ON INVARIANT DIFFERENTIAL IDEALS AND HOMOMORPHIC REPRESENTATIONS OF FUNCTIONAL DERIVATIONS IN DIFFERENTIAL RINGS

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Abstract. We analyze finitely-generated by some differntial-algebraic relationships differential ideals in functional rings, invariant with respect some specially constructed derivations and satisfying the corresponding Lie-algebraic relationships. Taking into account the finite-dimensionality of these ideals, we construct the suitably defined homomorphic Lax type representations of these derivations, which in some cases are reduced to constraints equivalent to differential-algebraic relationships on a generating function. The work in part generalizes the results devised before for proving integrability of the well known generalized hierarrchy of the Riemann type equations. We have also reformulated by means of the differential-algebraic terms the well known Dubrovin's integrability criterion of the classical Riemann equations, perturbed by means of some special terms from a suitably constructed differential ring.

1. Introduction

We begin with the functional ring $\mathcal{K} := C^{\infty}(\mathbb{R}\times\mathbb{R};\mathbb{R})$ of real-valued smooth functions on the spatial-temporal plane $\mathbb{R}\times\mathbb{R}$ and constructing the corresponding differential polynomial ring $\mathcal{K}\{u\} := \mathcal{K}[\Theta u]$ with respect to an arbitrarily yet fixed functional variable $u \in \mathcal{K}$, where Θ is the standard monoid of commuting to each other "*shifting*" derivations $\partial/\partial x$ and $\partial/\partial t$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential if $I\{u\} = \Theta I\{u\}$. On the invariant differential ring $\mathcal{K}\{u\}$ one can construct naturally another "*convective*" functional derivation $D_t := \partial/\partial t + u\partial/\partial x$, satisfying jointly with the "*shifting*" derivation $D_x := \partial/\partial x$ the Lie-commutator relationship

$$[D_t, D_x] = -(D_x u)\, D_x. \tag{1.1}$$

Then one can pose the following inverse problem:

Problem 1.1. *To describe the possible linear endomorphic representations* $D_x \to l(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$ and $D_t \to p(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$ *of the derivations* D_x and $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, respectively, satisfying the related differential-matrix *relationship*

$$D_t \circ l(u) - D_x \circ p(u) = -(D_x u)\, l(u), \tag{1.2}$$

imitating the Lie-commutator relationship (1.1) on suitably related vector spaces $\mathcal{K}\{u\}^N, N \in \mathbb{N}$.

It is easy to get convinced that for an arbitrarily chosen element $u \in \mathcal{K}$ there exists the unique linear endomorphic representation of the derivations $D_x \to l(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$ and $D_t \to p(u) : \mathcal{K}\{u\}^N \to \mathcal{K}\{u\}^N$, satisfying the differential-algebraic relationship (1.2) on the finite dimensional vector spaces $\mathcal{K}\{u\}^N, N \in \mathbb{N}$, and coinciding tautologically with the functional "*convective*" and "shifting" derivations $D_t := \partial/\partial t + u\partial/\partial x$ and $D_x := \partial/\partial x$, respectively. Nonetheless, if some additional differential-algebraic constraints are imposed on an element $u \in \mathcal{K}$, the Problem 1.1 becomes not trivial and solvable, as it was before demonstrated in [7, 8, 9, 10, 11], where there were constructed linear finite-dimensional matrix representations of the Lie-commutator relationship (1.1) in the naturally related functional vector spaces $\mathcal{K}\{u\}^N$ for the corresponding dimensions $N \in \mathbb{N}$. To make the approach, devised previously in [9] for constructing such finite-dimensional representations of the Lie-commutator relationship (1.1), more elaborated and practically feasible,

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we consider below in detail a new interesting enough differential-algebraic scheme, ensuing from the differential Riemann type relationship

$$D_t^n z(u) := (\partial/\partial t + u(x,t)\partial/\partial x)^n z(u) = 0, \tag{1.3}$$

imposed on an element $z(u)(x,t) \in \mathcal{K}$ for $n \in \mathbb{N}$ and an arbitrary $u \in \mathcal{K}$, which makes it possible to proceed from the ring $\mathcal{K}\{u\}$ to the Liouville type extended ring $\mathcal{K}_n\{z(u)\}$, $n \in \mathbb{N}$, generated by elements

$$\{z(u), D_t z(u), ..., D_t^{n-1} z(u) \in \mathcal{K}\{u\} : D_t^n z(u) = 0, u \in \mathcal{K}\} \tag{1.4}$$

and to pose the following slightly modified inverse problem 1.1:

Problem 1.2. *To describe the possible linear finite dimensional representations of the derivations* $D_x, D_t : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}$, $n \in \mathbb{N}$, satisfying the related homomorphic differential-algebraic relationship (1.2) *in the functional vector spaces* $\mathcal{K}_n\{z(u)\}^{m+n}$ for some suitably chosen $m \in \mathbb{N}$.

It is worth to remark here that the ring $\mathcal{K}_n\{z(u)\}$ can be considered as a coordinate set of an infinite dimensional manifold, and the Lie algebra of its derivations as the Lie algebra of vector fields of this manifold. Moreover, the devised below scheme of constructing finite-dimensional endomorphic representations of the Lie-commutator relationship (1.1), being in much motivated by the previous results, is strongly based on the suitably constructed finite-dimensional differential ideals $I_{n,m}^{(\alpha)}\{z(u)\} \subset \mathcal{K}_n\{z(u)\}, u \in \mathcal{K}$, parameterized by real numbers $\alpha \in \mathbb{R}\backslash\{0\}$, and their invariance properties subject to the "*convective*" derivation $D_t : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}, N \in \mathbb{N}$.

In particular, at $n = \emptyset$, $m = 2$ and $\alpha = 1/2$ we have stated a theorem, describing the corresponding differential-algebraic constraint on a fixed function $u \in \mathcal{K}$ and providing the related linear finite-dimensional endomorphic representations $D_x \to l(u) : \mathcal{K}_1\{z(u)\}^2 \to \mathcal{K}_1\{z(u)\}^2, D_t \to p(u) : \mathcal{K}_1\{z(u)\}^2 \to \mathcal{K}_1\{z(u)\}^2$ of our derivations $D_t, D_x : \mathcal{K}_1\{z(u)\} \to \mathcal{K}_1\{z(u)\}$, satisfying the differential - matrix relationship (1.2), imitating that of (1.1) in the two-dimensional functional vector spaces $\mathcal{K}_1\{z(u)\}^2$, and which can be rewritten in the following classical matrix commutator form:

$$D_t l[u] + u_x l[u] = [p(u), l(u)] + D_x p(u). \tag{1.5}$$

Theorem 1.3. *Let a function $u \in \mathcal{K}$ satisfy the following differential relationship:*

$$D_x D_t u + (D_x u)^2 - D_x^3 u = 0. \tag{1.6}$$

Then the following linear matrix endomorphisms

$$l(u) = \begin{pmatrix} 0 & 1 \\ 1/2 D_x u & 0 \end{pmatrix}, \quad p(u) = \begin{pmatrix} 1/2 D_x u & 0 \\ 1/2 D_x^2 u & -1/2 D_x u \end{pmatrix} \tag{1.7}$$

on the functional vector spaces $\mathcal{K}_1\{z(u)\}^2$ solve the related differential-matrix relationship (1.2).

This result is naturally generalized on the case of arbitrary $n \in N$ and formulated as the next theorem.

Theorem 1.4. *The differential-algebraic relationship $D_t^n z(u) = 0, n \in \mathbb{N}$, is equivalent to the differential-matrix relationship* (**??**) *for any element $u \in \mathcal{K}$ and a parameter $\lambda \in \mathbb{R}$ with the homomorphic matrix represenations $D_t \to p(u;\lambda) \in$ End$(\mathcal{K}_n\{z(u)\}^n)$, $D_x \to l(u;\lambda) \in$ End$(\mathcal{K}_n\{z(u)\}^n)$ of the "convective" and "shifting" $D_t, D_x : \mathcal{K}_n\{z(u)\} \to \mathcal{K}_n\{z(u)\}$ derivations, respectively, given by the matrix expressions (1.8) and (1.9):*

$$p(u;\lambda) = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ -\lambda & 0 & ... & 0 & 0 \\ ... & -\lambda & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & -\lambda & 0 \end{pmatrix} \tag{1.8}$$

and

(1.9)

$$l(u;\lambda)=\begin{pmatrix} \lambda D_xD_t^{n-2}z(u) & D_xD_t^{n-1}z(u) & ... & 0 & 0\\ 0 & \lambda D_xD_t^{n-2}z(u) & ... & 0 & 0\\ ... & ... & ... & (n-2)D_xD_t^{n-1}z(u) & 0\\ 0 & 0 & ... & \lambda D_xD_t^{n-2}z(u) & (n-1)D_xD_t^{n-1}z(u)\\ -n\lambda^n & -n\lambda^{n-1}z(u) & ... & -n\lambda^2D_xD_t^{n-3}z(u) & \lambda(1-n)D_xD_t^{n-2}z(u) \end{pmatrix}.$$

We have also considered the Dubrovin's integrability classification [26, 27, 28] of a general evolution equation

$$\begin{aligned} u_t+f(u)u_x &= \varepsilon[f_{21}(u)u_{xx}+f_{22}(u)u_x^2]+\\ &+\varepsilon^2[f_{31}(u)u_{xxx}+f_{32}(u)u_xu_{xx}+f_{33}u_x^3]+...+\\ &+\varepsilon^{N-1}[f_{N,\sigma}(u)\prod_{m=\overline{1,N}}(u_{jx})^{k_j}+...]:=F_{N,\varepsilon}(u), \end{aligned} \tag{1.10}$$

with graded homogeneous polynomials of the jest-variables $\{u_x,u_{xx},...,u_{kx}...\}\in J^\infty(\mathbb{R};\mathbb{R})$, where $f'(u)\neq 0$ for arbitrary $u\in\mathcal{K}:=C^\infty(\mathbb{R};\mathbb{R})$, and consisting in describing the set $\mathcal{F}$ of smooth functions $f_{j,\sigma}(u)$, $\sigma:=\{k_j\in\mathbb{N}:\sum_{j=\overline{1,N}}jk_j=N\}$, with a fixed natural integer $N\in\mathbb{N}$, for which the equation (1.10) reduces by means of the following transformation

$$v\to u+\sum_{k\in\mathbb{N}}\varepsilon^k\eta_k(u,u_x,u_{xx},...,u^{(m_k)})\in\exp\left(\mathcal{A}_\varepsilon(u)\right) \tag{1.11}$$

with finite orders $m_k\in\mathbb{N}, k\in\mathbb{N}$, being applied to an arbitrary Riemann type symmetry flow

$$v_s+h(v)v_x=0 \tag{1.12}$$

with respect to an evolution parameter $s\in\mathbb{R}$, reduces to the form

$$u_s+h(u)u_x=\sum_{k\in\mathbb{N}}\varepsilon^kh_k(u,x,u_{xx},...,u^{(k)}):=H_\varepsilon(u)\in\mathcal{A}_\varepsilon(u), \tag{1.13}$$

where $\mathcal{A}_\varepsilon(u)$ is a specially defined differential ring, depending on a chosen element $u\in\mathcal{K}$ and a free parameter ε. Having reformulated the Dubrovin's integrability criterion within the corresponding differential-algebraic tools, based on the basic "convexity" derivations $D_s^{(v)}:=\partial/\partial s+v\partial/\partial x$ and $D_s^{(h(v))}:=\partial/\partial s+h(v)\partial/\partial x$ with the common field of constants $Z(v)$ for any $v\in\mathcal{K}$, we successfully rederived this criterion.

The obtained results in some sense generalize the ones, obtained before in the works [10, 11], devoted to proving the Lax type integrability [5, 23, 24, 25] of a generalized Riemann type hierarchy of differential-algebraic relationships $D_t^nu=0$ for $u\in\mathcal{K}$ and natural $n\in N$.

An interesting problem and not analyzed here is to construct nontrivial finite-dimensional homomorphic representations $D_t\to p(u;\lambda)\in\mathrm{End}(\mathcal{K}_n\{z(u)\}^n)$, $D_x\to l(u;\lambda)\in\mathrm{End}(\mathcal{K}_n\{z(u)\}^n)$ of the "*convective*" and "shifting" $D_t,D_x:\mathcal{K}_n\{z(u)\}\to\mathcal{K}_n\{z(u)\}$ derivations, respectively, generated by f the general ideal (**??**) in $\mathcal{K}_n\{z(u)\}$ for arbitrary $n,m\in\mathbb{N}$.

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