THERMODYNAMICS OF SUBMERGED JETS: EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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System of hydrodynamic equations

$\rho = \rho(\mathbf{r}, t)$, $\mathbf{v} = \mathbf{v}(\mathbf{r}, t) = (v_1, v_2, v_3)$, $p = p(\mathbf{r}, t)$, $T = T(\mathbf{r}, t)$

The Navier-Stokes equations:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \eta \Delta \mathbf{v} + (\zeta + \frac{\eta}{3}) \operatorname{grad}(\operatorname{div} \mathbf{v}),$$

The continuity equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,$$

The equation of heat balance:

$$T \rho \frac{Ds}{Dt} = k \Delta T + \sigma_{ij} \frac{\partial v_i}{\partial x_j},$$

here $s = s(\mathbf{r}, t)$ is the specific entropy,

$$\sigma_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right) + \zeta \delta_{ij} \frac{\partial v_k}{\partial x_k} \quad \text{is the viscous stress tensor,}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v}, \nabla) \quad \text{is the material derivative.}$$

Classical results

Landau jet

$$\rho_0(\mathbf{v}, \nabla)\mathbf{v} - \eta\Delta\mathbf{v} = -\nabla p,$$

$$\operatorname{div}\mathbf{v} = 0,$$

$\mathbf{v} = \mathbf{v}(R, \theta)$ is velocity field, ρ_0 is density, $p = p(R, \theta)$ is pressure, η is viscosity.

- $T = T_0$

The velocity field:

$$v_R = \frac{2\eta}{\rho_0 R}\left(\frac{A^2 - 1}{(A - \cos\theta)^2} - 1\right), \quad v_\theta = -\frac{2\eta\sin\theta}{\rho_0 R(A - \cos\theta)}, \quad v_\varphi = 0.$$

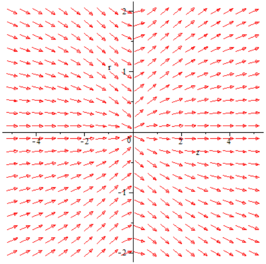

Fig.1 Landau jet

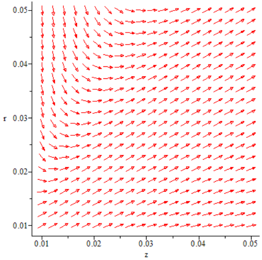

Fig.2 Rudenko jet

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Thermodynamic state (geometrical approach)

Valentin V. Lychagin has proposed the geometrical description of thermodynamic state as follows.
Let $\mathbb{R}^5(p, \rho, e, T, s)$ be a vector space equipped with contact 1-form:

$$\theta = \frac{1}{T}de - ds - \frac{p}{T\rho^2}d\rho.$$

Thermodynamic state is a 2-dimensional Legendrian manifold $L \subset \mathbb{R}^5(\rho, p, e, T, s)$ such that $\theta|_L = 0$.

The condition $\theta|_L = 0$ leads to the following relations if specific entropy is a function of specific energy and density $s = s(e, \rho)$:

$$p = -\rho^2 \frac{s_\rho}{s_e}, \quad T = \frac{1}{s_e}.$$

Thermodynamic state

To eliminate specific entropy s from the description of thermodynamic state we consider projection $\phi\colon \mathbb{R}^5 \to \mathbb{R}^4, \quad \phi\colon (\rho, p, e, T, s) \to (\rho, p, e, T)$. Then, $\mathbb{R}^4(\rho, p, e, T)$ is the symplectic space equipped with the structural 2-form

$$\Omega = d\theta = \frac{1}{T^2} de \wedge dT - \frac{1}{T\rho^2} dp \wedge d\rho + \frac{p}{T^2\rho} dT \wedge d\rho,$$

and $\overline{L} = \phi(L)$ is a Lagrangian manifold, such that $\Omega|_{\overline{L}} = 0$. Equivalently, **Thermodynamic state** is a 2-dimensional manifold $\overline{L} \subset \mathbb{R}^4(\rho, p, e, T)$ such that $\Omega|_{\overline{L}} = 0$. The manifold $\overline{L}$ can be defined by the two equations:

$$f(\rho, p, e, T) = 0, \quad g(\rho, p, e, T) = 0$$

and condition $\Omega|_{\overline{L}} = 0$ can be written as

$$[f, g]|_{\overline{L}} = 0,$$

here $[f, g]$ is the Poisson bracket: $[f, g]\Omega \wedge \Omega = df \wedge dg \wedge \Omega$.

Van der Waals gas

Real gases can be described by van der Waals equation:

$$f(\rho, p, e, T) = (p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT,$$

here a and b are the characteristics of the gas, R is the gas constant. To find out the second equation we assume that

$$g(\rho, p, e, T) = e - \beta(\rho, T)$$

and use the Poisson bracket:

$$[f, g]|_{\overline{L}} = 0 \Longrightarrow \beta(\rho, T) = -a\rho + E(T),$$

here $E(T)$ is an arbitrary smooth function. We define $E(T) = \alpha T$. Thus, the equations of state for the van der Waals gas are

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT = 0, \quad e - \alpha T + a\rho = 0$$

Entropy for the van der Waals gas

The equations of state for the van der Waals gas are

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT = 0, \quad e - \alpha T + a\rho = 0$$

Recall the expressions for specific entropy, pressure and temperature:

$$p = -\rho^2 \frac{s_\rho}{s_e}, \quad T = \frac{1}{s_e}.$$

The expressions above allow to represent the specific entropy as follows:

$$s(e, \rho) = \alpha \ln(e - e_0 + a\rho) + R \ln\left(\frac{1}{\rho} - b\right) + s_0.$$

Phase transitions for van der Waals gas

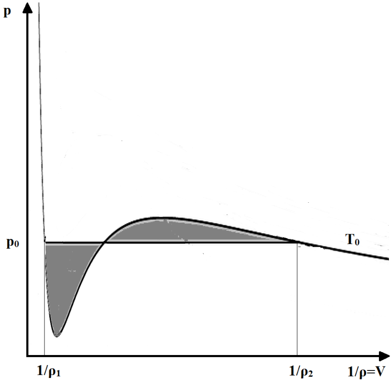


Fig.3 van der Waals isotherm

$$\mu(T_0, p_0, \rho_1) = \mu(T_0, p_0, \rho_2) = \mu_0,$$

$$\mu = e - Ts + \frac{p}{\rho}$$

System of equations for ρ_1 and ρ_2

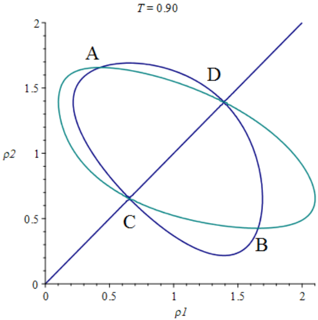
$$\mu_0 = \alpha T_0 - \rho_1 a - T_0 \left(\alpha \ln(\alpha T_0 - C_1) + R \ln\left(\frac{1}{\rho_1} - b\right)\right) + \frac{p_0}{\rho_1},$$
$$\mu_0 = \alpha T_0 - \rho_2 a - T_0 \left(\alpha \ln(\alpha T_0 - C_1) + R \ln\left(\frac{1}{\rho_1} - b\right)\right) + \frac{p_0}{\rho_2},$$
$$p_0 - p_0\rho_1 b + a\rho_1^2 - ab\rho_1^3 - \rho_1 R T_0 = 0,$$
$$p_0 - p_0\rho_2 b + a\rho_2^2 - ab\rho_2^3 - \rho_2 R T_0 = 0.$$

Eliminating μ_0 and p_0 from the system above we obtain the following equations:

$$(\rho_1 - \rho_2)(RT_0 - a(\rho_1 + \rho_2)(b\rho_1 - 1)(b\rho_2 - 1)) = 0,$$

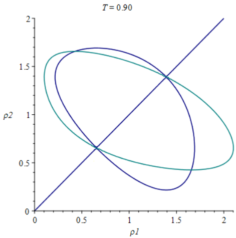
$$\rho_1 R T_0 (b\rho_2 - 1) \ln\left(\frac{\rho_1(1 - b\rho_2)}{\rho_2(1 - b\rho_1)}\right) - (\rho_1 - \rho_2)(a\rho_1(1 - b\rho_2) + ab\rho_2^2 + RT_0 - a\rho_2) = 0.$$

Solution for ρ_1 and ρ_2

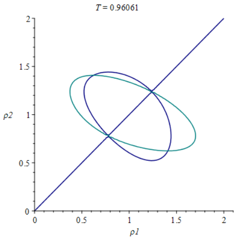


Solution

Dynamics of solution for ρ_1 and ρ_2



$T = 0.9\,T_{\text{crit}}$



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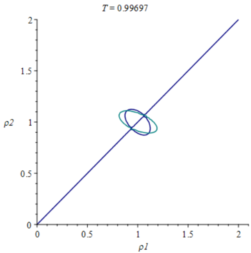

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Hydrodynamic equations for van der Waals gas

The Navier-Stokes equations:

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The continuity equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,$$

The equation of heat balance:

$$T\rho \frac{Ds}{Dt} = k\Delta T + \sigma_{ij} \frac{\partial v_i}{\partial x_j},$$

Equations of state (Legendrian manifold L):

$$s(e, \rho) = \alpha \ln(e - e_0 + a\rho) + R \ln\left(\frac{1}{\rho} - b\right) + s_0,$$

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT = 0, \quad e - \alpha T + a\rho = 0.$$

Asymptotic expansion for solution

$$
\begin{aligned}
\mathbf{v}(t,\mathbf{r}) &= \mathbf{v}_0(t,\mathbf{r}) + a\mathbf{v}_1(t,\mathbf{r}) + b\mathbf{v}_2(t,\mathbf{r}) + \dots, \\
\rho(t,\mathbf{r}) &= \rho_0(t,\mathbf{r}) + a\rho_1(t,\mathbf{r}) + b\rho_2(t,\mathbf{r}) + \dots, \\
e(t,\mathbf{r}) &= e_0(t,\mathbf{r}) + ae_1(t,\mathbf{r}) + be_2(t,\mathbf{r}) + \dots.
\end{aligned}
$$

The Navier-Stokes equations:

$$\rho_0\alpha\left(u_{0t}+u_0u_{0x}+\frac{R}{\alpha}e_{0x}\right)+R(e_{0x}-\beta)-\eta u_{0xx}=0,$$

The continuity equation:

$$\rho_{0t}+\rho_{0x}u_0+\rho u_{0x}=0,$$

The equation of heat balance:

$$\rho\alpha(e_{0t}+u_0e_{0x})-ke_{0xx}+R\rho u_{0x}(e_0-\beta)-\eta(u_{0x})^2=0.$$

Symmetry algebra of the system

The symmetry Lie algebra of the system $\mathrm{Sym}(\mathcal{E})$ is generated by the following vector fields:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_x + \partial_{u_0},$$

$$X_4 = t\partial_t + \rho_0\partial_{\rho_0} - u_0\partial_{u_0} + (2\beta - 2e_0)\partial_{e_0},$$

$$X_5 = x\partial_x - 2\rho_0\partial_{\rho_0} + u_0\partial_{u_0} + (2e_0 - 2\beta)\partial_{e_0}.$$

The Lie algebra structure is represented in the table below:

Field	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_2	X_1	0
X_2	0	0	0	0	X_2
X_3	$-X_2$	0	0	$-X_3$	X_3
X_4	$-X_1$	0	X_3	0	0
X_5	0	$-X_2$	$-X_3$	0	0

Invariant solution

$$X_5 = x\partial_x - 2\rho_0\partial_{\rho_0} + u_0\partial_{u_0} + (2e_0 - 2\beta)\partial_{e_0}.$$

The invariant solution has the following form

$$\rho_0(t,x) = \frac{C_2(t+t_0)}{x^2}, \quad u_0(t,x) = \frac{x}{t+t_0},$$

$$e_0(t,x) = \beta + \frac{\eta x^2}{(RC_2 - 2k)(t+t_0)^2} + x^2 C_1 (t+t_0)^{-2+2k/\alpha C_2 - R/\alpha}.$$

Areas of the same phases

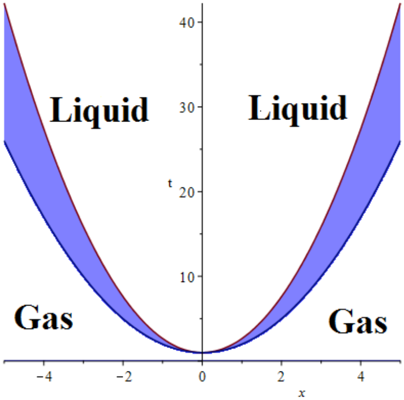

Fig.4 Phases

Thank you for attention!