Noether theorem for diffieties.

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INTRODUCTION

Noether theorem plays a fundamental role in mathematical physics. In the formulation [Vinogradov] (see also [Olver]) it gives a correspondence between characteristics of Noether (variational) symmetries and characteristics of conservation laws of a system of Euler-Lagrange equations. However, the statement of Noether theorem in this form does not allow answering the question of whether there is a similar correspondence for differential coverings over a system of Euler-Lagrange equations. The following question is: in what form a similar correspondence can be described for an isomorphic (reversible) covering.

1. BASIC NOTATIONS

Let $\pi\colon N_\pi \to M$ and $\xi\colon N_\xi \to M$ be a locally trivial smooth vector bundles over a smooth manifold M, $\dim M = n$, $\dim N_\pi = n + m$, $\dim N_\xi = n + m$.

Consider k-jet bundles $\pi_k\colon J^k(\pi) \to M$ and infinite-jet bundle $\pi_\infty\colon J^\infty(\pi) \to M$.

Denote by $\mathcal{F}(\pi)$ the algebra of smooth functions on $J^\infty(\pi)$. Let $\varkappa(\pi) = \Gamma(\pi_\infty^*(\pi))$ and $P(\pi) = \Gamma(\pi_\infty^*(\xi))$ be $\mathcal{F}(\pi)$-modules of sections of the corresponding induced bundles.

Let $U \subset M$ be a coordinate neighbourhood such that the bundle π becomes trivial over U. Choose local coordinates x^i in U and u^j – corresponding coordinates along the fibres of π over U. We introduce multi-index α as a sum $\alpha = \alpha_i x^i$, where $\alpha_i \in \mathbb{N} \cup \{0\}$. Put $|\alpha| = \sum_i \alpha_i$,

$$D_\alpha = D_{x^1}{}^{\alpha_1} \circ \ldots \circ D_{x^n}{}^{\alpha_n}, \quad u^i_\alpha = D_\alpha(u^i),$$
$$\theta^i_\alpha = du^i_\alpha - u^i_{\alpha + x^k} dx^k, \quad \mathbf{dx} = dx^1 \wedge \cdots \wedge dx^n\,,$$

where D_{x^i} are operators of total derivatives, D_0 is the identity operator.

Further we consider only adopted local coordinates on $J^\infty(\pi)$, i.e. local coordinates of the form x^i, u^j_α.

Consider the ideal $\mathcal{C}\Lambda^*(\pi)$ of the algebra $\Lambda^*(\pi)$ consisting of the Cartan forms (i.e., differential forms vanishing on the Cartan distribution). In local coordinates each Cartan form $\omega \in \mathcal{C}\Lambda^*(\pi)$ is a finite sum of the form $\omega = \theta^i_\alpha \wedge \omega^\alpha_i$, where $\omega^\alpha_i \in \Lambda^*(\pi)$.

Let $\mathcal{C}^p\Lambda^*(\pi)$ be a p-th power of the ideal $\mathcal{C}\Lambda^*(\pi)$. Consider $\Lambda^k_h(\pi) = \Lambda^k(\pi)/\mathcal{C}\Lambda^k(\pi)$ – module of horizontal k-forms on $J^\infty(\pi)$.

Groups $E_1^{p,q}(\pi)$ of the $\mathcal{C}$-spectral sequence can be described as follows. Each element of the group $E_1^{p,q}(\pi)$ is generated by a differential $(p+q)$-form $\omega \in \mathcal{C}^p\Lambda^{p+q}(\pi)$, such that $d\omega \in \mathcal{C}^{p+1}\Lambda^{p+q+1}(\pi)$. Then

$$\omega + \mathcal{C}^{p+1}\Lambda^{p+q}(\pi) + d(\mathcal{C}^p\Lambda^{p+q-1}(\pi)) \in E_1^{p,q}(\pi)\,.$$

The exterior differential induces homomorphisms $d_1^{p,q}\colon E_1^{p,q}(\pi) \to E_1^{p+1,q}(\pi)$.

Let $[W]_h \in \Lambda_h^k(\pi)$ denote the horizontal class of a differential form $W \in \Lambda^k(\pi)$ and $[W]_c \in E_1^{0,k}(\pi)$ denote its horizontal cohomology class. We will also use the notation $X(W)$ for the Lie derivative of a differential form $W \in \Lambda^*(\pi)$ with respect to a vector field X,

$$X(W) = i_X(dW) + d(i_X W)\,.$$

Here $i_X W$ is the interior product of X and W.

Consider the adjoint modules

$$\widehat{P}(\pi) = \mathrm{Hom}_{\mathcal{F}(\pi)}(P(\pi), \Lambda_h^n(\pi)), \qquad \widehat{\varkappa}(\pi) = \mathrm{Hom}_{\mathcal{F}(\pi)}(\varkappa(\pi), \Lambda_h^n(\pi))$$

and denote by $\mathcal{C}\mathrm{Diff}(I_1(\pi), I_2(\pi))$ a module of $\mathcal{C}$-differential operators from $I_1(\pi)$ to $I_2(\pi)$. In local coordinates $\mathcal{C}$-differential operators are linear operators in total derivatives.

Denote by $\ni_\varphi$ (or by $\boldsymbol{E}_\varphi$) an evolutionary derivation, corresponding to a section $\varphi \in \varkappa(\pi)$. Then in local coordinates for $\varphi = (\varphi^1, \ldots, \varphi^m)$

$$\ni_\varphi = D_\alpha(\varphi^j) \frac{\partial}{\partial u^j_\alpha} .$$

Consider l_F – linearization of a section $F \in \Gamma(\pi^*_K(\xi))$. Here $l_F(\varphi) = \ni_\varphi(F)$. Denote by l^*_F its adjoint operator. In local coordinates for arbitrary $\varphi \in \varkappa(\pi)$, $\psi \in P(\pi)$

$$l_{F\,ij}(\varphi^j) = \frac{\partial F_i}{\partial u^j_\alpha} D_\alpha(\varphi^j) , \qquad l^*_{F\,ij}(\psi^j) = (-1)^{|\alpha|} D_\alpha \Big(\frac{\partial F_j}{\partial u^i_\alpha} \psi^j \Big) .$$

Let $F \in \Gamma(\pi_K^*(\xi))$, $F = 0$ be a system of differential equations, where $\{F = 0\} \subset J^K(\pi)$ is a smooth submanifold. Let

$$\mathcal{E}\colon\ D_\alpha(F_i) = 0 \qquad \text{for all multi-indices } \alpha \text{ and } i = 1, \dots, m$$

be the infinite extension of the system of equations $F = 0$.

By regular system of equations we understand system of equations $\mathcal{E}$ such that
(1) $\mathcal{E}$ has no nontrivial consequences of zeroth order;
(2) a function $f \in \mathcal{F}(\pi)$ vanishes on $\mathcal{E}$, $f|_{\mathcal{E}} = 0$, if and only if $f = \Delta(F)$ for some $\mathcal{C}$-differential operator $\Delta\colon P(\pi) \to \mathcal{F}(\pi)$.

Put $l_{\mathcal{E}} = l_F|_{\mathcal{E}}$. From now on we consider only regular systems of equations.

Denote by $\mathbf{E}$ the Euler operator. Then $\mathbf{E}([L]_h) = d_1^{0,n}[L]_c$, $\mathbf{E}([L]_h) \in E_1^{1,n}(\pi)$. If $L \in \Lambda^n(\pi)$ is of the form $\mathtt{L}\,\mathbf{dx}$, then

$$\mathbf{E}([L]_h) = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial \mathtt{L}}{\partial u^i_\alpha}\Big)\theta^i_0 \wedge \mathbf{dx} + \mathcal{C}^2\Lambda^{n+1}(\pi) + d(\mathcal{C}\Lambda^n(\pi))\,.$$

Denote by $P(\mathcal{E})$ and $\varkappa(\mathcal{E})$ the restrictions of the modules $P(\pi)$ and $\varkappa(\pi)$ to the system of equations respectively. Denote by $\mathcal{F}(\mathcal{E})$ and $\Lambda^*(\mathcal{E})$ the restrictions of the algebras $\mathcal{F}(\pi)$ and $\Lambda^*(\pi)$ to the system of equations $\mathcal{E}$ respectively. Let $E_1^{p,q}(\mathcal{E})$ be the corresponding groups of the $\mathcal{C}$-spectral sequence for the system of equations $\mathcal{E}$.

Denote by $\mathcal{C}D(\mathcal{E})$ Cartan vector fields on $\mathcal{E}$ (i.e. vector fields of the form $\eta^i \bar{D}_{x^i}$, $\bar{D}_{x^i} = D_{x^i}|_{\mathcal{E}}$). Let $D_C(\mathcal{E})$ denote the Lie algebra of vector fields X, such that $[X, \mathcal{C}D(\mathcal{E})] \subset \mathcal{C}D(\mathcal{E})$ or

$$D_C(\mathcal{E}) = \{X \in D\mathcal{F}(\mathcal{E}) : \; X(\mathcal{C}\Lambda^*(\mathcal{E})) \subset \mathcal{C}\Lambda^*(\mathcal{E})\} \,.$$

Then $\mathcal{C}D(\mathcal{E})$ is an ideal of the Lie algebra $D_C(\mathcal{E})$.

By infinitesimal symmetries of the system of equations $\mathcal{E}$ we understand elements of the Lie quotient algebra

$$\mathrm{Sym}\,\mathcal{E} = D_C(\mathcal{E})/\mathcal{C}D(\mathcal{E}) \,.$$

Each symmetry of the system of equations $\mathcal{E}$ is generated by some evolutionary vector field $\ni_\varphi|_{\mathcal{E}}$ such that $\ni_\varphi(F)|_{\mathcal{E}} = 0$.

By conservation laws of the system of equations $\mathcal{E}$ we understand elements of the group $E_1^{0,\,n-1}(\mathcal{E}) = \bar{H}^{\,n-1}(\mathcal{E})$.

Map $\mathcal{V}$. For an arbitrary system of equations $\mathcal{E}$ there is a natural map $\mathcal{V}\colon \operatorname{Ker} l_{\mathcal{E}}^* \to E_1^{1,\,n-1}(\mathcal{E})$, which is defined as follows
1. For arbitrary $\psi \in \operatorname{Ker} l_{\mathcal{E}}^* \subset \widehat{P}(\mathcal{E})$ consider the operator $\chi \mapsto \langle \psi, l_{\mathcal{E}}(\chi)\rangle$, where $\langle\cdot,\cdot\rangle$ is the natural pairing between the module and its adjoint. Since $\psi \in \operatorname{Ker} l_{\mathcal{E}}^*$, then, according to the Green formula, there exists an operator $\nabla \in \mathcal{C}\mathrm{Diff}(\varkappa(\mathcal{E}), \Lambda_h^{n-1}(\mathcal{E}))$, such that

$$d_h(\nabla(\chi)) = \langle \psi, l_{\mathcal{E}}(\chi)\rangle \, .$$

2. One can restore the element $w \in \mathcal{C}\Lambda^1(\mathcal{E}) \otimes \Lambda_h^{n-1}(\mathcal{E})$ by the operator $\nabla \in \mathcal{C}\mathrm{Diff}(\varkappa(\mathcal{E}), \mathcal{F}(\mathcal{E})) \otimes \Lambda_h^{n-1}(\mathcal{E})$, using correspondence

$$\nabla_\omega(\chi) = \omega(\ni_\chi)\, .$$

This correspondence is an isomorphism of $\mathcal{C}\mathrm{Diff}(\varkappa(\mathcal{E}), \mathcal{F}(\mathcal{E}))/T$ and $\mathcal{C}\Lambda^1(\mathcal{E})$. Here T is submodule, consisting of operators of the form $\Delta \circ l_{\mathcal{E}}$. Then w generates the corresponding element $w' \in E_1^{1,\,n-1}(\mathcal{E})$. Put $\mathcal{V}(\psi) = w'$.

Remark 1.1. The considering map $\mathcal{V}$ appeared in [Vinogradov], from which follows the proof of its correctness. If ψ is a restriction of a characteristic of some conservation law $c \in \overline{H}^{\,n-1}(\mathcal{E})$, then $\mathcal{V}(\psi) = d_1^{0,\,n-1}(c)$. This follows from the identity

$$[\widetilde{\psi}^i|_{\mathcal{E}}\, l_{\mathcal{E}\,ij}(\varphi^j|_{\mathcal{E}})\,\mathbf{dx}]_h = [\ni_\varphi(\widetilde{\psi}^i F_i\,\mathbf{dx})]_h|_{\mathcal{E}}$$

and properties of the Lie derivative.

Remark 1.2. If a system of equations $\mathcal{E}$ is regular and l-normal, then $\mathcal{V}$ is an isomorphism [Vinogradov].

2. NOETHER THEOREM FOR DIFFIETIES

Consider the complex

$$
\begin{aligned}
0 \to \Lambda^1(\mathcal{E})/\mathcal{C}^2\Lambda^1(\mathcal{E}) \to \ldots \to \Lambda^n(\mathcal{E})/\mathcal{C}^2\Lambda^n(\mathcal{E}) \to \\
\to \Lambda^{n+1}(\mathcal{E})/\mathcal{C}^2\Lambda^{n+1}(\mathcal{E}) \to 0
\end{aligned}
\tag{2.1}
$$

and denote by $\overline{\overline{H}}^{\,i}(\mathcal{E})$ its i-th cohomology group. Here $\mathcal{C}^2\Lambda^1(\mathcal{E}) = 0$; the differentials of (2.1) are induced by the exterior differential d.

Definition. A group of generalized Lagrangians of the system of equations $\mathcal{E}$ is the group of horizontal n-forms $[L]_h \in \Lambda^n_h(\pi)$, such that $\mathbf{E}([L]_h)|_{\mathcal{E}} = 0$.

Denote by $\mathrm{EL}_{\mathcal{E}}$ the group of generalized Lagrangians of the system of equations $\mathcal{E}$.

Let $L \in \Lambda^n(\pi)$ be of the form

$$L = \mathrm{L}\,\mathbf{dx}\,.$$

Define a homomorphism $\mathcal{L}_{\mathcal{E}} \colon \mathrm{EL}_{\mathcal{E}} \to \overline{\overline{H}}^{\,n}(\mathcal{E})$ in the following way: let $\omega_L \in \mathcal{C}\Lambda^n(\pi)$ and $w_L \in \mathcal{C}^2\Lambda^{n+1}(\pi)$ be such that in local coordinates

$$d(L + \omega_L) = \frac{\delta \mathrm{L}}{\delta u^i}\,\theta^i_0 \wedge \mathbf{dx} + w_L\,, \tag{2.2}$$

where $\delta/\delta u^i$ is the variational derivative with respect to the i-th independent variable. Put

$$\mathcal{L}_{\mathcal{E}}([L]_h) = (L + \omega_L)|_{\mathcal{E}} + \mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))\,.$$

Remark 2.1. The differential form $L + \omega_L$ is a Lepagean equivalent of $[L]_c$.

Let us show the correctness of the map $\mathcal{L}_\mathcal{E}$. Let $\widetilde{\omega}_L \in \mathcal{C}\Lambda^n(\pi)$ and $\widetilde{w}_L \in \mathcal{C}^2\Lambda^{n+1}(\pi)$ be other differential forms, such that

$$d(L + \widetilde{\omega}_L) = \frac{\delta \mathrm{L}}{\delta u^i}\, \theta_0^i \wedge \mathbf{dx} + \widetilde{w}_L\,.$$

Then $d(\omega_L - \widetilde{\omega}_L) = w_L - \widetilde{w}_L \in \mathcal{C}^2\Lambda^{n+1}(\pi)$. The group $E_1^{1,\,n-1}(\pi)$ is trivial [Vinogradov], therefore

$$\omega_L - \widetilde{\omega}_L \in \mathcal{C}^2\Lambda^n(\pi) + d(\mathcal{C}\Lambda^{n-1}(\pi))\,.$$

Thus, $(\omega_L - \widetilde{\omega}_L)|_\mathcal{E}$ defines the trivial element of the group $\overline{\overline{H}}^{\,n}(\mathcal{E})$.

Further the group $\overline{\overline{H}}^{\,n}(\mathcal{E})$ will be called the group of internal generalized Lagrangians of the system of equations $\mathcal{E}$.

Remark 2.2. It is easy to verify that the definition of the homomorphism $\mathcal{L}_{\mathcal{E}}$ does not depend on the choice of a particular differential form $\widetilde{L} \in [L]_h$ in the relation (2.2) (instead of L). In addition, horizontal forms from the group $\mathrm{EL}_{\mathcal{E}}$, differing by a horizontal form from $\mathrm{Im}\, d_h$, correspond to the same elements of the group $\overline{\overline{H}}^{\,n}(\mathcal{E})$.

To construct the corresponding differential form $\omega_L \in \mathcal{C}\Lambda^n(\pi)$ it is convenient to use the Noether identity in the form

$$[\Im_\varphi(L)]_h - \left[\frac{\delta \mathrm{L}}{\delta u^i}\, \varphi^i \mathbf{dx}\right]_h = d_h[i_{\Im_\varphi}\Omega]_h\,, \tag{2.3}$$

where $\Omega \in \mathcal{C}\Lambda^n(\pi)$.

Proposition 2.1. *One can put* $\omega_L = \Omega$ *in the relation* (2.2), *where* Ω *is the differential form from the identity* (2.3).

Proof. Since $\Omega \in \mathcal{C}\Lambda^n(\pi)$, then $\ni_\varphi(\Omega) \in \mathcal{C}\Lambda^n(\pi)$. Thus

$$d_h[i_{\ni_\varphi}\Omega]_h = [di_{\ni_\varphi}\Omega]_h = [\ni_\varphi(\Omega) - i_{\ni_\varphi}d\Omega]_h = -[i_{\ni_\varphi}d\Omega]_h$$

and the identity (2.3) can be rewritten in the equivalent form

$$\begin{aligned} & [i_{\ni_\varphi}dL]_h - [di_{\ni_\varphi}\Omega]_h = \left[\frac{\delta \mathrm{L}}{\delta u^i}\,\varphi^i \mathbf{dx}\right]_h \quad \Leftrightarrow \\ \Leftrightarrow \quad & [i_{\ni_\varphi}d(L+\Omega)]_h = \left[\frac{\delta \mathrm{L}}{\delta u^i}\,\varphi^i \mathbf{dx}\right]_h \quad \Leftrightarrow \\ \Leftrightarrow \quad & i_{\ni_\varphi}\Big(d(L+\Omega) - \frac{\delta \mathrm{L}}{\delta u^i}\,\theta_0^i \wedge \mathbf{dx}\Big) \in \mathcal{C}\Lambda^n(\pi) \quad \Leftrightarrow \\ \Leftrightarrow \quad & d(L+\Omega) - \frac{\delta \mathrm{L}}{\delta u^i}\,\theta_0^i \wedge \mathbf{dx} \in \mathcal{C}^2\Lambda^{n+1}(\pi)\,. \end{aligned}$$

Let $q \in \overline{\overline{H}}^{\,n}(\mathcal{E})$. Define a map $\mathcal{N}_q \colon \mathrm{Sym}\,\mathcal{E} \to E_1^{1,\,n-1}(\mathcal{E})$ in the following way: let q be generated by the form $q_0 \in \Lambda^n(\mathcal{E})$. Consider a vector field X_0, such that $X_0 + \mathcal{C}D(\mathcal{E})$ is a symmetry of the system of equations $\mathcal{E}$. Put

$$\mathcal{N}_q(X) = i_{X_0} dq_0 + \mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E})) \in E_1^{1,\,n-1}(\mathcal{E})\,. \quad (2.4)$$

Due to the fact that

$$\begin{aligned} &i_{X_0} d(\mathcal{C}^2\Lambda^n(\mathcal{E})) \subset X_0(\mathcal{C}^2\Lambda^n(\mathcal{E})) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))\,,\\ &X_0(\mathcal{C}^2\Lambda^n(\mathcal{E})) \subset \mathcal{C}^2\Lambda^n(\mathcal{E}) \end{aligned}$$

the map $\mathcal{N}_q$ is well-defined.

The external differential d induces the homomorphism d_{cl} from the group of proper conservation laws $cl(\mathcal{E})$ into the group $E_1^{1,\,n-1}(\mathcal{E})$.

If the homomorphism d_{cl} is injective, then the map $\mathcal{N}_q$ induces the correspondence between some subspace in the Lie algebra $\mathrm{Sym}\,\mathcal{E}$ and the group of conservation laws of the system of equations $\mathcal{E}$. This holds true, for example, for l-normal regular systems of equations [Vinogradov].

Theorem 2.1. *If $\mathcal{E}$ is a regular system of Euler-Lagrange equations with the Lagrangian $[L]_h$ and the corresponding homomorphism d_{cl} is injective, then the internal generalized Lagrangian $\mathcal{L}_{\mathcal{E}}([L]_h)$ determines the correspondence given by Noether's theorem.*

Proof. Denote $\delta \mathrm{L}/\delta u^i = F_i$. Let an evolution vector field $\ni_\lambda$ on $J^\infty(\pi)$ generates the symmetry X of the corresponding system of equations $\mathcal{E}$. Then $\ni_\lambda$ is tangential to the system of equations $\mathcal{E}$ and

$$
\begin{aligned}
&d_h[i_{\ni_\varphi} i_{\ni_\lambda} d(L+\omega_L)]_h|_\mathcal{E} = [d i_{\ni_\varphi} i_{\ni_\lambda} d(L+\omega_L)]_h|_\mathcal{E} = \\
&= -[d i_{\ni_\lambda} i_{\ni_\varphi} d(L+\omega_L)]_h|_\mathcal{E} = -[\ni_\lambda(i_{\ni_\varphi} d(L+\omega_L))]_h|_\mathcal{E} + \\
&+[i_{\ni_\lambda} \ni_\varphi(d(L+\omega_L))]_h|_\mathcal{E} = [i_{\ni_\lambda} \ni_\varphi(F_i\,\theta_0^i \wedge \mathbf{dx} + w_L)]_h|_\mathcal{E} = \\
&= [i_{\ni_\lambda} \ni_\varphi(F_i\,\theta_0^i \wedge \mathbf{dx})]_h|_\mathcal{E} = [i_{\ni_\lambda} i_{\ni_\varphi} d(F_i\,\theta_0^i \wedge \mathbf{dx})]_h|_\mathcal{E} = \\
&= \left[i_{\ni_\lambda} i_{\ni_\varphi} \frac{\partial F_i}{\partial u^j_\alpha}\,\theta^j_\alpha \wedge \theta_0^i \wedge \mathbf{dx}\right]_h\Bigg|_\mathcal{E} = (\langle \lambda, l_F(\varphi)\rangle - \langle \varphi, l_F(\lambda)\rangle)|_\mathcal{E} = \\
&= \langle \lambda|_\mathcal{E}, l_\mathcal{E}(\varphi|_\mathcal{E})\rangle\,.
\end{aligned}
$$

Thus $\mathcal{V}(\lambda|_\mathcal{E})$ is generated by the differential form $i_{\ni_\lambda} d(L+\omega_L)|_\mathcal{E}$.

Remark 2.3. In general case, the correspondence $\mathcal{N}$ between symmetries and conservation laws of a system of Euler-Lagrange equations given by the Noether theorem can be generalized to diffieties as follows:

if X is a symmetry of the diffiety $\mathcal{O}$, $q \in \overline{\overline{H}}^{n}(\mathcal{O})$, then symmetry X corresponds to the conservation laws $d_{cl}^{-1}(\mathcal{N}_q(X))$, where $\mathcal{N}_q(X) \in E_1^{1,n-1}(\mathcal{O})$ is generated by the form $i_{X_0} dq_0$, as in (2.4). Here q_0 and X_0 are such that $q = q_0 + \mathcal{C}^2\Lambda^n(\mathcal{O}) + d(\Lambda^{n-1}(\mathcal{O}))$ and $X = X_0 + \mathcal{C}D(\mathcal{O})$.

3. CONCLUSION

The possibility of formulating the Noether theorem for diffieties emphasizes its deep geometric nature. The correspondence between symmetries and conservation laws, given by Noether theorem can be lifted to coverings. For coverings that are isomorphisms, the correspondence between symmetries and conservation laws given by Noether theorem is preserved. This correspondence is determined by the same element of the group of internal generalized Lagrangians. If a variational symmetry can be lifted to a covering, then its lift will be related to the lift of the corresponding conservation law. Such relation is given by the lift of the corresponding internal generalized Lagrangian.

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