Smooth local normal forms of hyperbolic Roussarie vector fields

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Introduction

In 1975, Robert Roussarie studied a special class of vector fields

$$\dot{x} = v, \quad \dot{y} = w, \quad \dot{z}_i = a_i v + b_i w, \quad i = 1, \ldots, n, \qquad (1)$$

where $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$.
Here v, w and a_i, b_i are C^∞-smooth functions on $(x, y, z) \in \mathbb{R}^{n+2}$.

The set of sing. points of (1) is a submanifold $S \subset \mathbb{R}^{n+2}$ of codimension 2, and the spectrum at every sing. point is

$$(\lambda_1, \lambda_2, 0, \ldots, 0).$$

The Roussarie condition: $\boxed{\lambda_1 + \lambda_2 = 0}$ at every sing. point.

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Further we deal with **orbital** normal forms, that is, it is allowed to multiply vector fields by non-vanishing functions.
Then there are three possible cases:

- **Real (hyperbolic) case**: $\lambda_{1,2} = \pm 1$
- **Imaginary (elliptic) case**: $\lambda_{1,2} = \pm\sqrt{-1}$
- **Zero (parabolic) case**: $\lambda_{1,2} = 0$

Roussarie investigated the real and imaginary cases. Under some genericity conditions, he obtained orbital normal forms (n. f.):

- **Real (hyperbolic) case**: C^∞-smooth n. f.
- **Imaginary (elliptic) case**: Formal n. f.

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- **Real (hyperbolic) case**: The germ of (1) is orbitally C^∞-smoothly equivalent to

$$\dot x = x, \quad \dot y = -y, \quad \dot z_1 = xy, \quad \dot z_i = 0, \quad i = 2, \dots, n$$

- **Imaginary (elliptic) case**: The germ of (1) is orbitally formally equivalent to

$$\dot x = y, \quad \dot y = -x, \quad \dot z_1 = x^2 + y^2, \quad \dot z_i = 0, \quad i = 2, \dots, n$$

Highly likely, the implication $\{$**formal** n. f. $\Rightarrow$ smooth n. f.$\}$ holds true for imaginary case as well. However, the proof of this conjecture is an open problem.

Hyperbolic Roussarie vector fields: Definition

From now on, we consider vector fields (named after Roussarie)

$$\dot{x} = v, \quad \dot{y} = w, \quad \dot{z}_i = a_i v + b_i w, \quad i = 1, \ldots, n, \tag{2}$$

where $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, satisfying the following conditions:

The spectrum at every sing. point is

$$(\lambda_1, \lambda_2, 0, \ldots, 0), \quad \mathrm{Re}\, \lambda_{1,2} \neq 0,$$

$$q\lambda_1 + p\lambda_2 = 0, \quad p, q \in \mathbb{Z}_+, \quad \gcd(p, q) = 1. \tag{3}$$

Hyperbolic Roussarie vector fields: Preliminary n. f.

Theorem (Preliminary n. f.)

Any Roussarie germ is C^∞-smoothly orbitally equivalent to

$$\dot{x} = px(1+\Phi_1(r,z)), \quad \dot{y} = qy(-1+\Phi_2(r,z)), \quad \dot{z}_i = r\Psi_i(r,z), \quad (4)$$

where $r = x^q y^p$ (resonant monomial), $\Phi_{1,2}$ and Ψ_i are smooth functions, $\Phi_{1,2}(0,0) = 0$.

Hyperbolic Roussarie vector fields: Preliminary n. f.

Construct the **quotient** vector field of (4).
First, in the (r, z)-space (4) generates the field

$$\begin{aligned}
\dot r = (\dot{x^q y^p}) = q x^{q-1} y^p \dot x + p x^q y^{p-1} \dot y = \\
q x^{q-1} y^p p x (1 + \Phi_1) + p x^q y^{p-1} q y (-1 + \Phi_2) = \\
pq r (\Phi_1 + \Phi_2), \\
\dot z_i = r \Psi_i(r, z), \quad i = 1, \ldots, n.
\end{aligned}$$

Reducing the common factor r, we get the **quotient** field:

$$\dot r = pq \Phi(r, z), \quad \dot z_i = \Psi_i(r, z), \quad i = 1, \ldots, n, \tag{5}$$

where $\Phi(r, z) = \Phi_1(r, z) + \Phi_2(r, z)$.

Hyperbolic Roussarie vector fields: Preliminary n. f.

The quotient vector field of (4):

$$\dot{r} = pq\Phi(r, z), \quad \dot{z}_i = \Psi_i(r, z), \quad i = 1, \dots, n, \tag{5}$$

where $\Phi(r, z) = \Phi_1(r, z) + \Phi_2(r, z)$.

Genericity condition: $\exists i \in \{1, \dots, n\}$ that $\Psi_i(0, 0) \neq 0$.

In other words, this means that 0 is not a sing. point of (5), since $\Phi(0, z) \equiv 0$ due to the condition that vector field (4) has the same resonance (3) at all sing. points.

Hyperbolic Roussarie vector fields: Final n. f.

Theorem (Final n. f.)

Any Roussarie germ satisfying the above genericity condition is C^∞-smoothly orbitally equivalent to

$$\dot{x} = px, \quad \dot{y} = -qy, \quad \dot{z}_1 = x^q y^p, \quad \dot{z}_i = 0, \quad i = 2, \ldots, n.$$

Moreover, it is C^{k-1}-smoothly orbitally equivalent to

$$\dot{x} = px, \quad \dot{y} = -qy, \quad \dot{z}_i = 0, \quad i = 1, \ldots, n, \tag{6}$$

where $k = \max\{p, q\}$. But not C^k-smoothly!

Hyperbolic Roussarie vector fields: Simplest example

Simplest example:

$$\dot{x} = x, \quad \dot{y} = -y, \quad \dot{z} = xy, \quad (x, y, z) \in \mathbb{R}^3.$$

Any saddle surface of this field has the form $z = -\frac{1}{2}F(x, y)$, where

$$F(x, y) = f(xy) + xy \ln\left|\frac{y}{x}\right|, \quad \text{if} \quad xy \neq 0,$$

and $F(x, y) = 0$, if $xy = 0$. Here f is an arbitrary continuous function.

Is it possible that $\exists f \, : \, F \in C^1$?

A very simple reasoning shows that **NO**. It can be $F \in C^0$ only.

Hyperbolic Roussarie vector fields: Simplest example

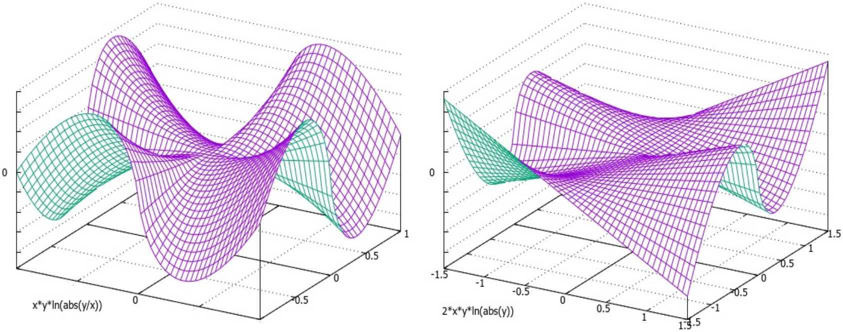

Figure: Two examples of C^0 saddle surfaces of the vector field
$\dot{x} = x$, $\dot{y} = -y$, $\dot{z} = xy$:
$z = -\frac{1}{2}xy \ln|y/x|$ (left) and $z = -xy \ln|y|$ (right).

Now consider the germ of the vector field

$$\dot{x} = px(1+\Phi_1(r,z)), \quad \dot{y} = qy(-1+\Phi_2(r,z)), \quad \dot{z}_i = r\Psi_i(r,z), \quad (4)$$

such that

$$\Psi_i(0,0) = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

The equivalent condition: the corresponding quotient vector field

$$\dot{r} = pq\Phi(r,z), \quad \dot{z}_i = \Psi_i(r,z), \quad i = 1, \ldots, n, \qquad (5)$$

$\Phi(r,z) = \Phi_1(r,z) + \Phi_2(r,z)$, vanishes at 0.

Hyperbolic Roussarie vector fields: Exceptional points

Assume that the spectrum of (5) is non-resonant. Then by the Sternberg–Chen theorem, the germ of vector field (5) is C^∞-smoothly orbitally equivalent to the linear field

$$\dot{r} = \mu_0 r, \quad \dot{z}_i = \Psi_i(z), \quad i = 1, \dots, n.$$

Here all $\Psi_i(z)$ are linear functions. Moreover, $\Psi_i(z) = \mu_i z_i$ or $\Psi_i(z) = \alpha_i z_i + \beta_i z_{i+1}$, where μ_i, α_i, β_i are reals.

Theorem

Any Roussarie germ satisfying the above conditions is C^∞-smoothly orbitally equivalent to

$$\dot{x} = px(1+\alpha r), \quad \dot{y} = -qy(1+\beta r), \quad \dot{z}_i = r\Psi_i(z), \quad r = x^q y^p,$$

where α, β are real numbers, $\Psi_i(z)$ are linear functions described above. The ratio $\alpha : \beta$ is invariant.