Extendable symplectic structures and the inverse problem of the calculus of variations for systems of equations in an extended Kovalevskaya form

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Symplectic structures of differential equations are similar to symplectic structures in Hamiltonian mechanics. They can be described in terms of variational analogs of differential 2-forms and in terms of equivalence classes of linear differential operators in total derivatives. Such operators are defined on infinite extensions of differential equations and they are allowed to be degenerate.

The well-known inverse problem of the calculus of variations can also be formulated in terms of linear differential operators in total derivatives. However, in this case such operators are defined on jets. Beside this, they must be nondegenerate in some sense. Fortunately, their restrictions to the corresponding differential equations are always related to symplectic structures.

Let $\xi \colon E \to X$ be a locally trivial smooth vector bundle over a smooth manifold X,

$$\dim X = n - 1\,, \quad \dim E = n - 1 + m\,.$$

Denote $M = X \times \mathbb{R}$, $N = E \times \mathbb{R}$. Consider the projection to the first multiplier

$$pr_X \colon X \times \mathbb{R} \to X$$

and denote

$$\pi = {pr_X}^*(\xi)\,.$$

Let $\varkappa(\pi) = \Gamma(\pi^*_\infty(\pi))$ be the module of sections of the corresponding induced bundle.

Choose local coordinates $x^1, \ldots, x^{n-1}$ on X and $u^1, \ldots, u^m$ along the fibers of ξ. Choose a global coordinate $t = x^n$ on $\mathbb{R}$. Then $x^1, \ldots, x^n$ are local coordinates on M.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. Here all α^i are non-negative integers. Denote by αx the formal sum:

$$\alpha x = \alpha_1 x^1 + \ldots + \alpha_n x^n = \alpha_i x^i \,.$$

Put $|\alpha| = \sum_i \alpha_i$,

$$D_{\alpha x} = {D_{x^1}}^{\alpha_1} \circ \ldots \circ {D_{x^n}}^{\alpha_n}, \quad u^i_{\alpha x} = D_{\alpha x}(u^i).$$

Here D_{x^i} are operators of total derivatives, D_0 is the identity operator. Further we consider only adapted local coordinates on $J^\infty(\pi)$, i.e. local coordinates of the form $x^1, ..., x^{n-1}, t = x^n, u^j_{\alpha x}$.

Let b be a vector $b = (b_1, \dots, b_m)$, where all b_i are positive integers. Further we assume that either $b_1 = \dots = b_m$, or a bundle ξ has a trivial structure group.

Let $\varphi \in \varkappa(\pi)$ be such a section, that its components φ^i depend on x^j, t and coordinates of the form $u^k_{\alpha x}$, for which holds $\alpha_n < b_k$. Then a system of equations

$$u^1_{b_1 t} = \varphi^1\,,$$
$$\dots\,,$$
$$u^m_{b_m t} = \varphi^m$$

is written in an extended Kovalevskaya form. We assume that section $F = u_{bt} - \varphi$ determines smooth submanifold $\{F = 0\} \subset J^K(\pi)$, where K is a maximal order of derivatives in F.

Let $\mathcal{E}$ be the infinite extension of the system of equations $u_{bt} = \varphi$

$$\mathcal{E}\colon\ D_{\alpha x}(u^i_{b_i t}) = D_{\alpha x}(\varphi^i) \qquad \text{for all} \ \ \alpha \text{ and } i = 1, \ldots, m\,. \tag{1}$$

Further we consider only systems of equations of the form (1).

By external with respect to $\mathcal{E}$ coordinates on $J^\infty(\pi)$ we will mean local coordinates on the left-hand side of equations from $\mathcal{E}$. Other local coordinates on $J^\infty(\pi)$ are internal with respect to $\mathcal{E}$.

Consider a projection $pr_M \colon M \times \mathbb{R} \to M$ and the induced bundle $\pi' = pr_M^*(\pi)$. Denote $M' = M \times \mathbb{R}$, $N' = N \times \mathbb{R}$. Here $\dim M' = n + 1$. Let a be a global coordinate on $\mathbb{R}$.

Denote k-jet of a section $h' \in \Gamma(\pi')$ at a point $(x, a) \in M \times \mathbb{R}$ by $[h']^k_{(x,a)}$. Define maps

$$f_k \colon J^k(\pi') \to J^k(\pi), \qquad g_k \colon {\pi'}_k^*(N') \to \pi_k^*(N)$$

in the following way:

$$f_k([h']^k_{(x,a)}) = [pr_N \circ h' \circ 0_M]^k_x, \qquad g_k(p', q') = (f_k(p'), pr_N(q')).$$

Here 0_M is a zero-section of the bundle pr_M, $(p', q') \in {\pi'}_k^*(N') \subset J^k(\pi') \times N'$.

Put

$$f = f_\infty, \qquad g = g_\infty.$$

Define a map $f' \colon \varkappa(\pi) \to \varkappa(\pi')$ in the following way: for each section $v \in \varkappa(\pi)$ there exists a unique section $v' \in \varkappa(\pi')$, such that

$$v \circ f = g \circ v'.$$

Then put $f'(v) = v'$.

Denote $f'(F)$ by F', an infinite extension of a system of equations $f'(u_{bt}) = f'(\varphi)$ by $\mathcal{E}'$.

Remark 1. A system of equations $f'(u_{bt}) = f'(\varphi)$ also has an extended Kovalevskaya form.

Since $D_a(F') = l_{F'}(u'_a)$, then a system of equations $\mathcal{E}'$ is similar to the tangent covering of $\mathcal{E}$. Here coordinates u'^i_a play role of fiber-wise coordinates q^i from the tangent covering

$$F = 0\,, \qquad l_F(q) = 0\,.$$

Denote by f_Λ a homomorphism from $\Lambda_h^n(\pi)$ to $\Lambda_h^{n+1}(\pi')$, which is induced by the map

$$\omega \mapsto da \wedge f^*(\omega)\,.$$

Let $A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ be a $\mathcal{C}$-differential operator. Denote by A' a unique $\mathcal{C}$-differential operator $A'\colon \varkappa(\pi') \to \widehat{\varkappa}(\pi')$, such that for any $\lambda, \mu \in \varkappa(\pi)$ holds

$$f_\Lambda(\langle A(\lambda), \mu\rangle) = \langle A'(f'(\lambda)), f'(\mu)\rangle\,.$$

Here $\langle\cdot,\cdot\rangle$ is the natural pairing between the module and its adjoint. Then $A'^* = (A^*)'$.

For any $G \in \widehat{\varkappa}(\pi)$ also define $G' \in \widehat{\varkappa}(\pi')$ from the identity

$$f_\Lambda(\langle G, \lambda\rangle) = \langle G', f'(\lambda)\rangle\,.$$

Further we use notations x'^i, a, u'^j for local coordinates on N'.

Functions u'^i_a determine a section $u'_a \in \varkappa(\pi')$. In local coordinates functions φ^i and $\varphi'^i = f'(\varphi)^i$ are identically equal up to primes of arguments.

In local coordinates for a $\mathcal{C}$-differential operator $A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ and any $\lambda, \mu \in \varkappa(\pi)$ holds

$$\langle A(\lambda), \mu\rangle = A^{\alpha}_{ij} D_{\alpha x}(\lambda^i)\mu^j dx^1 \wedge \ldots \wedge dx^n .$$

Then for A' and any $\lambda_1, \mu_1 \in \varkappa(\pi')$ holds

$$\langle A'(\lambda_1), \mu_1\rangle = A'^{\alpha}_{ij} D_{\alpha x'}(\lambda^i_1)\mu^j_1 \, da \wedge dx'^1 \wedge \ldots \wedge dx'^n .$$

Here for all i, j, α corresponding functions A^{α}_{ij} and A'^{α}_{ij} are identically equal up to primes of arguments.

Let $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$ be a $\mathcal{C}$-differential operator, such that

$$\Delta^* \circ l_{\mathcal{E}} = l_{\mathcal{E}}^* \circ \Delta\,.$$

One can consider an equivalence class of Δ modulo operators of the form $\square \circ l_{\mathcal{E}}$, where $\square = \square^*$.

A group of symplectic structures of a system of equations $\mathcal{E}$ consists of equivalence classes of operators, which are related to closed variational 2-forms.

Definition 1. A $\mathcal{C}$-differential operator $A^*\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ is said to be a variational operator for a system of equations $F = 0$ (or, equivalently, for its infinite extension $\mathcal{E}$), if for some Lagrangian L holds $A^*(F) = \mathbf{E}(L)$.

By a variational extension of an operator $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$ we will mean such an operator $A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$, that $A|_{\mathcal{E}} = \Delta$ and A^* is a variational operator for $\mathcal{E}$.

Definition 2. A symplectic structure of $\mathcal{E}$ is extendable to jets (or just extendable), if it is generated by some operator Δ, which admits some variational extension A.

An operator A from this definition is also an extension of the corresponding symplectic structure.

We will say, that an extension A of a symplectic structure of $\mathcal{E}$ (or a variational operator A^*) is nondegenerate, if the system of equations $A^*(F) = 0$ has the differential consequence $F = 0$. Then for some $\mathcal{C}$-differential operator $B\colon \widehat{\varkappa}(\pi) \to \varkappa(\pi)$ holds $F = B(A^*(F))$.

Example 1. Consider a one-dimensional shallow-water equations over uneven bottom in Euler's variables (SWE)

$$
\begin{aligned}
u_t + uu_x + \rho_x &= h'(x)\,, \\
\rho_t + u_x\rho + u\rho_x &= 0\,.
\end{aligned}
$$

One can introduce a nonlocal variable w by the relations

$$
\begin{aligned}
w_x &= -u\,, \\
w_t &= \frac{u^2}{2} + \rho - h(x)
\end{aligned}
$$

and consider the equation for potential $w = w(x,t)$ (SWP)

$$
w_{tt} - 2w_x w_{xt} + \Big(\frac{3}{2}w_x^2 - w_t - h(x)\Big)w_{xx} - h'(x)w_x = 0\,.
$$

It admits a variational operator $A_P^* = 1$.

One can also introduce a Lagrange's mass variable m by the relations

$$
\begin{aligned}
m_x &= \rho\,, \\
m_t &= -u\rho
\end{aligned}
$$

and consider the shallow water equation in Lagrange's variables for $m = m(x, t)$ (SWL)

$$
m_{tt} - \frac{2m_t m_{tx}}{m_x} + \Big(\frac{m_t^2}{m_x^2} - m_x\Big) m_{xx} + h'(x) m_x = 0\,.
$$

This equation admits a variational operator $A_L^* = m_x^{-1}$.

Using both mentioned conservation laws, we obtain a Bäcklund transformation (SWPL) of the equations SWP and SWL:

$$
\begin{aligned}
m_t &= m_x w_x\,,\\
w_t &= \frac{w_x^2}{2} + m_x - h(x)\,.
\end{aligned}
$$

The lifts of the obtained symplectic structures to SWPL coincide and are generated by the operator

$$
\Delta_{PL} = \begin{pmatrix} 0 & -\bar{D}_x \\ -\bar{D}_x & 0 \end{pmatrix}.
$$

Then consider the operator

$$
A_{PL} = \begin{pmatrix} 0 & -D_x \\ -D_x & 0 \end{pmatrix}.
$$

The operator A_{PL}^* is a (degenerate) variational operator for SWPL.

Example 2. Consider a trivial bundle $\pi\colon \mathbb{R}^5 \to \mathbb{R}^2$. Let x and t be global coordinates on a plane $\mathbb{R}^2$, u, v and w be coordinates along the fibers of π over $\mathbb{R}^2$.

Let $\mathcal{L}$ be a function, which depends on a finite number of arguments of the form $x, t, u, u_x, u_{xx}, u_{xxx}, \dots$ Denote $\mathbf{E}(\mathcal{L})$ by h, then $l_h = l_h^*$. Consider the following system of equations in an extended Kovalevskaya form

$$\begin{aligned} u_{tt} &= w, \\ v_{tt} &= u_{xx} - v_{xx} + h, \\ w_{tt} &= v_{xx} - h. \end{aligned}$$

Direct calculation shows, that the operator

$$\Delta = \begin{pmatrix} \overline{D}_x^2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfies the relation

$$\Delta^* \circ l_{\mathcal{E}} = l_{\mathcal{E}}^* \circ \Delta \, .$$

Consider the operator

$$A^* = \begin{pmatrix} D_x^2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

The operator A^* is a variational operator for the system of equations under consideration. The corresponding Lagrangian for $A^*(F)$ has the form

$$L = \mathcal{L} + \frac{u_{xt}^2}{2} + u_x v_x + u_x w_x - u_t w_t - \frac{v_t^2}{2} - v_t w_t - \frac{w_t^2}{2} - \frac{w^2}{2}.$$

Therefore Δ generates an extendable symplectic structure.

The operator A^* is a two-sided invertible $\mathcal{C}$-differential operator, which inversion is the operator

$$(A^*)^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 - D_x^2 & D_x^2 \\ 1 & D_x^2 & -D_x^2 \end{pmatrix}$$

Thus, system of equations $A^*(F) = 0$ has the consequence $F = 0$ and Δ generates an extendable symplectic structure with nondegenerate extension.

Define an operation

$$s_{\mathcal{E}'} : \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \widehat{\varkappa}(\pi)) \to \mathcal{C}\mathrm{Diff}(\varkappa(\pi'), \widehat{\varkappa}(\pi'))$$

as follows. Let $A \colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ be a $\mathcal{C}$-differential operator. Components of

$$A'(u'_a)|_{\mathcal{E}'}$$

can be considered as components of a unique element $\sigma \in \widehat{\varkappa}(\pi')$. Since components of σ are linear in total derivatives of u'^i_a, then for a unique $\mathcal{C}$-differential operator $\widetilde{\sigma} \colon \varkappa(\pi') \to \widehat{\varkappa}(\pi')$ holds $\sigma = \widetilde{\sigma}(u'_a)$. Put

$$s_{\mathcal{E}'}(A) = \widetilde{\sigma}\,.$$

Secondly define an operation

$$S_{\mathcal{E}}\colon \mathcal{C}\mathrm{Diff}(\varkappa(\pi),\widehat{\varkappa}(\pi)) \to \mathcal{C}\mathrm{Diff}(\varkappa(\pi),\widehat{\varkappa}(\pi))$$

as follows. Using local coordinates, it's easy to verify, that for a $\mathcal{C}$-differential operator $A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ there exists a unique $\mathcal{C}$-differential operator $B\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ of the form

$$\langle B(\lambda),\mu\rangle = B^{\alpha}_{ij} D_{\alpha x}(\lambda^i)\mu^j dx^1\wedge\ldots\wedge dx^n, \qquad B^{\alpha}_{ij} \neq 0 \text{ only if } \alpha_n < b_i\,,$$

such that holds

$$B' = s_{\mathcal{E}'}(A)\,.$$

Here components B^{α}_{ij} depend on internal with respect to $\mathcal{E}$ coordinates only. Put

$$S_{\mathcal{E}}(A) = B\,.$$

Therefore, $(S_{\mathcal{E}}(A))' = s_{\mathcal{E}'}(A)$.

Remark 2. For some $\mathcal{C}$-differential operators $B_1 \in \varkappa(\pi) \to \widehat{\varkappa}(\pi)$, $B_2 \colon \varkappa(\pi) \to \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \widehat{\varkappa}(\pi))$ holds

$$A'(u'_a) - s_{\mathcal{E}'}(A)(u'_a) = B'_1(D_a(F')) + B'_2(F')(u'_a)\,.$$

Since

$$D_a(F') = l_{F'}(u'_a)\,,$$

then also holds the relation

$$A - S_{\mathcal{E}}(A) = B_1 \circ l_F + B_2(F)\,.$$

Hence

$$A|_{\mathcal{E}} - S_{\mathcal{E}}(A)|_{\mathcal{E}} = B_1|_{\mathcal{E}} \circ l_{\mathcal{E}}\,.$$

Thirdly define an operation of a naive extension. Denote it by

$$e_J \colon \mathcal{C}\mathrm{Diff}(\varkappa(\mathcal{E}), \widehat{\varkappa}(\mathcal{E})) \to \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \widehat{\varkappa}(\pi))\,.$$

Let G be an element of the module $\widehat{\varkappa}(\pi)$, then holds a criterion

$$l_G = l_G^* \quad \Leftrightarrow \quad \exists\, \omega \in \Lambda_h^n(\pi')\colon\ \langle G', u_a' \rangle = d_h \omega \,. \qquad (2)$$

For any $\mathcal{C}$-differential operator $A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ holds

$$(A^*(F))' = A'^*(F')\,.$$

Then (2) can be generalized in the following way: put $G = A^*(F)$ in (2); using Green formula, one can obtain, that

$$l_{A^*(F)} = l_{A^*(F)}^* \quad \Leftrightarrow \quad \exists\, \omega \in \Lambda_h^n(\pi')\colon\ \langle A'(u_a'), F' \rangle = d_h \omega$$

The criterion (2) can be rewritten in the form

$$G \in \operatorname{Im} \mathbf{E} \quad \Leftrightarrow \quad \exists\, \omega \in \Lambda_h^n(\pi') \colon \ \langle u_a', G' \rangle = d_h \omega \,.$$

It is related to a Noether identity on $J^\infty(\pi)$

$$\langle q, \mathbf{E}(L) \rangle = E_q(L) + d_h w \,.$$

Here E_q is an evolution vector field, w is some horizontal $(n-1)$-form.

Recall, that if a conservation law of the system of equations $\mathcal{E}'$ has some global characteristic, then it also has a global characteristic, which components depend on internal with respect to $\mathcal{E}'$ coordinates only, therefore

$$l_{A^*(F)} = l^*_{A^*(F)} \quad \Rightarrow \quad \exists\, \omega \in \Lambda^n_h(\pi')\colon\ \langle s_{\mathcal{E}'}(A)(u'_a), F' \rangle = d_h\omega \quad \Rightarrow$$
$$\Rightarrow \ \ l_{S_{\mathcal{E}}(A)^*(F)} = l^*_{S_{\mathcal{E}}(A)^*(F)} \,.$$

Thus, we obtain the following theorem.

Theorem 1. *If A^* is a variational operator for a system of equations $\mathcal{E}$ in an extended Kovalevskaya form, then $S_{\mathcal{E}}(A)^*$ is also a variational operator for $\mathcal{E}$.*

Corollary 1. *A $\mathcal{C}$-differential operator $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$ of the form*

$$\langle \Delta(\lambda), \mu \rangle = \Delta^{\alpha}_{ij} \overline{D}_{\alpha x}(\lambda^{i}) \mu^{j} dx^{1} \wedge \ldots \wedge dx^{n}, \qquad \Delta^{\alpha}_{ij} \neq 0 \text{ only if } \alpha_n < b_i \tag{3}$$

generates an extendable symplectic structure of $\mathcal{E}$ if and only if it satisfies the system of equations

$$l_{e_J(\Delta)^*(F)} = l^{*}_{e_J(\Delta)^*(F)} \tag{4}$$

Corollary 2. *Each extendable symplectic structure of a system of equations $\mathcal{E}$ in an extended Kovalevskaya form is generated by some operator of the form (3).*

The following proposition shows that one can obtain extensions of a trivial symplectic structure, using Lagrangians of a special form.

Proposition 1. *Let* $\mathcal{E}$ be a system of equations in an extended Kovalevskaya form and $\nabla\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ be a $\mathcal{C}$-differential operator. Then for some extension $\widetilde{A}$ of a trivial symplectic structure holds

$$\widetilde{A}^*(F) = \mathbf{E}(\langle\nabla(F), F\rangle)\,.$$

However, in the context of the inverse problem of the calculus of variations it's important to find nondegenerate extensions of extendable symplectic structures (if it's possible).

Let A^* be a nondegenerate variational operator for a system of equations $F = 0$. Denote

$$\Delta = S_{\mathcal{E}}(A)|_{\mathcal{E}} \, .$$

For some $\mathcal{C}$-differential operator $B \colon \widehat{\varkappa}(\pi) \to \varkappa(\pi)$ holds

$$F = B(A^*(F))$$

and hence

$$l_{\mathcal{E}} = B|_{\mathcal{E}} \circ A^*|_{\mathcal{E}} \circ l_{\mathcal{E}} \, .$$

System of equations $\mathcal{E}$ is l-normal, then we obtain the relation

$$B|_{\mathcal{E}} \circ A^*|_{\mathcal{E}} = \mathrm{id}_{\varkappa(\mathcal{E})} \, . \tag{5}$$

Since the operator $A|_{\mathcal{E}}$ generates a symplectic structure for $\mathcal{E}$, then there exists some self-adjoint $\mathcal{C}$-differential operator $\Box$, such that holds $A|_{\mathcal{E}} = \Delta + \Box \circ l_{\mathcal{E}}$.

Thus, we obtain the following relation for $\nabla = B|_{\mathcal{E}}$

$$\nabla \circ (\Delta^* + l^*_{\mathcal{E}} \circ \square) = \mathrm{id}_{\varkappa(\mathcal{E})} \tag{6}$$

and hence

$$\nabla \circ (\Delta^* + l^*_{\mathcal{E}} \circ \square) \circ l_{\mathcal{E}} \circ \nabla^* = l_{\mathcal{E}} \circ \nabla^* .$$

Operator on the left-hand side is self-adjoint, then $l_{\mathcal{E}} \circ \nabla^* = \nabla \circ l^*_{\mathcal{E}}$. From (6) it also follows that

$$(\Delta + \square \circ l_{\mathcal{E}}) \circ \nabla^* = \mathrm{id}_{\widehat{\varkappa}(\mathcal{E})} ,$$

then

$$\Delta \circ \nabla^* + \square \circ \nabla \circ l^*_{\mathcal{E}} = \mathrm{id}_{\widehat{\varkappa}(\mathcal{E})} . \tag{7}$$

The lift of the relation (7) to the cotangent covering becomes

$$\widetilde{\Delta}(\widetilde{\nabla}^*(p)) = p .$$

Components of the homomorphism $\widetilde{\nabla}^*(p)$ are linear in p and its derivatives. Then for some $\mathcal{C}$-differential operator $\zeta\colon \widehat{\varkappa}(\mathcal{E}) \to \varkappa(\mathcal{E})$ of the form

$$\langle \chi, \zeta(\psi) \rangle = \zeta^{\alpha\, ij} \overline{D}_{\alpha x}(\psi_i)\chi_j \, dx^1 \wedge \ldots \wedge dx^n, \quad \zeta^{\alpha\, ij} \neq 0 \text{ only if } \alpha_n < b_i$$

holds $\widetilde{\nabla}^*(p) = \widetilde{\zeta}(p)$ and hence

$$\widetilde{\Delta}(\widetilde{\zeta}(p)) = p \,. \tag{8}$$

Thus, we obtained a necessary condition (8) for a symplectic structure to have a nondegenerate extension.

Now assume additionally, that the initial system of equations

$$F = 0$$

is a system of evolution equations. Then the relation $\widetilde{\Delta}(\widetilde{\zeta}(p)) = p$ implies the relation

$$e_J(\Delta) \circ e_J(\zeta) = \mathrm{id}_{\widehat{\varkappa}(\pi)}$$

and hence

$$e_J(\zeta^*) \circ e_J(\Delta^*) = \mathrm{id}_{\varkappa(\pi)} \,. \qquad (9)$$

Since Δ is a skew-adjoint operator: $\Delta^* = -\Delta$, then one can rewrite the relation (9) in the form

$$e_J(-\zeta^*) \circ e_J(\Delta) = \mathrm{id}_{\varkappa(\pi)} \,.$$

Therefore, we obtain the following theorem

Theorem 2. *Let* A^* be a nondegenerate variational operator for a system of evolution equations $\mathcal{E}$. Then $S_{\mathcal{E}}(A)^*$ is a two-sided invertible variational operator for $\mathcal{E}$ with a $\mathcal{C}$-differential inversion.

Thus, symplectic structures with nondegenerate extensions for a system of evolution equations $\mathcal{E}$ are completely characterized by two-sided invertible $\mathcal{C}$-differential operators of the form

$$\langle \Delta(\lambda), \mu \rangle = \Delta_{ij}^{\alpha} \overline{D}_{\alpha x}(\lambda^i) \mu^j dx^1 \wedge \ldots \wedge dx^n, \qquad \Delta_{ij}^{\alpha} \neq 0 \text{ only if } \alpha_n < b_i$$

which satisfy the system of equations

$$l_{e_J(\Delta)^*(F)} = l^*_{e_J(\Delta)^*(F)} \,.$$

If $F=0$ is a scalar equation in an extended Kovalevskaya form, then for composition of any $\mathcal{C}$-differential operators $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$ and $\nabla\colon \widehat{\varkappa}(\mathcal{E}) \to \varkappa(\mathcal{E})$ holds

$$\mathrm{ord}\,(\Delta \circ \nabla) = \mathrm{ord}\,(\Delta) + \mathrm{ord}\,(\nabla)\,.$$

Hence, in this case we obtain from the relation

$$B|_{\mathcal{E}} \circ A^*|_{\mathcal{E}} = \mathrm{id}_{\varkappa(\mathcal{E})}\,,$$

that for each nondegenerate variational operator A^* holds

$$\mathrm{ord}\,A^*|_{\mathcal{E}} = 0\,.$$

Therefore, $S_{\mathcal{E}}(A)^* = e_J(A|_{\mathcal{E}})^*$ is a zero order nondegenerate variational operator for $\mathcal{E}$ (i.e. variational multiplier). Here a component of $S_{\mathcal{E}}(A)^*$ depends on internal with respect to $\mathcal{E}$ coordinates only.

Thus, if a scalar equation in an extended Kovalevskaya form admits a variational formulation, than b is even number. In this case each symplectic structure with nondegenerate extension has a zero order nondegenerate extension.

Application of the result of M. Alonso to prime systems led to a complete description of extendable symplectic structures for systems of equations in an extended Kovalevskaya form. Beside this, the obtained results allowed to describe all extendable symplectic structures with nondegenerate extensions for systems of evolution equations. The obtained description is related to the problem of constructing for a given $\mathcal{C}$-differential operator its $\mathcal{C}$-differential inversion. However, in general a calculation of extendable symplectic structures of all orders for a system of equations in an extended Kovalevskaya form is a non-trivial problem by itself.

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