The Nonpolynomial Conservation Laws in Generalized Riemann Equation

A. K. Prikarpatsky, J. Golenia, O.D. Artemovych,
M. Pavlov, Z. Popowicz*

* Institute of Theoretical Physics, University of Wroclaw, pl. Maxa Borna 9
WROCŁAW

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Gas dynamic

$$
\begin{aligned}
u_t + uu_x + \frac{1}{\rho} P_x &= 0 \\
\rho_t + (\rho u)_x &= 0 \\
s_t + us_x &= 0
\end{aligned}
$$

where $u =>$ veliocity $\rho =>$ density,
$s =>$entropy , $P(\rho, s) =>$ pressure.

Examples

1.) Polytropic gas

$$P = \rho^{\gamma}, \qquad \gamma \neq 0, 1$$

2.) Chaplygin gas

$$P = \frac{-1}{\rho}$$

3.) Invisible Nondissipative Dark Matter of Universe
Gurevich - Zybin (1988)

$$
\begin{aligned}
P &= -\frac{1}{2}(\partial^{-1}\nu)^2 \\
u_t &= -uu_x + \partial^{-1}\nu \\
\nu_t &= -(u\nu)_x
\end{aligned}
$$

When $\nu = w_x$ then

$$
\begin{aligned}
u_t &= -uu_x + w \\
w_t &= -uw_x
\end{aligned}
$$

Riemann Equation

$$\begin{aligned} u_t &= -uu_x \\ \mathcal{D}_t u &= (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})u = 0 \end{aligned}$$

Generalized Riemann Equation

$$\mathcal{D}_t^N = 0, \qquad N = 1, 2, 3, \dots$$

For $N = 2$ we have Gurevich - Zybin equation because

$$\begin{aligned} \mathcal{D}^2 u &= \mathcal{D}(v = \mathcal{D}u) = 0 \\ v = \mathcal{D}u \Rightarrow \quad u_t &= v - uu_x \\ \mathcal{D}v = 0 \Rightarrow \quad v_t &= -uv_x \end{aligned}$$

while for $N = 4$

For $N = 4$ we have

$$\begin{aligned} u_t &= v - uu_x \\ v_t &= w - uv_x \\ w_t &= z_x - uw_x \\ z_t &= -uz_x \end{aligned}$$

$$\begin{aligned} N = 2 \qquad &\Rightarrow \qquad w = 0, z = 0 \\ N = 3 \qquad &\Rightarrow \qquad z = 0 \end{aligned}$$

What we can say on the integrability of these models?

The Hamiltonian structure for $N = 2$ have been investigated by A. Das, J. Brunelli (2004) and M. Pavlov (2004)

Hamiltonian structure, zero-curvature condition for $N = 2$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = P_1 \delta H_3 = P_2 \delta H_2 = P_3 \delta H_1$$

where $\delta H = (\frac{\delta H}{\delta u}, \frac{\delta H}{\delta v})^t$

$$P_1 = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} \partial^{-1} & -u_x \\ u_x & -\partial v - v\partial \end{pmatrix}$$

$$P_3 = P_2 P_1^{-1} P_2 = R P_2$$

$H_1 = \int dx v, \qquad H_2 = \int dx uv, \qquad H_3 = \frac{1}{2} \int dx [u^2 v + (\partial^{-1} v)^2]$

from which follows

$$\delta H_3 = P_1^{-1} P_2 \delta H_2 = R^T \delta H_2 = (R^T)^2 \delta H_1$$

R^T is recursion operator

Recursion operator

$$\delta H_{n+1} = (R^T)^n \delta H_1 \qquad n = 1, 2, \ldots$$

For example

$$H_4 = \int dx \Big[\tfrac{1}{6} u^3 v - uv(\partial^{-1} v) - \tfrac{1}{2} u(\partial^{-1})^2\Big]$$

For $n < 1$ we obtain the so called inverse hierarchy

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = P_1 \delta H_1 = P_2 \delta H_0 = 0$$

$$H_0 = 2 \int dx \Big(v - \tfrac{1}{2} u_x^2\Big)^{\frac{1}{2}}$$

is the Casimir function for P_2 and H_1 is the Casimir for P_1

$$\delta H_0 = P_2^{-1} P_1 \delta H_1 = R^{-1} \delta H_1$$

$$\delta H_{-n} = R^{-n} \delta H_1, \qquad n = 0, 1, 2, \ldots$$

$$H_{-2} = -\int dx u_{xx}(v - \tfrac{1}{2}u_x^2)^{\frac{1}{2}}$$

It is not the end of story because

$$\hat{H} = \int dx \left[u_{xx} v_{xx} - u_{xxx} v_x\right]^{\frac{1}{6}}$$

$$\hat{G} = \int \left[u_x(u_{xx} v - u_x v_x) + v_x v\right]^{\frac{1}{4}}$$

are conserved quantities which are not connected with the recursion operator.

Theorem

If

$$h_t = k u_x h + u h_x$$

where k is an arbitray number then

$$H = \int dx \quad h^{\frac{1}{k}}$$

is a conserved quantity

Theorem: If

$$h_{j,t} = \lambda(uh_j)_x, \qquad \forall j$$

where λ is an arbitray constant then

$$H_{i,j} = \sum_{n\in Z_+} \int dx h_i^{2^n} h_j^{1-2^n}$$

is a conserved quantity.

Method: Dimensional analysis and Computer Algebra
$[u] = 2, [v] = 5, [\partial_x] = 1, [\partial_t] = 3, [u_x] = 3, \ldots$

Anstatz:

$$h_n = Polynomial(u, v, u_x, v_x, ..)$$

Zero - Curvature Condition

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda u_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_t = \begin{pmatrix} -u\partial_x & 0 \\ -\lambda & -u\partial_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

$\Psi_{x,t} = \Psi_{t,x} \quad => \quad u_t = v - uu_x \quad v_t = -uv_x$

Let $\Phi = \Psi_2$

$$\begin{aligned} \Phi_{xx} &= (\lambda u_{xx} + 2\lambda^2(u_x^2 - v))\Phi \\ \Phi_t &= -(u + \frac{1}{2\lambda})\Phi_x + u_x\Phi \end{aligned}$$

It very well known spectral problem

$$\Phi_{xx} = (\lambda^2\eta^2 + \lambda u_{xx})\Phi$$
$$\Phi_t = a(\eta, u, \lambda)\Phi_x - \frac{1}{2}a_x\Phi$$

Two-component Harry-Dym equation (Fordy, Antonowicz)

$$a = \frac{2\lambda}{\eta}, \qquad s = u_{xx}$$
$$\eta_t = \partial_x(\frac{s}{\eta^2}), \qquad s_t = \partial^3(\frac{1}{\eta})$$

Two - component Hunter - Saxton equation

$$a = \frac{1}{2\lambda} - u,$$
$$\eta_t + \partial_x(\eta u) = 0$$
$$u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = \frac{1}{2}\eta^2$$

For $N = 3$ new problems

$$u_t = v - uu_x, \quad v_t = z - uv_x, \quad z_t = -uz_x$$

Conserved quantities (polynomial , nonpolynomial)

$$H_n = \int dx\; z^n(vu_x - v_x - \frac{n+2}{n+1}z), \quad G = \int dx\; (z^2 u_x - 2zvv_x)$$

$$H = \int dx\; (v_{xx}u_x - v_x u_{xx} - z_{xx})^{\frac{1}{3}}$$

$$H = \int dx\; (-2u_{xxx}u_x z_x + u_{xxx}v_x^2 + 6u_{xx}^2 z_x - 6u_{xx}u_x z_{xx} - 3u_{xx}v_{xx}v_x + 2u_x^2 z_{xxx} - u_x v_{xxx}v_x + 3u_x v_{xx}^2 + 3v_{xxx}z_x - 3v_x z_{xxx})^{\frac{1}{5}}$$

Zero - curvature condition

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}_x = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r & -3\lambda & \lambda u_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}_t = \begin{pmatrix} -u\partial_x & 0 & 0 \\ \lambda & -u\partial_x & 0 \\ 0 & 1 & -u\partial_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$$

$$\begin{aligned} r_t + (uv)_x &= 1, \qquad\qquad r = r_1 + r_0 \\ r_{0,t} + (ur_0)_x &= 0 \end{aligned}$$

$$r_0 = z_x v - z v_x, \quad r_0 = 2zz_x, \quad r_0 = (\frac{1}{2}v^2 - uz)_x$$

The method of solution

1.) Eliminate 1

$$\frac{\partial}{\partial t}(\frac{v_{nx}}{z_{nx}}) = 1 + ..., \qquad n = 0, 1, 2, \dots$$

2.) Dimensional analysis

$[u] = 2, [v] = 5, [z] = 8, [\partial_x] = 1, [\partial_t] = 3$

$$[\frac{v}{z}] = [\frac{vz}{z^2}] = \frac{[13]}{[16]} \qquad => \qquad [r] = \frac{[13]}{[16]} \approx \frac{58}{161}$$

then our equation reduces to **11 722** nonlinear algebriac equations only and it is impossible to use the CA and Groebner basis to solve it.

The simpler ansatz where [16] is without derivatives give us

$$r_1 = \frac{-u_x u + 3v}{3z} + \frac{-2u_x v^2 + 2v_x uv - z_x u^2}{6z^2}$$

The next ansatz $r = \frac{[9]}{[[12]]}$ where [[12]] denotes that only first derivatives can occurs produce

$$r_1 = \frac{2v_x - u_x^2}{2z_x}, \qquad r_1 = \frac{2u_x^3 - 6u_x v_x + 9z_x}{2u_x z_x - v_x^2}$$

Hamiltonian structure

$$\begin{pmatrix} u \\ v \\ v \end{pmatrix}_t = \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1} & \partial^{-1}z_x \\ 0 & z_x\partial^{-1} & 0 \end{pmatrix} \delta H$$

where $H = \int dx \ \ (u_x v - z)$

N=4

$$
\begin{aligned}
u_t &= v - uu_x, \qquad v_t = w - uv_x \\
w_t &= z - uw_x, \qquad z_t = -uz_x
\end{aligned}
$$

We have a lot of conserved quantities

$$
\begin{aligned}
H &= \int dx\,(vw_x - uz_x), \qquad H = \int dx\,(w^2 - 2vz) \\
H &= \int dx\,(w_x^2 - 2v_x z_x)^{\frac{1}{2}} \\
H &= \int dx\,(u_{xx}z_x - u_x z_{xx} + v_x w_{xx} - v_{xx} w_x)^{\frac{1}{3}}
\end{aligned}
$$

We have only zero - curvature condition

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}_x = \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}_t = \begin{pmatrix} -u\partial_x & 0 & 0 & 0 \\ \lambda & -u\partial_x & 0 & 0 \\ 0 & \lambda & -u\partial_x & 0 \\ 0 & 0 & \lambda & -u\partial_x \end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$$

where

$$r_{1,t} = -(r_1 u)_x + 1, \qquad r_{2,t} = -(r_2 u)_x + r_1$$

Solutions r_1, r_2

1.

$$r_1 = \frac{1}{10z}(-4u_x v + v_x u + 10w) + \frac{1}{10z^2}(-3u_x w^2 + 3v_x vw - w_x uw - w_x v^2 + z_x uv)$$

$$r_2 = \frac{1}{5z}(2v - u_x u) + \frac{1}{20z}(6v_x uv - 12u_x vw + 6v_x v^2 - 8w_x uv + 5z_x u^2 + 6w^2)$$

2.

$$r_1 = \frac{1}{5z_x}(5w_x - u_x v_x) + \frac{1}{5z_x}(v_x^2 w_x - u_x w_x)$$

$$r_2 = \frac{1}{5z_x}(9v_x - 5u_x^2) + \frac{1}{15z_x}(v_x^3 - 6w_x^2)$$