/_{\langle \phi = 0 \rangle}$ where

$$\phi = x_1 x_2 x_3 + \epsilon_1^{(d)} x_1^2 + \epsilon_2^{(d)} x_2^2 + \epsilon_3^{(d)} x_3^2 + \omega_1^{(d)} x_1 + \omega_2^{(d)} x_2 + \omega_3^{(d)} x_3 + \omega_4^{(d)} = 0, \tag{5}$$

According to Saito and Van der Put, the monodromy manifolds $\mathcal{M}^{(d)}$ have all the form of M_ϕ

Here d is an index running on the list of the extended Dynkin diagrams $\widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8$ and the parameters $\epsilon_i^{(d)}$, $\omega_i^{(d)}$, $i = 1, 2, 3$ are given by:

$$
\begin{aligned}
\epsilon_1^{(d)} &= \begin{cases} 1 & \text{for } d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \\ 0 & \text{for } d = \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8, \end{cases} \\
\epsilon_2^{(d)} &= \begin{cases} 1 & \text{for } d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8 \\ 0 & \text{for } d = \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8, \end{cases} \\
\epsilon_3^{(d)} &= \begin{cases} 1 & \text{for } d = \widetilde{D}_4, \\ 0 & \text{for } d = \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8. \end{cases}
\end{aligned} \tag{6}
$$

The coefficients $\omega^{(d)}$ are defined by:

$$\begin{aligned}
\omega_1^{(d)} &= -G_1^{(d)} G_\infty^{(d)} - \epsilon_1^{(d)} G_2^{(d)} G_3^{(d)}, \\
\omega_2^{(d)} &= -G_2^{(d)} G_\infty^{(d)} - \epsilon_2^{(d)} G_1^{(d)} G_3^{(d)}, \\
\omega_3^{(d)} &= -G_3^{(d)} G_\infty^{(d)} - \epsilon_3^{(d)} G_1^{(d)} G_2^{(d)}, \\
\omega_4^{(d)} &= \epsilon_2^{(d)} \epsilon_3^{(d)} \left(G_1^{(d)} \right)^2 + \epsilon_1^{(d)} \epsilon_3^{(d)} \left(G_2^{(d)} \right)^2 + \epsilon_1^{(d)} \epsilon_2^{(d)} \left(G_3^{(d)} \right)^2 + \\
&\left(G_\infty^{(d)} \right)^2 + G_1^{(d)} G_2^{(d)} G_3^{(d)} G_\infty^{(d)} - 4 \epsilon_1^{(d)} \epsilon_2^{(d)} \epsilon_3^{(d)},
\end{aligned} \tag{7}$$

Here $G_1^{(d)}, G_2^{(d)}, G_3^{(d)}, G_\infty^{(d)}$ are some constants related to the parameters appearing in the Painlevé equations as follows:

$$G_1^{(d)} = \begin{cases} 2\cos\pi\theta_0 & d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{E}_6 \\ e^{-\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^{*} \\ e^{-i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 1 & d = \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_8 \\ e^{\frac{i\pi(\theta_0+\theta_\infty)}{2}} + e^{\frac{-i\pi(\theta_0+\theta_\infty)}{2}} & d = \widetilde{D}_6, \end{cases}$$

$$G_2^{(d)} = \begin{cases} 2\cos\pi\theta_1 & d = \widetilde{D}_4, \widetilde{D}_5, \\ 2\cos\pi\theta_\infty & d = \widetilde{E}_6 \\ e^{-\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^{*} \\ e^{i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 1 & d = \widetilde{D}_8, \widetilde{E}_8 \\ e^{\frac{i\pi(\theta_0-\theta_\infty)}{2}} + e^{\frac{i\pi(-\theta_0+\theta_\infty)}{2}} & d = \widetilde{D}_6 \end{cases}$$

$$G_3^{(d)} = \begin{cases} 2\cos\pi\theta_t & d = \widetilde{D}_4, \\ 1 & d = \widetilde{D}_5, \widetilde{D}_7 \\ 2\cos\pi\theta_\infty & d = \widetilde{E}_6 \\ e^{-\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^* \\ e^{-i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 0 & d = \widetilde{D}_6, \widetilde{D}_8, \widetilde{E}_8 \end{cases}$$

$$G_\infty^{(d)} = \begin{cases} 2\cos\pi\theta_\infty & d = \widetilde{D}_4, \widetilde{D}_5, \widetilde{E}_6 \\ e^{\frac{i\pi(\theta_0+1)}{2}} & d = \widetilde{E}_7^* \\ e^{i\pi\theta_0} & d = \widetilde{E}_7^{**} \\ 1 & d = \widetilde{D}_8, \widetilde{E}_8 \\ e^{\frac{i\pi(\theta_0+\theta_\infty)}{2}} & d = \widetilde{D}_6 \\ 0 & d = \widetilde{D}_7 \end{cases}$$

This family of cubics is a variety $M_\phi = \{(\bar{x}, \bar{\omega}) \in \mathbb{C}^3 \times \Omega) : \phi(\bar{x}, \bar{\omega}) = 0\}$ where $\bar{x} = (x_1, x_2, x_3), \quad \bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and the "$\bar{x}-$forgetful" projection $\pi : M_\phi \to \Omega : \pi(\bar{x}, \bar{\omega}) = \bar{\omega}$. This projection defines a family of affine cubics with generically non–singular fibres $\pi^{-1}(\bar{\omega})$ The cubic surface $M_{\phi_{\bar{\omega}}}$ has a volume form $\vartheta_{\bar{\omega}}$ given by the Poincaré residue formulae:

$$\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial \phi_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial \phi_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial \phi_{\bar{\omega}})/(\partial x_2)}. \qquad (8)$$

The volume form is a holomorphic 2-form on the non-singular part of $M_{\phi_{\bar{\omega}}}$ and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

$$\{x_1, x_2\}_{\bar{\omega}} = \frac{\partial \phi_{\bar{\omega}}}{\partial x_3} \tag{9}$$

The other brackets are defined by circular transposition of x_1, x_2, x_3. For $(i, j, k) = (1, 2, 3)$:

$$\{x_i, x_j\}_{\bar{\omega}} = \frac{\partial \phi_{\bar{\omega}}}{\partial x_k} = x_i x_j + 2\epsilon_i^d x_k + \omega_i^d \tag{10}$$

and the volume form (8) reads as

$$\vartheta_{\bar{\omega}} = \frac{dx_i \wedge dx_j}{(\partial \phi_{\bar{\omega}})/(\partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2\epsilon_i^d x_k + \omega_i^d)}. \tag{11}$$

Observe that for any $\phi \in \mathbb{C}[x_1, x_2, x_3]$ the following formulae define a Poisson bracket on $\mathbb{C}[x_1, x_2, x_3]$:

$$\{x_i, x_{i+1}\} = \frac{\partial \phi}{\partial x_{i+2}}, \qquad x_{i+3} = x_i, \tag{12}$$

and ϕ itself is a central element for this bracket, so that the quotient space

$$M_\phi := \mathbb{C}[x_1, x_2, x_3]/_{\langle \phi = 0 \rangle}$$

inherits the Poisson algebra structure [Nambu $\sim$ 70].
Today I am going to quantize it.

Affine Cubic as it is:

- In singularity theory - the universal unfolding of the D_4 singularity.

Affine Cubic as it is:

- In singularity theory - the universal unfolding of the D_4 singularity.
- Oblomkov: the quantisation of the affine cubic surface M_ϕ coincides with spherical subalgebra of the generalised rank 1 double affine Hecke algebra H (or Cherednick algebra of type $C_1 C_1^\nu$)

Affine Cubic as it is:

- In singularity theory - the universal unfolding of the D_4 singularity.
- Oblomkov: the quantisation of the affine cubic surface M_ϕ coincides with spherical subalgebra of the generalised rank 1 double affine Hecke algebra H (or Cherednick algebra of type $C_1 C_1^\nu$)
- It appears in the Teichmüller theory of a Riemann sphere with 4 holes/m points of arbitrary order. [Chekhov-Mazzocco]

Affine Cubic as it is:

- In singularity theory - the universal unfolding of the D_4 singularity.
- Oblomkov: the quantisation of the affine cubic surface M_ϕ coincides with spherical subalgebra of the generalised rank 1 double affine Hecke algebra H (or Cherednick algebra of type $C_1 C_1^\nu$)
- It appears in the Teichmüller theory of a Riemann sphere with 4 holes/m points of arbitrary order. [Chekhov-Mazzocco]
- In the theory of the PVI equation - the manifold of monodromy data. [Jimbo] The confluences are monodromy manifolds of all other P-eqs [Saito van der Put]

Affine Cubic as it is:

- In singularity theory - the universal unfolding of the D_4 singularity.
- Oblomkov: the quantisation of the affine cubic surface M_ϕ coincides with spherical subalgebra of the generalised rank 1 double affine Hecke algebra H (or Cherednick algebra of type $C_1 C_1^\nu$)
- It appears in the Teichmüller theory of a Riemann sphere with 4 holes/m points of arbitrary order. [Chekhov-Mazzocco]
- In the theory of the PVI equation - the manifold of monodromy data. [Jimbo] The confluences are monodromy manifolds of all other P-eqs [Saito van der Put]
- In algebraic geometry - projective completion:

$$\overline{M}_{\tilde{\phi}} := \{(u, v, w, t) \in \mathbb{P}^3 \,| x_1^2 t + x_2^2 t + x_3^2 t - x_1 x_2 x_3 +$$

$$+\omega_3 x_1 t^2 + \omega_2 x_2 t^2 + \omega_3 x_3 t^2 + \omega_4 t^3 = 0\}$$

is a del Pezzo surface of degree three and differs from it by three smooth lines at infinity forming a triangle [Oblomkov] $t = 0, \qquad x_1 x_2 x_3 = 0.$

Singularities

Dynkin	Painlevé equations	Surface singularity type
$\widetilde{D}_4$	P_{VI}	D_4
$\widetilde{D}_5$	P_V	A_3
$\widetilde{D}_6$	deg $P_V{=}P_{III}(\widetilde{D}_6)$	A_1
$\widetilde{D}_6$	$P_{III}(\widetilde{D}_6)$	A_1
$\widetilde{D}_7$	$P_{III}(\widetilde{D}_7)$	non-singular
$\widetilde{D}_8$	$P_{III}(\widetilde{D}_8)$	non-singular
$\widetilde{E}_6$	P_{IV}	A_2
$\widetilde{E}_7^*$	$P_{II}(FN)$	A_1
$\widetilde{E}_7^{**}$	$P_{II}(MJ)$	A_1
$\widetilde{E}_8$	P_I	non-singular

Table:

The meaning of the table: for each Painlevé equation from the first column there is at least one singular fibre with singularity of the type given in the second column of the table.

Singularity Theory

A singularity of a function $f(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_n)$, is an isolated critical point $\mathbf{x_0}$, i.e. $\mathrm{d}f = 0$. Arnol'd classified all these up to analytic coordinate tarnsformations, what he called right equivalence.

Simple singularities are called Kleinian singularities.

$$A_k : x_1^{k+1} + x_2^2 + \ldots, x_n^2,$$

$$D_k : x_1(x_1^{k-2} + x_2^2) + x_3^2 + \ldots, x_n^2,$$

and so on. On $\mathbb{C}^3$ the can all be recasted in the form:

$$x_1^p + x_2^q + x_3^r + ax_1x_2x_3, \qquad a \neq 0, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

D_4 corresponds to $p = q = r = 2$.

$\widetilde{D}_4$

Show that the cubic of PVI is diffeomorphic to the versal unfolding of D_4 and map this cubic to the Arnol'd form:

- shift all variables by $x_i \to x_i + 2$, $i = 1, 2, 3$ to obtain

$$x_1^2+x_2^2+x_3^2+2x_1x_2+2x_2x_3+2x_1x_3+x_1x_2x_3+\widetilde{\omega}_1x_1+\widetilde{\omega}_2x_2+\widetilde{\omega}_3x_3+\widetilde{\omega}_4 = 0 \qquad (13)$$

  where

$$\widetilde{\omega}_i = \omega_i+8, \quad \text{for } i = 1, 2, 3, \qquad \widetilde{\omega}_4 = \omega_4+2(\omega_1+\omega_2+\omega_3)+20.$$

- use the following diffeomorphism around the origin:

$$x \to x - \frac{1}{2}y, \quad y \to x + \frac{1}{2}x, \quad z \to z + \frac{y^2}{8} - 2x - \frac{x^2}{2} - \frac{\widetilde{\omega}_3}{2}$$

- The new cubic (up to a Morse singularity and after a shift $x \to x - \frac{\omega_3}{4}$) becomes the versal unfolding of a D_4 singularity in Arnol'd form:

$$-2x_1^3 + \frac{x_1x_2^2}{2} + \widehat{\omega}_1x_1 + \widehat{\omega}_2x_2 + \widehat{\omega}_3x_1^2 + \widehat{\omega}_4.$$

Here

$$\widehat{\omega}_1 = \omega_1 + \omega_2 - 8 - 4\omega_3 - \frac{\omega_3^2}{8}, \quad \widehat{\omega}_2 = \frac{\omega_2 - \omega_1}{2},$$
$$\widehat{\omega}_3 = 8 + \omega_3, \quad \widehat{\omega}_4 = \omega_4 + 2\omega_3 - \frac{\omega_3(\omega_1 + \omega_2 - \omega_3)}{4} + 4.$$

The above formulae show that the versal unfolding parameters $\widehat{\omega}_1, \ldots, \widehat{\omega}_4$ are independent as long as $\omega_1, \ldots, \omega_4$ are.

Braid group action

Dubrovin-Mazzocco: the procedure of analytic continuation of a local solution to the Painlevé VI corresponds to the following action of the braid group on the monodromy manifold:

$$\beta_1 : \begin{array}{lcl} x_1 & \to & -x_1 - x_2 x_3 - \omega_1, \\ x_2 & \to & x_3, \\ x_3 & \to & x_2, \end{array} \tag{14}$$

$$\beta_2 : \begin{array}{lcl} x_1 & \to & x_3, \\ x_2 & \to & -x_2 - x_1 x_2 - \omega_2, \\ x_3 & \to & x_1, \end{array} \tag{15}$$

$$\beta_3 : \begin{array}{lcl} x_1 & \to & x_2, \\ x_2 & \to & x_1, \\ x_3 & \to & -x_3 - x_1 x_2 - \omega_3. \end{array} \tag{16}$$

Note that two of these are enough to generate the whole braid group.

Theorem

*(*M. Mazzocco -V.R.*) When $G_\infty = 2$ (geometrically this means that we have a puncture at infinity), the action of the braid group coincides with a tagged cluster algebra structure of* Chekhov-M.Shapiro.

In order to see this let us compose each braid with a Okamoto symmetry in order to obtain the following

$$\widetilde{\beta}_i: \begin{array}{ccl} x_i & \to & -x_i - x_j x_k - \omega_i, \quad j,k \neq i, \\ x_j & \to & x_j, \quad \text{for} \quad j \neq i \end{array} \tag{17}$$

For the cubic (5) this transformation acquires a cluster flavour:

$$\widetilde{\beta}_i: \; x_i x_i' = x_j^2 + x_k^2 + \omega_j x_j + \omega_k x_k + \omega_4 \quad j,k \neq i. \tag{18}$$

Indeed let us introduce the shifted variables:

$$y_i := x_i - G_i, \quad i = 1, 2, 3,$$

they satisfy the tagged cluster algebra relation:

$$\mu_i : \; y_i y_i' = y_j^2 + y_k^2 + G_i y_j y_k \quad j, k \neq i. \tag{19}$$

Note that tagged cluster algebras satisfy the Laurent phenomenon. In particular this result implies that procedure of analytic continuation of the solutions to the Painlevé VI satisfies the Laurent phenomenon: if we start from a local solution corresponding to the point (y_1^0, y_2^0, y_3^0) on the shifted Painlevé cubic

$$y_1 y_2 y_3 + y_1^2 + y_2^2 + y_3^2 + G_1 y_2 y_3 + G_2 y_1 y_3 + G_3 y_1 y_2 = 0$$

any other branch of that solution will corresponds to points (y_1, y_2, y_3) on the same cubic such that each y_i is a Laurent polynomial of the initial (y_1^0, y_2^0, y_3^0).

Quantisation

Theorem

*(*M. Mazzocco-V.R*)*
Denote by X_1, X_2, X_3 the quantum Hermitian operators corresponding to x_1, x_2, x_3 as above. The quantum commutation relations are:

$$q^{\frac{1}{2}} X_i X_{i+1} - q^{-\frac{1}{2}} X_{i+1} X_i = \left(\frac{1}{q} - q\right) \epsilon_k^{(d)} X_k + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \omega_k^{(d)} \quad (20)$$

where $\epsilon_i^{(d)}$ and $\omega_i^{(d)}$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^{\frac{1}{2}} X_3 X_1 X_2 + q X_3^2 + q^{-1} \epsilon_1^{(d)} X_1^2 + q \epsilon_2^{(d)} X_2^2 +$$

$$q^{-\frac{1}{2}} \epsilon_3^{(d)} + \omega_3 X_3 + q^{\frac{1}{2}} \omega_1^{(d)} X_1 + q^{\frac{1}{2}} \omega_2^{(d)} X_2 + \omega_4^{(d)} = 0.$$

PII cubic and Sklyanin algebra

$$F(x, y, z) = xyz + x + y + z = 0 \tag{21}$$

- PII cubic relation which geometrically describes a 2-dimensional affine variety $S \subset \mathbb{C}^3$. We suppose that (x, y, z) is a "generic" point in S and consider an algebra Q_4 defined by

$$Q_4(x, y, z) := \mathbb{C} < x_0, x_1, x_2, x_3 > / J$$

- J is the bilateral ideal generated by six quadratic relations involving x, y, z :

$$[x_0, x_1] - x\{x_2, x_3\}, \quad \{x_0, x_1\} - [x_2, x_3];$$
$$[x_0, x_2] - y\{x_2, x_3\}, \quad \{x_0, x_2\} - [x_3, x_1];$$
$$[x_0, x_3] - z\{x_2, x_3\}, \quad \{x_0, x_3\} - [x_1, x_2].$$

- $Q_4(x, y, z)$ is an associative graded algebra which was introduced by Sklyanin as an "elliptic deformation" of the polynomial algebra in four variables;
- This a Koszul Calabi-Yau algebra;
- The isomorphism classes of $Q_4(x, y, z)$ are in one-to-one correspondence with the orbifold S/Σ_3 (Shedler et al). (The symmetric group of order 3 isomorphically acts by cyclic permutations of (x, y, z) and by cyclic permutations of (x_1, x_2, x_3) with fixed x_0 : the latter operation permutes $(x, y, z) \to (z, x, y)$.)

The "elliptic nature" of the parameters (x, y, z) is clarified by Sklyanin with help of an uniformization of the surface S using four Jacobi theta-functions $\theta_{\alpha\beta} \mid (\alpha\beta) = (00), (01), (10), (11)$. They are quasi-periodic holomorphic functions on $\mathbb{C}$ which are related to the elliptic curve $\mathcal{E} := \mathbb{C}/\Gamma_\tau$ with $\Gamma_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ where $\tau \in \mathbb{C}, \quad \Im\tau > 0$. The only zero of $\theta_{\alpha\beta}$ in the fundamental parallelogram is at the point $\frac{\alpha+1}{2}\tau + \frac{1-\beta}{2}$.

This uniformization reads as follows: fix $\eta \in \mathbb{C}$ which is not of order 4 in $\mathcal{E}$: $4\eta \neq 0$ then

$$x = \left(\frac{\theta_{11}(\eta)\theta_{00}(\eta)}{\theta_{01}(\eta)\theta_{10}(\eta)}\right)^2, \quad y = -\left(\frac{\theta_{11}(\eta)\theta_{01}(\eta)}{\theta_{00}(\eta)\theta_{10}(\eta)}\right)^2, \quad z = \left(\frac{\theta_{11}(\eta)\theta_{10}(\eta)}{\theta_{01}(\eta)\theta_{00}(\eta)}\right)^2 \tag{22}$$

Proposition

The defining relation of the affine cubic surface (21) is one of the classical duplication identities for Jacobi theta-functions ([Whittaker-Watson], p. 488):

$$\theta_{11}(\eta)^4 + \theta_{00}(\eta)^4 = \theta_{01}(\eta)^4 + \theta_{10}(\eta)^4.$$

Links and open problems

- There are various links to Sklyanin algebras and their degenerations;
- (Non-)commutative potentials and NC SUSY Yang-Mills;
- Toric charcter varieties, their "uniformization" ("toric theta-functions");
- Deformations of cubics.
- Interesting and intriguing problems are related to a construction of NC cubic surfaces and their relations to NC cluster algebras.

THANKS FOR YOUR ATTENTION!