SHOCK WAVES IN EULER FLOWS OF GASES

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Equations

We consider 1-dimensional non-stationary flows of gases:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho(u_t + u u_x) = -p_x, \end{cases} \tag{1}$$

where $u(t,x)$ is the velocity, $\rho(t,x)$ is the density, $p(t,x)$ is the pressure. System (1) becomes complete if one has

- Equations of state

$$\widehat{L} = \{s = S(\rho, e),\ p = P(\rho, e),\ T = T(\rho, e)\},$$

  where s is entropy, e is inner energy, T is temperature.
- Thermodynamic process $I \subset \widehat{L}$

$$\widehat{L} \supset I = \{s = s(\rho),\ p = p(\rho),\ T = T(\rho),\ e = e(\rho)\},$$

  ρ is the coordinate on I.

Consider the space $(\mathbb{R}^5, \theta)$ with coordinates (s, e, ρ, p, T), the contact form

$$\theta = T^{-1}de - ds - pT^{-1}\rho^{-2}d\rho$$

and the differential quadratic form

$$\kappa = d(T^{-1}) \cdot de - \rho^{-2}d(pT^{-1}) \cdot d\rho.$$

Thermodynamic state is a Legendrian manifold $\widehat{L} \subset (\mathbb{R}^5, \theta)$, such that $\theta|_{\widehat{L}} = 0$. If (e, ρ) are coordinates on $\widehat{L}$, then

$$\widehat{L} = \left\{ s = S(e, \rho),\ T = \frac{1}{S_e},\ p = -\rho^2 \frac{S_\rho}{S_e} \right\}$$

for a given function $S(e, \rho)$.
Applicable states: $\kappa|_{\widehat{L}} < 0$.

Thermodynamics

Consider projection $\pi \colon \mathbb{R}^5 \to \mathbb{R}^4$, $\pi(p, T, e, \rho, s) = (p, T, e, \rho)$. Then, **thermodynamic state** is a Lagrangian manifold $\pi(\widehat{L}) = L \subset (\mathbb{R}^4, \Omega)$, such that $\Omega|_L = 0$, where Ω is the symplectic form

$$\Omega = d\theta = d(T^{-1}) \wedge de - d(pT^{-1}\rho^{-2}) \wedge d\rho.$$

In the symplectic space $(\mathbb{R}^4, \Omega)$ the Lagrangian manifold L is given by **state equations**:

$$L = \{f(p, T, e, \rho) = 0,\ g(p, T, e, \rho) = 0\},$$

such that the Poisson bracket $[f, g]$ with respect to the structure form Ω

$$[f, g]\ \Omega \wedge \Omega = df \wedge dg \wedge \Omega$$

vanishes on L:

$$[f, g] = 0 \text{ on } L.$$

Let us choose (T, ρ) as coordinates on L, i.e.

$$L = \{p = P(T, \rho),\ e = E(T, \rho)\}.$$

The condition $[f, g] = 0$ on L leads to the equation $(-\rho^{-2}T^{-1}P)_T = (T^{-2}E)_\rho$, and therefore the following theorem is valid:

Theorem

The Lagrangian manifold L is given by means of the Massieu-Planck potential $\phi(\rho, T)$

$$p = -\rho^2 T\phi_\rho, \quad e = T^2\phi_T.$$

The restriction of the quadratic form κ on L is

$$\kappa|_L = -(2T^{-1}\phi_T + \phi_{TT})dT \cdot dT + (2\rho^{-1}\phi_\rho + \phi_{\rho\rho})d\rho \cdot d\rho.$$

Thus the applicability condition $\kappa|_L < 0$ is equivalent to

$$e_T > 0, \quad p_\rho > 0$$

Examples

- Ideal gas

$$\widehat{L} = \left\{ p = R\rho T,\ e = \frac{n}{2}RT,\ s = R\ln\left(\frac{T^{n/2}}{\rho}\right) \right\},$$

Differential quadratic form $\kappa|_{\widehat{L}}$

$$\kappa|_{\widehat{L}} = -\frac{Rn}{2}\frac{dT^2}{T^2} - R\rho^{-2}d\rho^2,$$

Both inequalities $e_T > 0$, $p_\rho > 0$ are satisfied.

- van der Waals gas

$$\widehat{L} = \left\{ p = \frac{8T\rho}{3-\rho} - 3\rho^2,\ e = \frac{4nT}{3} - 3\rho,\ s = \ln\left(T^{4n/3}(3\rho^{-1}-1)^{8/3}\right) \right\},$$

Differential quadratic form $\kappa|_{\widehat{L}}$

$$\kappa|_{\widehat{L}} = -\frac{Rn}{2}\frac{dT^2}{T^2} - \frac{9R\rho^2(-\rho^3 + 6\rho^2 + 4T - 9\rho)}{4T(\rho-3)^2}d\rho^2,$$

Inequality $e_T > 0$ is satisfied, while $p_\rho > 0$ is not everywhere.

Equations and forms

Let $E = J^0(t, x, u, \rho)$, $M = \mathbb{R}^2(t, x)$.

$$\mathcal{E} = \begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x + \dfrac{p'(\rho)}{\rho}\rho_x = 0, \end{cases} \qquad \begin{cases} \omega_1 = \rho dt \wedge du + u dt \wedge d\rho - dx \wedge d\rho, \\ \omega_2 = u dt \wedge du + \dfrac{p'(\rho)}{\rho} dt \wedge d\rho - dx \wedge du, \end{cases}$$

$\omega_1, \omega_2 \in \Omega^2(E)$. Any differential 2-form $\omega \in \Omega^2(E)$ generates an operator

$$\Delta_\omega \colon C^\infty(M) \to \Omega^2(M), \quad \Delta_\omega(f) = \omega|_{\Gamma^0(f)},$$

where $\Gamma^0(f) \subset E$ is a graph of the vector-function f. The system $\mathcal{E}$ can be written as

$$\Delta_{\omega_1}(f) = 0, \quad \Delta_{\omega_2}(f) = 0,$$

where $f = (u(t, x), \rho(t, x))$.
A 2-dimensional manifold $N \subset E$ is said to be a *(multivalued) solution* of $\mathcal{E}$, if $\omega_1|_N = \omega_2|_N = 0$.

Type of equations

Let $q = dt \wedge dx \wedge du \wedge d\rho$ be a volume form on E. Introduce a bilinear operator

$$P\colon \Omega^2(E) \times \Omega^2(E) \to C^\infty(E)$$

by the following way

$$\alpha_1 \wedge \alpha_2 = P(\alpha_1, \alpha_2)q, \quad \alpha_i \in \Omega^2(E).$$

Introduce the notation $P_\omega = \|P(\omega_i, \omega_j)\|$, $i, j = 1, 2$. Then, system $\mathcal{E}$

$$\Delta_{\omega_1}(f) = 0, \quad \Delta_{\omega_2}(f) = 0,$$

is said to be *hyperbolic*, if $\det(P_\omega) < 0$, *elliptic*, if $\det(P_\omega) > 0$ and *parabolic*, if $\det(P_\omega) = 0$.

Straightforward computations show:

$$P_\omega = \begin{pmatrix} 2\rho & 0 \\ 0 & -2\rho^{-1}p'(\rho) \end{pmatrix},$$

Recall that $\kappa|_{\widehat{L}} < 0 \iff e_T > 0, \quad p_\rho > 0.$

Theorem

If a thermodynamic process curve $l \subset \widehat{L}$ lies in an applicable domain, then the system $\mathcal{E}$ is hyperbolic.

Other forms

Obviously, 2-forms

$$\widehat{\omega_1} = a_{11}\omega_1 + a_{12}\omega_2,$$
$$\widehat{\omega_2} = a_{21}\omega_1 + a_{22}\omega_2,$$

where $a_{11}a_{22} - a_{12}a_{21} \neq 0$, define the Euler system as well. In a hyperbolic case one can choose a_{ij} in such a way that

$$\omega_1 \wedge \omega_2 = 0, \quad \omega_1 \wedge \omega_1 = -\omega_2 \wedge \omega_2. \tag{2}$$

Theorem

Let $\mathcal{E}$ be a system of hyperbolic type. Then, it can be given by 2-forms

$$\omega_1 = A(\rho)(\rho dt \wedge du + u dt \wedge d\rho - dx \wedge d\rho),$$
$$\omega_2 = u dt \wedge du + \rho A^2(\rho) dt \wedge d\rho - dx \wedge du,$$

where $A(\rho) = \rho^{-1}\sqrt{p'(\rho)}$, and 2-forms ω_1, ω_2 satisfy relations (2).

Field of endomorphisms

$$\begin{aligned}\omega_1 &= A(\rho)(\rho dt\wedge du + udt\wedge d\rho - dx\wedge d\rho),\\ \omega_2 &= udt\wedge du + \rho A^2(\rho)dt\wedge d\rho - dx\wedge du.\end{aligned}$$

Consider a linear operator

$$A_\omega\colon D(E)\to D(E),\quad X\rfloor\omega_2 = A_\omega(X)\rfloor\omega_1.$$

Let $\langle\partial_t,\partial_x,\partial_\rho,\partial_u\rangle$ be a basis in $D(E)$. Then, in this basis the matrix W of the operator A_ω will be

$$W=\frac{1}{\rho A(\rho)}\begin{pmatrix} u & -1 & 0 & 0\\ u^2-\rho^2A^2(\rho) & -u & 0 & 0\\ 0 & 0 & 0 & \rho A^2(\rho)\\ 0 & 0 & \rho & 0\end{pmatrix},$$

and moreover $A_\omega^2=\mathrm{id}.$

Characteristic distributions

Let $a \in E$, $\mathcal{C}_{\pm}(a)$ be eigenspaces of the operator $A_\omega(a)$, then

$$T_a E = \mathcal{C}_-(a) \oplus \mathcal{C}_+(a),$$

and *characteristic distributions* $\mathcal{C}_+ = \langle X_+, Y_+ \rangle$ and $\mathcal{C}_- = \langle X_-, Y_- \rangle$ are generated by vector fields

$$\begin{aligned} X_\pm &= \pm A(\rho)\partial_u + \partial_\rho, \\ Y_\pm &= (\mp \rho A(\rho) + u)^{-1}\partial_t + \partial_x. \end{aligned}$$

Theorem

Distributions $\mathcal{C}_+$ and $\mathcal{C}_-$ are integrable if and only if

$$p(\rho) = c_0 \rho^3 + c_1, \qquad (3)$$

where c_0 and c_1 are constants.

Solutions

N is a (multivalued) solution of the system $\mathcal{E}$, $\dim N = 2$, if and only if $\forall a \in N$

$$T_a N = h_+(a) \oplus h_-(a),$$

where $h_\pm(a) = \mathcal{C}_\pm(a) \cap T_a N$ are *characteristic directions*, $\dim h_\pm(a) = 1$. One-parametric family of solutions is defined by some 3-dimensional manifold $N_1 \subset E$, with integrable distribution $h = \langle h_+, h_- \rangle$. Let

$$N_1 = \{F(t, x, \rho, u) = 0\}$$

and let $V_\pm \in \mathcal{C}_\pm$ be tangent to N_1 vector fields. They have the form

$$V_\pm = (F_u A(\rho) \pm F_\rho)\partial_t + ((uF_u - \rho F_\rho)A(\rho) \pm (uF_\rho - \rho A^2(\rho)F_u))\partial_x - \\ - (F_t + uF_x \mp \rho A(\rho)F_x)\partial_u + (\rho A(\rho)F_x \mp (F_t + uF_x))\partial_\rho.$$

One needs to find such a function $F(t, x, \rho, u)$, that the distribution $V = \langle V_-, V_+ \rangle$ is integrable, i.e. $V_- \wedge V_+ \wedge [V_-, V_+] = 0$.

Solution

Consider a particular case $F(t,x,\rho,u) = f(t,\rho,u)$. Then the condition $V_- \wedge V_+ \wedge [V_-, V_+] = 0$ leads to the following equation for $f(t,\rho,u)$:

$$-2\rho A^2 f_u f_t f_{ut} - \rho f_\rho^2 f_{tt} - \rho f_t^2 f_{\rho\rho} + \rho A^2 \left(f_u^2 f_{tt} + f_t^2 f_{uu}\right) + 2\rho f_\rho f_t f_{\rho t} - 2 f_\rho f_t^2 = 0,$$

one of its solutions is

$$f(u,\rho,t) = \alpha_0 + \alpha_1 \rho + \alpha_2 \rho t - u(\rho + \alpha_3),$$

where α_j are constants.
Let (t,x,ρ) be coordinates on N_1. Then restrictions $h_\pm$ of the vector fields $V_\pm$ on N_1 will take the form

$$h_\pm = \frac{A(\rho+\alpha_3)^2 \mp \alpha_3(\alpha_1 + t\alpha_2) \pm \alpha_0}{(\rho+\alpha_3)^2}\partial_t \pm \frac{\alpha_2 \rho}{\rho+\alpha_3}\partial_\rho + \\ + \frac{\left(A(\rho+\alpha_3)^2 \mp \alpha_3(\alpha_1 + t\alpha_2) \pm \alpha_0\right)\left(\mp A\rho(\rho+\alpha_3) + \rho(t\alpha_2 + \alpha_1) + \alpha_0\right)}{(\rho+\alpha_3)^3}\partial_x .$$

Solution

The distribution $h = \langle h_+, h_- \rangle$ is integrable, and solutions are given by levels of its integral:

$$\int \frac{A^2(\rho+\alpha_3)}{\alpha_2} d\rho + \frac{(\alpha_1\alpha_3 - \alpha_0)^2}{2\alpha_2(\rho+\alpha_3)^2} + \\ + \frac{2x(\rho+\alpha_3)^2 - 2t\left(\rho^2\alpha_1 + \alpha_3(\alpha_0 + 2\rho\alpha_1)\right) - \rho\alpha_2(\rho+2\alpha_3)t^2}{2(\rho+\alpha_3)^2} = 0.$$

The velocity is found from the relation
$f(u,\rho,t) = \alpha_0 + \alpha_1\rho + \alpha_2\rho t - u(\rho+\alpha_3) = 0$, i.e.

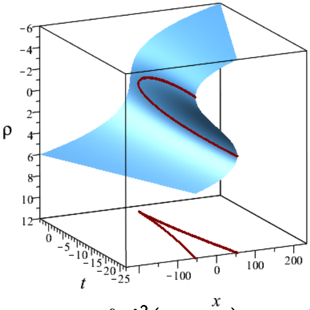
$$u = \frac{\alpha_2\rho t + \alpha_1\rho + \alpha_0}{\rho+\alpha_3}.$$

The solution obtained is

1. multivalued
2. valid for any thermodynamic state and any process.

Caustics

Singularities of projection of N to $\mathbb{R}^2(t,x)$ are given by the relation $x_\rho = 0$. Such a curve is called *caustic*.



By choosing ρ as a coordinate on the caustic, one can write its equation in (t,x) by the following way:

$$t(\rho) = \frac{\pm A(\rho+\alpha_3)^2 - \alpha_1\alpha_3 + \alpha_0}{\alpha_3\alpha_2},$$

$$x(\rho) = -\int \frac{A^2(\rho+\alpha_3)}{\alpha_2}\,d\rho + \frac{\rho(\rho+2\alpha_3)(\rho+\alpha_3)^2A^2 - \alpha_3^2\alpha_1^2 + \alpha_0^2 \pm 2\alpha_0(\rho+\alpha_3)^2A}{2\alpha_3^2\alpha_2}.$$

To construct a discontinued solution from a multivalued one we need a conservation law.
Let us choose (t, ρ) as coordinates on N:

$$u(\rho, t) = \frac{\alpha_2 \rho t + \alpha_1 \rho + \alpha_0}{\rho + \alpha_3}, \quad x(\rho, t) = -\int \frac{A^2(\rho + \alpha_3)}{\alpha_2} d\rho + \\ + \frac{\alpha_2^2 t^2 \rho(\rho + 2\alpha_3) + 2t\alpha_2 \left(\alpha_3(\alpha_0 + 2\alpha_1 \rho) + \alpha_1 \rho^2\right) - (\alpha_0 - \alpha_1 \alpha_3)^2}{2\alpha_2(\rho + \alpha_3)^2},$$

Conservation law: $d\Theta|_N = 0 \Rightarrow \Theta|_N = dH$.
Let N_+ and N_- be the two branches of the multivalued solution and let $\gamma \in \mathbb{R}^2(t, x)$ be a discontinuity line. Then the condition $(\Theta|_{N_+})|_\gamma = (\Theta|_{N_-})\big|_\gamma$ implies that $(H|_{N_-})\big|_\gamma = (H|_{N_+})|_\gamma + \mathrm{const}$. Since caustics intersect in a single point, one has $\mathrm{const} = 0$.
Discontinuity points ρ_+ and ρ_- are found from relations

$$H(\rho_+, t) = H(\rho_-, t), \quad x(\rho_+, t) = x(\rho_-, t)$$

.

Conservation laws

- Conservation of mass $\Theta_1|_N = H_\rho d\rho + H_t dt$

$$\Theta_1|_N = \rho x_\rho d\rho + (x_t - \rho u)dt.$$

- Conservation of momentum $\Theta_2|_N = G_\rho d\rho + G_t dt$

$$\Theta_2|_N = u x_\rho d\rho + \left(u x_t - \frac{u^2}{2} - \rho A^2 \right) dt.$$

Conservation of mass

The potential $H(\rho, t)$ in case of mass conservation law is found from an overdetermined system

$$H_\rho = \rho x_\rho, \quad H_t = \rho(x_t - u(\rho, t)),$$

and is equal to

$$H(\rho, t) = \frac{(\alpha_2 \rho t - \alpha_1 \alpha_3 + \alpha_0)\left(\alpha_1 \alpha_3^2 + \alpha_3((t\alpha_2 + 2\alpha_1)\rho - \alpha_0) - 2\rho\alpha_0\right)}{2\alpha_2(\rho + \alpha_3)^2} - \\ - \int \frac{\rho(\rho + \alpha_3)A^2}{\alpha_2} d\rho.$$

Example: ideal gas

State equations

$$\widehat{L} = \left\{ p = R\rho T,\ e = \frac{n}{2} RT,\ s = R \ln\left(\frac{T^{n/2}}{\rho}\right) \right\},$$

Differential quadratic form $\kappa|_{\widehat{L}}$

$$\kappa|_{\widehat{L}} = -\frac{Rn}{2}\frac{dT^2}{T^2} - R\rho^{-2} d\rho^2,$$

Therefore, the system $\mathcal{E}$ in case of ideal gases is hyperbolic for any process $I \subset \widehat{L}$.

Adiabatic process, $s = s_0 = \mathrm{const}$:

$$T(\rho) = \exp\left(\frac{2s_0}{Rn}\right)\rho^{2/n}, \quad p(\rho) = R\exp\left(\frac{2s_0}{Rn}\right)\rho^{2/n+1}.$$

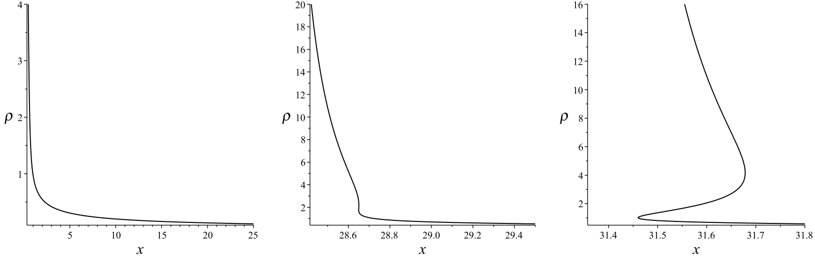
The function $A(\rho) = \rho^{-1}\sqrt{p'(\rho)} = A_0\rho^m$, where

$$A_0 = \sqrt{R\left(1+\frac{2}{n}\right)\exp\left(\frac{2s_0}{Rn}\right)}, \quad m = \frac{1}{n} - 1.$$

Example: ideal gas

Solution for the density:

$$\frac{A_0^2\rho^{2m+1}(2m\rho+2m\alpha_3+\rho+2\alpha_3)}{2\alpha_2(m+1)(2m+1)}+\frac{(\alpha_1\alpha_3-\alpha_0)^2}{2\alpha_2(\rho+\alpha_3)^2}+$$
$$+\frac{2x(\rho+\alpha_3)^2-2t\left(\rho^2\alpha_1+\alpha_3(\alpha_0+2\rho\alpha_1)\right)-\rho\alpha_2(\rho+2\alpha_3)t^2}{2(\rho+\alpha_3)^2}=0.$$



Density graph for $n=3$ at moments $t=0$, $t=6.58$, $t=7$

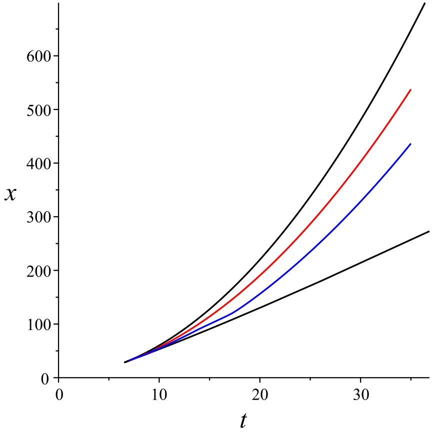
Caustics

$$x(\rho) = -\frac{A_0^2}{\alpha_2}\left(\frac{\rho^{2m+2}}{2m+2} + \frac{\rho^{2m+1}}{2m+1}\right) + $$
$$+ \frac{\rho(\rho+2\alpha_3)(\rho+\alpha_3)^2 A_0^2 \rho^{2m} - \alpha_3^2\alpha_1^2 + \alpha_0^2 \pm 2\alpha_0(\rho+\alpha_3)^2 A_0 \rho^m}{2\alpha_3^2\alpha_2},$$
$$t(\rho) = \frac{\pm A_0 \rho^m (\rho+\alpha_3)^2 - \alpha_1\alpha_3 + \alpha_0}{\alpha_3\alpha_2}.$$

Potential $H(\rho, t)$

$$H(\rho,t) = \frac{(\alpha_2\rho t - \alpha_1\alpha_3 + \alpha_0)\left(\alpha_1\alpha_3^2 + \alpha_3((t\alpha_2 + 2\alpha_1)\rho - \alpha_0) - 2\rho\alpha_0\right)}{2\alpha_2(\rho+\alpha_3)^2} -$$
$$-\frac{A_0^2}{\alpha_2}\left(\frac{\rho^{2m+3}}{2m+3} + \frac{\alpha_3\rho^{2m+2}}{2m+2}\right).$$

Caustics and discontinuity line



Caustic (black) and shock wave front: conservation of mass — red, conservation of momentum — blue, $n = 3$

Thank you for attention!