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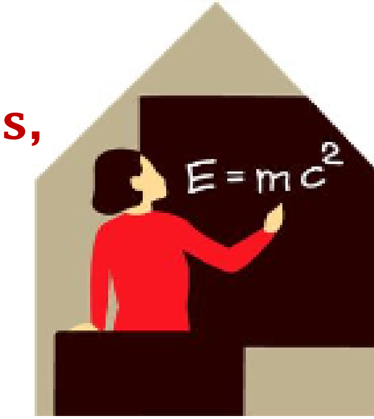

ON INVARIANT DIFFERENTIAL IDEALS AND ENDOMORPHIC REPRESENTATIONS OF FUNCTIONAL DERIVATIONS IN DIFFERENTIAL RINGS

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2. Differential-algebraic problems setting

2.1. The Dubrovin's integrability classification. Let $\mathcal{A}_\varepsilon(u) := \mathcal{K}(u)[u_1, u_2, ..., u_k, ...][[\varepsilon]]$ be a specially defined differential ring, depending on a chosen element $u \in \mathcal{K}$ and a free parameter ε. The Dubrovin's integrability classification [19, 20, 21] of a general evolution equation

(2.1)

$$u_t + f(u)u_x = \varepsilon[f_{21}(u)u_{xx} + f_{22}(u)u_x^2]+$$

$$+\varepsilon^2[f_{31}(u)u_{xxx} + f_{32}(u)u_x u_{xx} + f_{33}u_x^3] + ...+$$

$$+\varepsilon^{N-1}[f_{N,\sigma}(u) \prod_{m=\overline{1,N}} (u_{jx})^{k_j} + ...] := F_{N,\varepsilon}(u)$$

with graded homogeneous polynomials of the jet-variables $\{u_x, u_{xx}, ..., u_{kx} ...\} \in J^\infty(\mathbb{R};\mathbb{R})$, where $f'(u) \neq 0$ for arbitrary $u \in \mathcal{K} := C^\infty(\mathbb{R};\mathbb{R})$, consists in describing the set $\mathcal{F}$ of smooth functions $f_{j,\sigma}(u)$, $\sigma := \{k_j \in \mathbb{N} : \sum_{j=\overline{1,N}} jk_j = N\}$, with a fixed natural integer $N \in \mathbb{N}$, for which the equation (2.1) reduces by means of the following transformation

(2.2)

$$v \to u+\sum_{k\in\mathbb{N}} \varepsilon^k \eta_k(u, u_x, u_{xx}, ..., u^{(m_k)}) := u+\eta_\varepsilon(u) \in \exp \mathcal{A}_\varepsilon$$

with finite orders $m_k \in \mathbb{N}, k \in \mathbb{N}$, to the form

$$v_s + f(v)v_x = 0, \tag{2.3}$$

and when the transformation (2.2) is applied to an arbitrary Riemann type symmetry flow

$$v_s + h(v)v_x = 0 \tag{2.4}$$

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and when the transformation (2.2) is applied to an arbitrary Riemann type symmetry flow

$$v_s + h(v)v_x = 0 \tag{2.4}$$

with respect to an evolution parameter $s \in \mathbb{R}$, the latter reduces to the canonical form

(2.5)

$$u_s + h(u)u_x = \sum_{k \in \mathbb{N}} \varepsilon^k h_k(u, x, u_{xx}, ..., u^{(k)}) := H_\varepsilon(u) \in \mathcal{A}_\varepsilon(u).$$

In the Dubrovin's works there was formulated the following interesting differential-algebraic *integrability criterion*:

Definition 3.1. The evolution equation (3.1) is defined to be formally integrable, iff the corresponding inverse to (3.2) transformation

(3.5)
$$v \to u + \sum_{k \in \mathbb{N}} \varepsilon^k \eta_k(u, u_x, u_{xx}, ..., u^{(m_k)}) := u + \eta_\varepsilon(u) \in \exp(\mathcal{A}_\varepsilon(u))$$

with finite orders $m_k \in \mathbb{N}, k \in \mathbb{N}$, being applied to an arbitrary Riemann type symmetry flow

(3.6)
$$v_s + h(v)v_x = 0$$

$$v_s + h(v)v_x = 0 \tag{3.6}$$

with respect to an evolution parameter $s \in \mathbb{R}$, reduces to the form

$$u_s+h(u)u_x = \sum_{k\in\mathbb{N}} \varepsilon^k h_k(u, x, u_{xx}, ..., u^{(k)}) := H_\varepsilon(u) \in \mathcal{A}_\varepsilon(u) \tag{3.7}$$

with uniform homogeneous differential polynomials $w_k(u, x, u_{xx}, ..., u^{(k)}) \in \mathcal{K}(u)\ [u_1, u_2, ..., u_k, ...], u \in \mathcal{K},$ of the order $k \in \mathbb{N}$.

Remark 3.2. In their works B. Dubrovin and his collaborators applied this scheme to the equation

$$u_t + uu_x = \varepsilon^3[f_{31}(u)u_{xxx} + f_{32}(u)u_xu_{xx} + f_{33}(u)u_x^3] \tag{3.8}$$

and described [21] the complete classification of its integrable representatives.

The main Dubrovin's motivation which led him to formulation the criterion above was based on a fact that all flows (3.6) are integrable and commuting to each other for arbitrary smooth mapping $h : \mathcal{K} \rightarrow \mathcal{K}$ of the functional ring $\mathcal{K}$. In what will follow below we will reinterpret this criterion in somewhat simpler differential-algebraic terms, having observed that the left hand side expressions of the equations (3.1) and (3.7) are deeply related with functional "*convective*" derivations $D_t^{(f)} := \partial/\partial t + f(\circ)\partial/\partial x$ and $D_t^{(h)} := \partial/\partial t + h(\circ)\partial/\partial x$, respectively. Moreover, the reducing isomorphism (3.2) of rings $\mathcal{A}_\varepsilon(u) := \mathcal{K}(u)[u_1, u_2, ..., u_k, ...][[\varepsilon]]$ and $\mathcal{A}_\varepsilon(v) := \mathcal{K}(v)\ [v_1, v_2, ..., v_k, ...][[\varepsilon]]$ is, in reality, defined on the set $Z_f = \{v = u + \eta_\varepsilon(u) \in \exp \mathcal{A}_\varepsilon(u), u \in \mathcal{K}\}$ of constants of the derivation $D_t^{(f)} := \partial/\partial t + f(\circ)\partial/\partial x$, that is $D_t^{(f)} v = 0, v \in Z_f$. In particular, we can also observe is $D_t^{(f)} v = 0, v \in Z_f$. In particular, we can also observe that the featuring ingredient of the Dubrovin's integrability criterion above consists in checking that for any smooth mapping $h : \mathcal{K} \to \mathcal{K}$ the inverse to (3.2) transformation (3.5), maps the set $Z_f \subset \exp \mathcal{A}_\varepsilon$ of constants of the derivation $D_s^{(f)} := \partial/\partial s + f(\circ)\partial/\partial x$, $s \in \mathbb{R}$, into the differential expression $D_s^{(h)} \tilde{u} := \partial \tilde{u}/\partial s + h(\tilde{u})\partial \tilde{u}/\partial x$, perturbed by means of some differential term from the ring $\mathcal{A}_\varepsilon(\tilde{u})$. As a logical inference from the properties mentioned above we can easily deduce the following reasonings.

perturbed by means of some differential term from the ring $\mathcal{A}_\varepsilon(\tilde{u})$. As a logical inference from the properties mentioned above we can easily deduce the following reasonings, formulated by means of the differential algebraic tools. Namely, based on the element $v \in Z_f$, one can construct for every $N \in \mathbb{N}$ a subset $I_{\varepsilon,N} \subset \mathcal{A}_\varepsilon(u)$:

(3.9)

$$I_{\varepsilon,N}(v) := \{ \sum_{j=\overline{1,N}} a_{j,\varepsilon}(u) D_x^j v : v = u + \eta_\varepsilon(u) \in Z_f, a_{j,\varepsilon}(u) \in \mathcal{A}_\varepsilon(u)\}$$

of the differential ring $\mathcal{A}_\varepsilon(u)$, which proves to be its ideal, invariant with respect to the "convective" differentiation $D_s^{(f)} := \partial/\partial t + f(v)\partial/\partial x, v \in Z_f$, as $D_t^{(f)} v = 0$ and

(3.10)

$$D_t^{(f)} D_x^k v = D_x \left(D_t^{(f)} D_x^{k-1} v \right) - (D_x f(v)) D_x^k v, ..., D_t^{(f)} D_x v = -(D_x f(v)) D_x v$$

for all $k \in \mathbb{N}$. Consider now a smooth invertible transformation $\xi^{(h)} : \mathcal{K} \to \mathcal{K}$, $\xi'_{(h)} \neq 0$, satisfying the condition $f \circ \xi_{(h)} = h$, where $h : \mathcal{K} \to \mathcal{K}$ is any smooth and invertible mapping, and calculate the expression

(3.11)

$$D_s^{(f)}|_{v \to \xi_{(h)}(w)}(\xi_{(h)}(w)) = 0 \Longrightarrow \xi'_{(h)}(w)(\partial w/\partial s +$$

$$+ f \circ \xi_{(h)}(w)\ \partial w/\partial x) = \ \xi'_{(h)}(w) D_s^{(h)} w = 0,$$

meaning, in particular, that this transformation maps the set of constants $Z_h \subset \exp \mathcal{A}_\varepsilon$ of the derivation $D_s^{(h)} := \partial/\partial s + h(v)\partial/\partial x$ into the set of constants $Z_f \subset \exp \mathcal{A}_\varepsilon$, $v = \xi_{(h)}(w)$, of the derivation $D_s^{(f)} := \partial/\partial s + f(w)\partial/\partial x$, where $w = \tilde{u} + \eta_\varepsilon(\tilde{u}) \in \exp \mathcal{A}_\varepsilon$ for $\tilde{u} \in \mathcal{K}$. The latter makes it tion $D_s^{(f)} := \partial/\partial s + f(w)\partial/\partial x$, where $w = \tilde{u} + \eta_\varepsilon(\tilde{u}) \in \exp \mathcal{A}_\varepsilon$ for $\tilde{u} \in \mathcal{K}$. The latter makes it possible to state that the corresponding to (3.9) ideal

(3.12)

$$\tilde{I}_{\varepsilon,N}(w) := \{ \sum_{j=\overline{1,N}} \tilde{a}_{j,\varepsilon}(\tilde{u}) D_x^j w : w = \tilde{u} + \eta_\varepsilon(\tilde{u}) \in Z_h, \tilde{a}_{j,\varepsilon}(\tilde{u}), \tilde{b}_{k,\varepsilon}(\tilde{u}) \in \mathcal{A}_\varepsilon(\tilde{u})\},$$

is also invariant with respect to the derivation $D_s^{(h)} := \partial/\partial s + h(w)\partial/\partial x$, $w \in Z_h$, for any $h : \mathcal{K} \to \mathcal{K}$. Moreover, as the invariance of the ideal (3.9) takes place iff there holds the equivalence

$$D_s^{(f)}(u + \eta_\varepsilon(u)) = 0 \Longleftrightarrow D_s^{(f)} u = F_{N.\varepsilon}(u), \tag{3.13}$$

the equivalence

$$D_s^{(f)}(u+\eta_\varepsilon(u))=0 \Longleftrightarrow D_s^{(f)}u=F_{N.\varepsilon}(u), \tag{3.13}$$

analogously the invariance of the ideal (3.12) takes place iff there holds the equivalence

$$D_s^{(h)}(u+\eta_\varepsilon(u))=0 \Longleftrightarrow D_s^{(h)}u=H_\varepsilon(u), \tag{3.14}$$

what coincides exactly with the B. Dubrovin's criterion (3.7). Thus, one can summarize the reasonings above as the following theorem.

Theorem 3.3. *Let $f \in \mathcal{K}$ and the invertible smooth mapping $\xi_{(h)} : \mathcal{K} \to \mathcal{K}$ be defined via the composition $f \circ \xi_{(h)} = h$, where $h : \mathcal{K} \to \mathcal{K}$ is any invertible smooth mapping. Then the evolution flow (3.1) is integrable, iff the set $Z_f := \{v := u + \eta_\varepsilon(u) \in \exp \mathcal{A}_\varepsilon, u \in \mathcal{K}\}$ of constants of the derivation $D_s^{(f)} := \partial/\partial s + f(v)\partial/\partial x$, $s \in \mathbb{R}$, coincides modulo the mapping $\xi_{(h)} : \mathcal{K} \to \mathcal{K}$ with the set of constants $Z_h = \{w := \tilde{u} + \eta_\varepsilon(\tilde{u}) \in \exp \mathcal{A}_\varepsilon, \tilde{u} \in \mathcal{K}\}$ of the derivation $D_s^{(h)} := \partial/\partial s + h(\circ)\partial/\partial x$, where $\tilde{u} + \eta_\varepsilon(\tilde{u}) = \xi_{(h)}^{-1}(u + \eta_\varepsilon(u))$, $u + \eta_\varepsilon(u) \in Z_f$. Moreover, the corresponding ideals $I_\varepsilon(v) \in \mathcal{A}_\varepsilon(u)$ (3.9) and $\tilde{I}_\varepsilon(w) \in \mathcal{A}_\varepsilon(\tilde{u})$ (3.12) are invariant iff there hold, , respectively, the equivalences (3.13) and (3.14).*

The Lie-algebraic commutator relationship

$$[\mathrm{D}_t, \mathrm{D}_x] = -(D_x u)\mathrm{D}_x$$

and its endomorphic representations

2.2. Endomorphic representations of the Lie-algebraic commutator relationship $[D_t, D_x] = -(D_x u) D_x$**.** We have also considered the functional ring $\mathcal{K} := C^{\infty}(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ of real-valued smooth functions on the spatial-temporal plane $\mathbb{R} \times \mathbb{R}$ and the corresponding differential polynomial ring $\mathcal{K}\{u\} := \mathcal{K}[\Theta u]$ with respect to an arbitrary yet fixed function variable $u \in \mathcal{K}$, where Θ is the standard monoid of commuting to each other "*shifting*" derivations $\partial/\partial x$ and $\partial/\partial t$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential [6, 10, 28, 27, 29] if $I\{u\} = \Theta I\{u\}$.

On the invariant differential ring $\mathcal{K}\{u\}$ one can construct naturally another "*convective*" functional derivation $D_t := \partial/\partial t + u\partial/\partial x$, satisfying jointly with the derivation $D_x := \partial/\partial x$ the Lie-commutator relationship

$$[D_t, D_x] = -(D_x u)\, D_x. \tag{2.6}$$

Then one can pose the following inverse problem:

Problem 2.2. *To describe the possible linear endomorphic representations* $D_x \to l(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$ and $D_t \to p(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$ *of the derivations* D_x and $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, respectively, satisfying the related differential-matrix *relationship*

$$D_t \circ l(u) - D_x \circ p(u) = -(D_x u)\, l(u), \tag{2.7}$$

$$D_t \circ l(u) - D_x \circ p(u) = -(D_x u)\, l(u), \tag{2.7}$$

imitating the Lie-commutator relationship (2.6) on suitably related vector spaces $\mathcal{K}\{u\}^N, N \in \mathbb{N}$.

It is easy to get convinced that for an arbitrarily chosen element $u \in \mathcal{K}$ there exists the unique linear endomorphic representation of the derivations $D_x \to l(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$ and $D_t \to p(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$, satisfying the differential-algebraic relationship (2.7) on the finite dimensional vector spaces $\mathcal{K}\{u\}^N, N \in \mathbb{N}$, and coinciding tautologically with the functional "*convective*" and "*shifting*" derivations $D_t := \partial/\partial t + u\partial/\partial x$ and $D_x := \partial/\partial x$, respectively. Nonetheless, if some coinciding tautologically with the functional "*convective*" and "*shifting*" derivations $D_t := \partial/\partial t + u\partial/\partial x$ and $D_x := \partial/\partial x$, respectively. Nonetheless, if some additional differential-algebraic constraints are imposed on an element $u \in \mathcal{K}$, the Problem 2.2 becomes not trivial and solvable, as it was before demonstrated in [31, 32, 33, 34, 35], where there were constructed linear finite-dimensional matrix representations of the Lie-commutator relationship (2.6) in the naturally related functional vector spaces $\mathcal{K}\{u\}^N$ for the corresponding dimensions $N \in \mathbb{N}$.

To make the approach, proposed previously in [33] for constructing such finite-dimensional representations of the Lie-commutator relationship (2.6), more elaborated and practically feasible, we considered below in detail a new interesting enough differential-algebraic scheme, ensuing from the differential Riemann type relationship

$$D_t^n z(u) := (\partial/\partial t + u(x,t)\partial/\partial x)^n z(u) = 0, \tag{2.8}$$

imposed on an element $z(u) \in \mathcal{K}$ for $n \in \mathbb{N}$ and an arbitrary $u \in \mathcal{K}$. It makes it possible to proceed from the ring $\mathcal{K}\{u\}$ to the Liouville type extended ring $\mathcal{K}_n\{z(u)\}$, $n \in \mathbb{N}$, generated by elements

$$\{z(u), D_t z(u), ..., D_t^{n-1} z(u) \ \in \mathcal{K}\{u\} : D_t^n z(u) = 0, u \in \mathcal{K}\} \tag{2.9}$$

and to pose the following slightly modified problem 2.2:

(2.9)

$$\{z(u), D_t z(u), \dots, D_t^{n-1} z(u) \in \mathcal{K}\{u\} : D_t^n z(u) = 0, u \in \mathcal{K}\}$$

and to pose the following slightly modified problem 2.2:

Problem 2.3. *To describe the possible linear finite dimensional representations of the derivations* $D_x, D_t : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}$, $n \in \mathbb{N}$, satisfying the related endomorphic differential-algebraic relationship (2.7) *in the functional vector spaces* $\mathcal{K}_n\{z(u)\}^m$ for some suitably chosen $m \in \mathbb{N}$.

Proceed now to the before formulated Problem and consider the functional element $z^{(\alpha;n)} := \left(D_x D_t^{n-1} z(u)\right)^{-\alpha} \in \mathcal{K}_n\{z(u)\}, n \in \mathbb{N},$ for $\alpha \in \mathbb{R}\backslash\{0\},$ generating for any $m \in \mathbb{N}$ the finite-dimensional "*tangent*" ideal

(4.1)

$$I_{n,m}^{(\alpha)}\{z(u)\} := \left\{\sum_{k=0}^{n-1} g_k D_t^k z + \sum_{j=0}^{m-1} f_j (-D_x)^{m-j-1} z^{(\alpha;n)} :\right.$$

$$\left. g_k, f_j \in \mathcal{K}_n\{z(u)\}, k = \overline{0, n-1}, j = \overline{0, m-1}, D_t^n z(u) = 0, u \in \mathcal{K}\right\}$$

in the differential ring $\mathcal{K}_n\{z(u)\}$. The next lemma holds.

Lemma 4.1. *The ideal (4.1) is invariant with respect to the "convective" derivation* $D_t : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}$, $n \in \mathbb{N}$, *that is* $D_t I_{n,m}^{(\alpha)}\{z(u)\} \subset I_{n,m}^{(\alpha)}\{z(u)\}$, $\alpha \in \mathbb{R}\backslash\{0\}$ *for any* $m \in \mathbb{N}$ *at any* $u \in \mathcal{K}$.

Proof. Observe that, owing to the basic differential relationship $D_t\left(D_t^{n-1}z(u)\right) = 0$, $n \in \mathbb{N}$, for $\alpha \in \mathbb{R}\backslash\{0\}$, one easily obtains a set of recurrent expressions:

(4.2)

$$D_t(D_t^k z) = D_t^{k+1} z, \quad D_t z^{(\alpha;n)} = \alpha(D_x u) z^{(\alpha;n)},$$
$$D_t\left(D_x z^{(\alpha;n)}\right) = (\alpha - D_x u)\, D_x z^{(\alpha;n)}, \ldots,$$
$$D_t\left(D_x^{j+1} z^{(\alpha;n)}\right) = D_x\left(D_t D_x{}^j z^{(\alpha;n)}\right) - (D_x u)\left(D_x{}^{j+1} z^{(\alpha)}\right), \ldots$$

for $j = \overline{0,m}$, respectively, guaranteeing the invariance of the "*tangent*" ideal (4.1) with respect to the "*convective*" derivation $D_t : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}$ for any $n, m \in \mathbb{N}$ at any $u \in \mathcal{K}$. $\square$

Now let us consider the simplest case $n = 1$, $m = 2$, for which the invariant ideal equals

(4.3)

$$I_{1,2}^{(\alpha)}\{z(u)\} = \{f_0 D_x z^{(\alpha;1)} - f_1 z^{(\alpha;1)} : f_0, f_1 \in \mathcal{K}_1\{z(u)\}\}.$$

Based on Lemma 4.1, one can easily find the kernel Ker $D_t \subset I_{1,2}^{(\alpha)}\{z(u)\}$ of the derivation $D_t : \mathcal{K}_1\{z(u)\} \to \mathcal{K}_1\{z(u)\}$, suitably reduced on the invariant ideal (4.3):

(4.4)

$$\text{Ker } D_t = \{f_0 D_x z^{(\alpha;1)} - f_1 z^{(\alpha;1)} \in I_{1,2}^{(\alpha)}\{z(u)\} :$$

$$D_t f_0 = (1-\alpha) f_0 D_x u, \quad D_t f_1 = \alpha f_0 D_x^2 u - \alpha f_1 D_x u\}.$$

The latter makes it possible to construct a finite-dimensional linear endomorphic representation $D_t \to p(u) : \mathcal{K}_1\{z(u)\}^2 \to \mathcal{K}_1\{z(u)\}^2$ of the "*convective*" derivation $D_t : \mathcal{K}_1\{z(u)\} \to \mathcal{K}_1\{z(u)\}$ in the equivalent matrix form:

$$p(u) = \begin{pmatrix} (1-\alpha)D_x u & 0 \\ \alpha D_x^2 u & -\alpha D_x u \end{pmatrix} \tag{4.5}$$

for any $\alpha \in \mathbb{R}\backslash\{0\}$.

Concerning the related finite-dimensional linear endomorphic representation $D_x \to l(u) : \mathcal{K}_1\{z(u)\}^2 \to \mathcal{K}_1\{z(u)\}^2$ of the "*shifting*" derivation $D_x : \mathcal{K}_1\{z(u)\} \to \mathcal{K}_1\{z(u)\}$, we easily obtain that it satisfies the endomorphic differential-matrix relationship

$$D_t \circ l(u) - D_x \circ p(u) = -(D_x u)\, l(u), \tag{4.6}$$

mimicking the Lie-commutator relationship (2.6), and which we will analyze in more detail using the functional structure of the kernel subspace (4.4). The latter can be derived the following way: first we observe that, as follows from (4.4), the element $\tilde{f}_1 := D_x f_0 \in \mathcal{K}\{z(u)\}$ satisfies the differential relationship

$$D_t \tilde{f}_1 = (1-\alpha) f_0 D_x^2 u - \alpha \tilde{f}_1 D_x u, \tag{4.7}$$

$$D_t\tilde{f}_1 = (1-\alpha)f_0 D_x^2 u - \alpha\tilde{f}_1 D_x u, \tag{4.7}$$

which exactly coincides with that on the element $f_1 \in \mathcal{K}_1\{z(u)\}$:

$$D_t f_1 = \ \alpha f_0 D_x^2 u - \alpha f_1 D_x u \tag{4.8}$$

at $\alpha = 1/2$, simply meaning that at $\alpha = 1/2$ the element $\tilde{f}_1 = f_1 \in \mathcal{K}_1\{z(u)\}$ and the linear endomorphic representations of the derivations $D_x, D_t : \mathcal{K}_1\{z(u)\} \to \mathcal{K}_1\{z(u)\}$ can be realized with respect to the linearly related to each other basis

(4.9)

$$\{f_0, f_1 = D_x f_0 \in \mathcal{K}_1\{z(u)\} : D_t f_0 = 1/2 f_0 D_x u,$$

$$D_t f_1 = \ 1/2(f_0 D_x^2 u - f_1 D_x u)\}.$$

Really, taking into account the above relationships (4.9) and the Lie-commutator relationship (2.6), one easily obtains that

$$D_x f_0 = f_1, \quad D_x f_1 = 1/2v(u)\ f_0, \tag{4.10}$$

if the additional functional constraint

$$D_t v(u) + 2v(u) D_x u - D_x^3 v(u) = 0 \tag{4.11}$$

is imposed on an element $v(u) \in \mathcal{K}_1\{z(u)\}$. In particular, if we put, by definition, $v(u) := u \in \mathcal{K}_1\{z(u)\}$, then the differential-algebraic relationship (4.11) reduces to

$$D_t u + 2u D_x u - D_x^3 u = 0, \tag{4.12}$$

which is equiavlent to the well known nonlinear Korteweg-de Vries dynamical system

$$u_t = -3uu_x + u_{3x} \tag{4.13}$$

$$u_t = -3uu_x + u_{3x} \tag{4.13}$$

on a suitably chosen functional manifold $M \subset \mathcal{K}$. On the other hand, if we put, by definition, $v(u) := D_x u \in \mathcal{K}_1\{z(u)\}$, then the differential-algebraic relationship (4.11) reduces, respectively, to the next differential-algebraic relationhip

$$D_x D_t u + (D_x u)^2 - D_x^3 u = 0. \tag{4.14}$$

Simultaneosuly, the obtained above differential-functional relationships (4.10) give rise right away to the following matrix representation

$$l(u) = \begin{pmatrix} 0 & 1 \\ 1/2 D_x u & 0 \end{pmatrix} \tag{4.15}$$

on the linear functional vector space $\mathcal{K}_1\{z(u)\}^2$ of the derivation $D_x : \mathcal{K}_1\{z(u)\} \rightarrow \mathcal{K}_1\{z(u)\}$, satisfying jointly with the matrix on the linear functional vector space $\mathcal{K}_1\{z(u)\}^2$ of the derivation $D_x : \mathcal{K}_1\{z(u)\} \to \mathcal{K}_1\{z(u)\}$, satisfying jointly with the matrix representation (4.5) the related differential-matrix endomorphic relationship (4.6), solving the posed above Problem 2.3. Thus, the results obtained above can be formulated as the following theorem.

Theorem 4.2. *Let a function $u \in \mathcal{K}$ satisfy the following differential relationship:*

$$D_x D_t u + (D_x u)^2 - D_x^3 u = 0. \tag{4.16}$$

Then the following linear matrix endomorphisms

(4.17)

$$l(u) = \begin{pmatrix} 0 & 1 \\ 1/2 D_x u & 0 \end{pmatrix}, \quad p(u) = \begin{pmatrix} 1/2 D_x u & 0 \\ 1/2 D_x^2 u & -1/2 D_x u \end{pmatrix}$$

on the functional vector space $\mathcal{K}_1\{z(u)\}^2$ solve the differential-matrix endomorphic relationship (4.6), imitating the Lie-commutator relationship (2.6).

Remark 4.3. It is worth to mention here that the differential-functional constraint (4.11) on an element $u \in \mathcal{K}$, rewritten in the following equivalent partial differential equation form

$$u_{tx} + uu_{xx} + 2u_x^2 - u_{xxx} = 0, \tag{4.18}$$

is well known in fluid mechanics [41, 42, 43, 44] as the generalized Proudman-Johnson equation and was recently analyzed in the work [45] by means of differential-geometric tools, and where there was constructed its differential covering [53, 54] in the nonlinear form

$$\begin{aligned} q_x &= -q^2 + 1/2u_x, \\ q_t &= uq^2 - u_x q + 1/2(u_{xx} - uu_x) \end{aligned} \tag{4.19}$$

by means of a smooth functional element $q \in \mathcal{K}_1\{z(u)\}$. Yet, as

$$
\begin{aligned}
(4.19) \qquad q_x &= -q^2 + 1/2u_x, \\
q_t &= uq^2 - u_x q + 1/2(u_{xx} - uu_x)
\end{aligned}
$$

by means of a smooth functional element $q \in \mathcal{K}_1\{z(u)\}$. Yet, as is easily to check, the differential covering (4.19) reduces via the substitution $q = D_x(\ln f_0)$, $f_0 \in \mathcal{K}_1\{z(u)\}$, to the obtained above linear differential relationships (4.9) and (4.10).

We can here observe that the construction above can be easily generalized to the case of a derivation $D_t^{(\varphi)} := \partial/\partial t + \varphi(u)\partial/\partial x$ in the suitably Liouville type extended ring $\mathcal{K}_n^{(\varphi)}\{z(u)\}, n \in \mathbb{N}$, where $\varphi(u) \in \mathcal{K}\{u\}$ and $u \in \mathcal{K}$ is such that for some element $z(u) \in \mathcal{K}$ there holds the next constraint

$$
(4.20) \qquad D_t^{(\varphi)n} z(u) := (\partial/\partial t + \varphi(u)\partial/\partial x)^n z(u) = 0.
$$

(4.20) $$D_t^{(\varphi)n} z(u) := (\partial/\partial t + \varphi(u)\partial/\partial x)^n z(u) = 0.$$

The derivation $D_t^{(\varphi)} : \mathcal{K}_n^{(\varphi)}\{z(u)\} \to \mathcal{K}_n^{(\varphi)}\{z(u)\}$ satisfies the following analog of the Lie algebraic commutative cndition (2.6):

(4.21) $$[D_t^{(\varphi)}, D_x] = -(D_x\varphi)\, D_x.$$

Then one can similarly construct the set

(4.22)

$$I_{n,m}^{(\varphi;\alpha)}\{z(u)\} := \Big\{ \sum_{k=0}^{n-1} g_k D_t^{(\varphi)k} z + \sum_{j=0}^{m-1} f_j (-D_x)^{m-j-1} z^{(\alpha;n)} :$$

$$g_k, f_j \in \mathcal{K}_n^{(\varphi)}\{z(u)\}, k = \overline{0, n-1}, j = \overline{0, m-1}, D_t^n z(u) = 0, u \in \mathcal{K} \Big\}$$

Then one can similarly construct the set

(4.22)

$$I_{n,m}^{(\varphi;\alpha)}\{z(u)\} := \Big\{\sum_{k=0}^{n-1} g_k D_t^{(\varphi)k} z + \sum_{j=0}^{m-1} f_j(-D_x)^{m-j-1} z^{(\alpha;n)} :$$

$$g_k, f_j \in \mathcal{K}_n^{(\varphi)}\{z(u)\}, k=\overline{0,n-1}, j=\overline{0,m-1}, D_t^n z(u)=0, u\in\mathcal{K}\Big\}$$

in the differential ring $\mathcal{K}_n^{(\varphi)}\{z(u)\}$, where, by definition, $z^{(\alpha;n)} := \left(D_x D_t^{(\varphi)(n-1)} z\right)^{-\alpha}$, $\alpha \in \mathbb{R}\backslash\{0\}$. The next lemma holds.

Lemma 4.4. *The ideal (4.1) is invariant with respect to the "convective" derivation* $D_t^{(\varphi)} : \mathcal{K}_n^{(\varphi)}\{z(u)\} \to \mathcal{K}_n^{(\varphi)}\{z(u)\}$, $n\in\mathbb{N}$, *that is* $D_t^{(\varphi)} I_{n,m}^{(\alpha)}\{z(u)\} \subset I_{n,m}^{(\alpha)}\{z(u)\}$, $\alpha\in\mathbb{R}\backslash\{0\}$ *for any* $m\in\mathbb{N}$ *at any* $u\in\mathcal{K}$.

Lemma 4.4. *The ideal (4.1) is invariant with respect to the "convective" derivation* $D_t^{(\varphi)}: \mathcal{K}_n^{(\varphi)}\{z(u)\} \to \mathcal{K}_n^{(\varphi)}\{z(u)\}$, $n \in \mathbb{N}$, *that is* $D_t^{(\varphi)} I_{n,m}^{(\alpha)}\{z(u)\} \subset I_{n,m}^{(\alpha)}\{z(u)\}$, $\alpha \in \mathbb{R}\backslash\{0\}$ *for any* $m \in \mathbb{N}$ *at any* $u \in \mathcal{K}$.

Proof. Observe that, owing to the basic differential relationship $D_t^{(\varphi)}\left(D_t^{(\varphi)(n-1)} z(u)\right) = 0$, $n \in \mathbb{N}$, for $\alpha \in \mathbb{R}\backslash\{0\}$, one easily obtains a set of recurrent expressions:

(4.23)

$$D_t^{(\varphi)}(D_x^{(\varphi)k} z) = D_t^{(\varphi)(k+1)} z, \quad D_t^{(\varphi)} z^{(\alpha;n)} = \alpha(D_x\varphi) z^{(\alpha;n)},$$
$$D_t^{(\varphi)}\left(D_x z^{(\alpha;n)}\right) = (\alpha - D_x\varphi)\, D_x z^{(\alpha;n)}, \ldots,$$
$$D_t^{(\varphi)}\left(D_x^{j+1} z^{(\alpha;n)}\right) = D_x\left(D_t^{(\varphi)} D_x{}^j z^{(\alpha;n)}\right) - (D_x\varphi)\left(D_x{}^{j+1} z^{(\alpha)}\right), \ldots$$

for $j = \overline{0, m}$, respectively, guaranteeing the invariance of the "*tangent*" ideal (4.1) with respect to the "*convective*" derivation $D_t : \mathcal{K}_n^{(\varphi)}\{z(u)\} \to \mathcal{K}_n^{(\varphi)}\{z(u)\}$ for any $n, m \in \mathbb{N}$ at any $u \in \mathcal{K}$. $\square$

As before let us proceed to the simplest case $n = 1$, $m = 2$, for which the invariant ideal equals

(4.24)

$$I_{1,2}^{(\varphi;\alpha)}\{z(u)\} = \{f_0 D_x z^{(\alpha;1)} - f_1 z^{(\alpha;1)} : f_0, f_1 \in \mathcal{K}_1^{(\varphi)}\{z(u)\}\}.$$

Based on Lemma 4.4, one can easily find the kernel Ker $D_t^{(\varphi)} \subset I_{1,2}^{(\varphi;\alpha)}\{z(u)\}$ of the derivation $D_t^{(\varphi)} : \mathcal{K}_n^{(\varphi)}\{z(u)\} \to \mathcal{K}_n^{(\varphi)}\{z(u)\}$, suitably reduced on the invariant ideal (4.3):

(4.25)

$$\mathrm{Ker}\ D_t^{(\varphi)} = \{f_0 D_x z^{(\alpha;1)} - f_1 z^{(\alpha;1)} \in I_{1,2}^{(\varphi;\alpha)}\{z(u)\} :$$

$$D_t^{(\varphi)} f_0 = (1-\alpha) f_0 D_x \varphi, \quad D_t^{(\varphi)} f_1 = \alpha f_0 D_x \varphi - \alpha f_1 D_x \varphi\}.$$

The latter makes it possible to construct a finite-dimensional linear endomorphic representation $D_t^{(\varphi)} \to p^{(\varphi)}(u) : \mathcal{K}_1^{(\varphi)}\{z(u)\}^2 \to \mathcal{K}_1^{(\varphi)}\{z(u)\}^2$ of the "*convective*" derivation $D_t^{(\varphi)} : \mathcal{K}_1^{(\varphi)}\{z(u)\} \to \mathcal{K}_1^{(\varphi)}\{z(u)\}$ in the equivalent matrix form:

$$p^{(\varphi)}(u) = \begin{pmatrix} (1-\alpha)D_x\varphi & 0 \\ \alpha D_x^2\varphi & -\alpha D_x\varphi \end{pmatrix} \tag{4.26}$$

for any $\alpha \in \mathbb{R}\backslash\{0\}$.

Concerning the related finite-dimensional linear endomorphic representation $D_x \to l^{(\varphi)}(u) : \mathcal{K}_1^{(\varphi)}\{z(u)\}^2 \to \mathcal{K}_1^{(\varphi)}\{z(u)\}^2$ of the "*shifting*" derivation $D_x : \mathcal{K}_1^{(\varphi)}\{z(u)\} \to \mathcal{K}_1^{(\varphi)}\{z(u)\}$, we easily obtain that it satisfies the endomorphic differential-matrix relationship

$$D_t^{(\varphi)} \circ l^{(\varphi)}(u) - D_x \circ p^{(\varphi)}(u) = -(D_x\varphi)\, l^{(\varphi)}(u), \tag{4.27}$$

$$D_t^{(\varphi)} \circ l^{(\varphi)}(u) - D_x \circ p^{(\varphi)}(u) = -(D_x\varphi)\, l^{(\varphi)}(u), \tag{4.27}$$

mimicking the Lie-commutator relationship (2.6), and which we will analyze in more detail using the functional structure of the kernel subspace (4.4). The latter can be derived the following way: first we observe that, as follows from (4.4), the element $\tilde{f}_1 := D_x f_0 \in \mathcal{K}_1^{(\varphi)}\{z(u)\}$ satisfies the differential relationship

$$D_t^{(\varphi)} \tilde{f}_1 = (1-\alpha) f_0 D_x\varphi - \alpha \tilde{f}_1 D_x\varphi, \tag{4.28}$$

which exactly coincides with that on the element $f_1 \in \mathcal{K}_1^{(\varphi)}\{z(u)\}$:

$$D_t^{(\varphi)} f_1 = \ \alpha f_0 D_x^2\varphi - \alpha f_1 D_x\varphi \tag{4.29}$$

at $\alpha = 1/2$, simply meaning that the element $\tilde{f}_1 = f_1 \in \mathcal{K}_1^{(\varphi)}\{z(u)\}$

$$D_t^{(\varphi)} f_1 = \alpha f_0 D_x^2 \varphi - \alpha f_1 D_x \varphi \tag{4.29}$$

at $\alpha = 1/2$, simply meaning that the element $\tilde{f}_1 = f_1 \in \mathcal{K}_1^{(\varphi)}\{z(u)\}$ and the linear endomorphic representations of the derivations $D_x, D_t^{(\varphi)} : \mathcal{K}_1^{(\varphi)}\{z(u)\} \to \mathcal{K}_1^{(\varphi)}\{z(u)\}$ can be realized with respect to the linearly related to each other basis

$$\begin{gathered} \{f_0, f_1 = D_x f_0 \in \mathcal{K}_1^{(\varphi)}\{z(u)\} : D_t^{(\varphi)} f_0 = 1/2 f_0 D_x \varphi, \\ D_t^{(\varphi)} f_1 = 1/2(f_0 D_x^2 \varphi - f_1 D_x \varphi)\}. \end{gathered} \tag{4.30}$$

Really, taking into account the above relationships (4.9) and the Lie-commutator relationship (2.6), one easily obtains that

$$D_x f_0 = f_1, \quad D_x f_1 = 1/2v(u)\ f_0, \tag{4.31}$$

if the additional functional constraint

$$D_t^{(\varphi)} v(u) + 2v(u) D_x\varphi - D_x^3\varphi(u) = 0 \tag{4.32}$$

holds. In particular, if to assume that the element $v(u) = u + \lambda$, where $\lambda \in \mathbb{R}$ is an arbitrary parameter, then the relationship (4.32) is rewritten as

$$\partial u/\partial t = (D_x^3 - uD_x - D_x u)\varphi - 2\lambda D_x\varphi, \tag{4.33}$$

naturally meaning that the element $\varphi(u) := \varphi(u;\lambda) \in \mathcal{K}\{u\}$

$$\partial u/\partial t = (D_x^3 - uD_x - D_x u)\varphi - 2\lambda D_x \varphi, \tag{4.33}$$

naturally meaning that the element $\varphi(u) := \varphi(u;\lambda) \in \mathcal{K}\{u\}$ should *a priori* depend on $\lambda \in \mathbb{R}$. Moreover, as the left hand side of (4.33) does not depend on on $\lambda \in \mathbb{R}$ the right hand side is linear $\lambda \in \mathbb{R}$, whence one easily derives that the following polynomial representation

$$\varphi(u;\lambda) \to \varphi_m(u;\lambda) = \sum_{j \in \overline{0,m}} \varphi_j(u)\lambda^{m-j} \tag{4.34}$$

solves (4.33) if the coefficients $\varphi_j(u) \in \mathcal{K}\{u\}, j = \overline{0,m}$, satisfy the following recurrent relationships

$$(D_x^3 - uD_x - D_x u)\varphi_j(u) = 2D_x \varphi_{j+1}(u) \tag{4.35}$$

$$(D_x^3 - uD_x - D_x u)\varphi_j(u) = 2D_x\varphi_{j+1}(u) \tag{4.35}$$

under the initial condition $\varphi_0 = 1$, as should be, evidently, $D_x\varphi_0(u) = 0$ for any $u \in \mathcal{K}$. In particular, we find that $\varphi_2(u) = -1/2u$, $\varphi_3(u) = 1/4(3u^2 - D_x^2 u)$... etc. Whence the expression (4.33) obtains for any $m \in \mathbb{N}$ the following reduced form of the nonlinear dynamical system

$$\partial u/\partial t = (D_x^3 - uD_x - D_x u)\varphi_m(u) \tag{4.36}$$

on the functional manifold $M \subset \mathcal{K}$.

Remark 4.5. It is worth to mention here that since the differential operator $\eta := \left(D_x^3 - uD_x - D_x u\right) : T^*(M) \to T(M)$ is skew-symmetric and Poissonian [46, 47, 23, 51] on the functional manifold M, it allows to restate the well known fact [52, 49, 50] that nonlinear dynamical system (4.36) is Hamiltonian with respect to the Hamiltonian function $H_m := \int_0^1 d\mu(\varphi_m(\mu u)|u)$, $m \in \mathbb{N}$, where $(\cdot|\cdot)$ denotes the standard bilinear convolution form on the product $T^*(M) \times T(M)$. Moreover, as the differential operator $\vartheta := D_x : T^*(M) \to T(M)$ is also skew-symmetric, Poissonian on M and compatible with the operator $\eta : T^*(M) \to T(M)$, one derives that the whole infinite hierarchy of Hamiltonian dynamical systems (4.36) for all $m \in \mathbb{N}$ is commuting to each other, being equivalent to its integrability.

The stated above existence of the Lax type representation (4.6) for the partial differential equation (4.18) nonetheless can not ensure its integrability, as the latter still requires the existence of an infinite hierarchy of nontrivial and functionally independent conservation laws, what the corresponding to (4.18) evolution flow fails to possess. To demonstrate this we will make use of the gradient-holonomic integrability scheme [15, 26], which first consists in finding special asymptotical solutions to the following Lax-Noether functional equation

$$\varphi_t + K^{',*}[u]\varphi = 0, \tag{4.37}$$

where, by definition, $\varphi \in T^*(M)^*, K^{',*}[u] : T^*(M) \to T^*(M)$ denotes the adjoint expression of the Frechet derivative $K'[u] : T(M) \to T(M)$ with respect to the natural bilinear form $T^*(M) \times T(M) \to \mathbb{R}$ for a vector field $K : M \to T(M)$ on a suitably defined functional manifold $M \subset C^{\infty}(\mathbb{R};\mathbb{R})$, determined by means of the following analytic expression:

$$u_t = -uu_x + u_{xx} - \partial^{-1}u_x^2 := K[u]. \tag{4.38}$$

As the related with the vector field (4.38) operator $K^{',*}[u] = uD_x + D_x^2 - 2D_x^{-1}u_xD_x$, it is easy to find the asymptotic as $|\lambda| \to \infty$ solution $\varphi = \exp\left[-\lambda^2 t + D_x^{-1}\sigma(u;\lambda)\right]$ to the functional equation (4.37), where

$$\sigma(u;\lambda) \sim \sum_{j \in \mathbb{Z}_+ \cup \{-1\}} \sigma_j[u]\lambda^{-j}, \tag{4.39}$$

$$\sigma(u;\lambda) \sim \sum_{j\in\mathbb{Z}_+\cup\{-1\}} \sigma_j[u]\lambda^{-j}, \tag{4.39}$$

and whose coefficients satisfy the foillowing recurrent differential-functional relationships:

$$\begin{aligned} -\delta_{j,-2} + D_x^{-1}\sigma_{j,t} + u\sigma_j + \sigma_{j,x} + \sum_{k\in\mathbb{Z}_+\cup\{-1\}} \sigma_{j-k}\sigma_k - 2a_j &= 0, \\ a_j, x + \sum_{k\in\mathbb{Z}_+\cup\{-1\}} \sigma_{j-k}a_k - u_x\sigma_j &= 0 \end{aligned} \tag{4.40}$$

for $j \in \mathbb{Z}_+ \cup \{-2,-1\}$. Since the quantities $\int_{\mathbb{R}} \sigma_j[u]dx, j \in \mathbb{Z}_+$, for $j \in \mathbb{Z}_+ \cup \{-2, -1\}$. Since the quantities $\int_{\mathbb{R}} \sigma_j[u]dx, j \in \mathbb{Z}_+$, should be, by construction, conservation laws for the vector field (4.38), we easily obtain from (4.40) that $a_{-1} = 0, \sigma_{-1} = 1; a_0 = u_x, \sigma_0 = -u/2$. To obtain the next functional elements a_1 and σ_{-1} one needs to state that the quantity $\int_{\mathbb{R}} \sigma_0[u]dx$ is conserved along the vector field (4.38), yet one easily obtains that

$$\frac{d}{dt}\int_{\mathbb{R}} \sigma_0[u]dx \neq 0, \tag{4.41}$$

thus meaning that the asymptotic expression (4.39) does not generates an infinite hierarchy of nontrivial conservation laws for the vector field (4.38), thus contradicting its Lax type integrability.

Moreover, it possessess, respectively, no Poissonian structure on the functional manifold, subject to which the vector field (4.38) could be represented as a Hamiltoninan system. The obtained result we formulate as the following proposition.

Proposition 4.6. *The nonlinear evolution flow (4.38) does not generate a Lax type integrable Hamiltonian dynamical system on the functional manifold M.*

Proceed now to the case $n = 2, m = \emptyset$, when the D_t- invariant two-dimensional ideal (4.1) in the differential ring $\mathcal{K}_2\{z(u)\}$ is presented as

Proceed now to the case $n = 2, m = \emptyset$, when the D_t − invariant two-dimensional ideal (4.1) in the differential ring $\mathcal{K}_2\{z(u)\}$ is presented as

$$I^{(\alpha)}_{2,\emptyset}\{z(u)\} := \{\lambda g_0 z(u) + g_1 D_t z(u) \ :$$

(4.42)

$$g_0, g_1 \in \mathcal{K}_2\{z(u)\}, D_t^2 z(u) = 0, \lambda \in \mathbb{R}, u \in \mathcal{K}\},$$

and construct the related representation of the endomorphic differential-matrix relationship (4.6). First we need to construct the kernel Ker D_t of the "*convective*" derivation $D_t : \mathcal{K}_2\{z(u)\} \to \mathcal{K}_2\{z(u)\}$, reduced on the ideal (4.42) above to

(4.43) $\quad$ $\text{Ker } D_t = \{g_0, g_1 \in \mathcal{K}_2\{z(u) : D_t g_0 = 0, D_t g_1 = -\lambda g_0\}.$

The latter produces the endomorphic matrix representation $D_t \to p(u;\lambda) : \mathcal{K}_2\{z(u)\}^2 \to \mathcal{K}_2\{z(u)\}^2$ of the "*convective*" $D_t : \mathcal{K}_2\{z(u)\}$

The latter produces the endomorphic matrix representation $D_t \to p(u;\lambda) : \mathcal{K}_2\{z(u)\}^2 \to \mathcal{K}_2\{z(u)\}^2$ of the "*convective*" $D_t : \mathcal{K}_2\{z(u)\} \to \mathcal{K}_2\{z(u)\}$ in the functional vector space $\mathcal{K}_2\{z(u)\}^2$, where

$$p(u;\lambda) = \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix} \tag{4.44}$$

for an arbitrary element $u \in \mathcal{K}$, depending on an arbitrary "spectral" parameter $\lambda \in \mathbb{R}$. To obtain the corresponding endomorphic matrix representation $D_x \to l(u;\lambda) : \mathcal{K}_2\{z(u)\}^2 \to \mathcal{K}_2\{z(u)\}^2$ of the "*shifting*" derivation $D_x : \mathcal{K}_2\{z(u)\} \to \mathcal{K}_2\{z(u)\}$, we will solve the corresponding matrix relationship (4.6), rewriting it in the following simplified matrix form:

$$D_t a(u;\lambda) = [p(u;\lambda), a(u;\lambda)], \tag{4.45}$$

$$D_t a(u;\lambda) = [p(u;\lambda), a(u;\lambda)], \tag{4.45}$$

where we put, by definition, $l(u;\lambda) := D_x a(u;\lambda), \lambda \in \mathbb{R}, u \in \mathcal{K}$. The differential-matrix relationship possesses the following simple solution

$$a(u;\lambda) = \begin{pmatrix} \lambda z(u) & D_t z(u) \\ -2\lambda^2 x & -\lambda z(u) \end{pmatrix}, \tag{4.46}$$

giving rise to the following matrix $l(u;\lambda) \in \mathrm{End}(\mathcal{K}_2\{z(u)^2)$, representing the derivation $D_x : \mathcal{K}_2\{z(u)\} \to \mathcal{K}_2\{z(u)\}$:

$$l(u;\lambda) = \begin{pmatrix} \lambda D_x z(u) & D_x D_t z(u) \\ -2\lambda^2 & -\lambda D_x z(u) \end{pmatrix}, \tag{4.47}$$

$$l(u;\lambda) = \begin{pmatrix} \lambda D_x z(u) & D_x D_t z(u) \\ -2\lambda^2 & -\lambda D_x z(u) \end{pmatrix}, \tag{4.47}$$

depending on an arbitrary "*spectral*" parameter $\lambda \in \mathbb{R}$. The result obtained above we can formulate as the following theorem.

Theorem 4.7. *The differential-algebraic relationship $D_t^2 z(u) = 0$, where $D_t = \partial/\partial t + u\partial/\partial x$ is the functional "convective" derivation of the differential ring $\mathcal{K}_2\{z(u)$, is equivalent to the differential-matrix relationship (4.6) for any element $u \in \mathcal{K}$ and a parameter $\lambda \in \mathbb{R}$ with the endomorphic matrix representations $D_t \to p(u;\lambda) \in \mathrm{End}(\mathcal{K}_2\{z(u)\}^2)$, $D_x \to l(u;\lambda) \in \mathrm{End}(\mathcal{K}_2\{z(u)\}^2)$ of the "convective" and "shifting" $D_t, D_x : \mathcal{K}_2\{z(u)\} \to \mathcal{K}_2\{z(u)\}$ derivations, respectively, given by the matrix expressions (4.44) and (4.47).*

It is easy to observe that the above result is naturally generalized on the case of the invariant differential ideal (4.1) for arbitrary $n \in \mathbb{N}$, $m = \emptyset$:

$$
\begin{aligned}
I_{n,\emptyset}^{(\alpha)}\{z(u)\} := \Big\{ \textstyle\sum_{k=0}^{n-1} g_k D_t^k z(u) : g_k \in \mathcal{K}_n\{z(u)\}, \\
k = \overline{0, n-1}, D_t^n z(u) = 0, u \in \mathcal{K} \Big\}
\end{aligned}
\tag{4.48}
$$

in the differential ring $\mathcal{K}_n\{z(u)\}$, whose details we drop out. Namely, the following theorem holds.

in the differential ring $\mathcal{K}_n\{z(u)\}$, whose details we drop out. Namely, the following theorem holds.

Theorem 4.8. *The differential-algebraic relationship* $D_t^n z(u) = 0, n \in \mathbb{N}$, *is equivalent to the differential-matrix relationships (4.6) for any element* $u \in \mathcal{K}$ *and a parameter* $\lambda \in \mathbb{R}$ *with the endomorphic matrix representations* $D_t \to p(u;\lambda) \in \mathrm{End}(\mathcal{K}_n\{z(u)\}^n)$, $D_x \to l(u;\lambda) \in \mathrm{End}(\mathcal{K}_n\{z(u)\}^n)$ *of the "convective" and "shifting"* $D_t, D_x : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}$ *derivations, respectively,* *given by the matrix expressions*

$$p(u;\lambda) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ -\lambda & 0 & \dots & 0 & 0 \\ \dots & -\lambda & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -\lambda & 0 \end{pmatrix} \tag{4.49}$$

and

(4.50)

$$l(u;\lambda) = \begin{pmatrix} \lambda D_x D_t^{n-2} z(u) & D_x D_t^{n-1} z(u) & \dots & 0 & 0 \\ 0 & \lambda D_x D_t^{n-2} z(u) & \dots & 0 & 0 \\ \dots & \dots & \dots & (n-2) D_x D_t^{n-1} z(u) & 0 \\ 0 & 0 & \dots & \lambda D_x D_t^{n-2} z(u) & (n-1) D_x D_t^{n-1} z(u) \\ -n\lambda^n & -n\lambda^{n-1} z(u) & \dots & -n\lambda^2 D_x D_t^{n-3} z(u) & \lambda(1-n) D_x D_t^{n-2} z(u) \end{pmatrix}$$

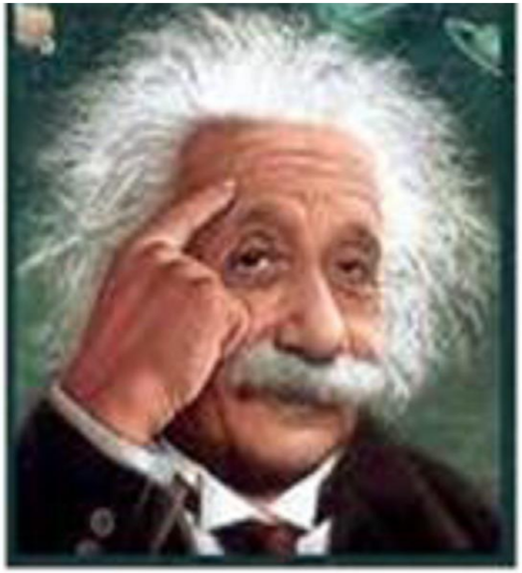
The obtained results in some sense generalize the ones, obtained before in the works [35, 36], devoted to proving the Lax type integrability [26, 23, 48, 25] of a generalized Riemann type hierarchy of differential-algebraic relationships $D_t^n u = 0$ for $u \in \mathcal{K}$ and natural $n \in N$.

An interesting problem and not analyzed here is to construct nontrivial finite-dimensional endomorphic representations $D_t \rightarrow p(u;\lambda) \in \operatorname{End}(\mathcal{K}_n\{z(u)\}^n)$, $D_x \rightarrow l(u;\lambda) \in \operatorname{End}(\mathcal{K}_n\{z(u)\}^n)$ of the "*convective*" and "*shifting*" $D_t, D_x : \mathcal{K}_n\{z(u)\} \rightarrow \mathcal{K}_n\{z(u)\}$ derivations, respectively, generated by the general ideal (4.1) in $\mathcal{K}_n\{z(u)\}$ for arbitrary $n, m \in \mathbb{N}$.

5. Conclusion.

We presented a short review of new approaches to studying finitely-generated differential ideals related to some differential-algebraic constraints and invariant with respect to specially constructed derivations in functional rings, satisfying some Lie-algebraic relationships. In particular, we have succeeded in constructing endomorphic representations of these derivations, equivalent to a kind of the Lax type representation, reducing to some differential-algebraic relationships on a generating function. We have also realgebraic relationships on a generating function. We have also reformulated using the differential-algebraic terms the well known Dubrovin's integrability criterion of the classical Riemann equations, perturbed by means of some special elements from a suitably constructed differential ring, and shown that this criterion is firmly based on the differential properties of the corresponding common set of constants, generated by the suitably defined derivations.

Dziękuję serdecznie za uwagę!



Thanks for your attention!

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