THE GENERALIZED PEAKON EQUATIONS

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Talk based on the common work with
N. Li and Q.P.Liu
Department of Mathematics, China University of Mining and Technology, Beijing P.R.China

Plan:

1.) Comassa-Holm and Degasperis-Procesi equation

2.) Method of generalizations:
  - a.) Scalar,
  - b.) Lax pair.

3.) 4 component case:
  - a.) Bi-Hamiltonian structure,
  - b.) Hierarchy.

4.) Reduction:
  - a.) Equation,
  - b.) Spectral problem.

5.) Conserved quantities.

A four-component Camassa-Holm type hierarchy
Nianhua Li, Q. P. Liu, Z. Popowicz
arXiv:1310.1781

Camassa - Holm equation

$$
\begin{aligned}
u_t - u_{xxt} &= \frac{1}{2}\left(-3u^2 + 2uu_{xx} + u_x^2\right)_x \\
u_t &+ \frac{1}{2}\partial\left(u^2 + G \divideontimes (2u^2 + u_x^2)\right) = 0 \\
f \divideontimes g &= \int dy\ f(y)g(x-y), \qquad G(x) = \frac{1}{2}e^{-|x|}
\end{aligned}
$$

Tautological solutions

$$
\begin{aligned}
u_t - u_{xxt} &= \alpha uu_x + \beta u_x u_{xx} + \gamma uu_{xxx} \\
u(x,t) &= c_1(t)e^x + c_2(t)e^{-x} \\
&\alpha + \beta + \gamma = 0
\end{aligned}
$$

From the physical point of view (Novikov due to symmetry) only

A.) Comassa-Holm $\alpha = -3, \ \beta = 2, \ \gamma = 1,$
B.) Degasperis-Procesi $\alpha = -4, \ \beta = 3, \ \gamma = 1$

Peakon Solutions of Camassa-Holm

$$m_t = -um_x - 2mu_x, \qquad m = u - u_{xx}.$$

The scalar spectral problem is

$$\Psi_{xx} = (\frac{1}{4} - \lambda m)\Psi$$

One peakon

$$\begin{aligned} u(x,t) &= p(t)e^{-|x-q(t)|} = p(t)e^{-|x-ct-c_o|} \\ u_x &= -sgn(x-q)u, \quad m = 2\delta(x-q)u \\ p_t &= 0, \quad , q_t = 0, \Rightarrow q = ct + c_o, \quad p = c. \end{aligned}$$

Degasperis-Procesi

$$\begin{aligned} u_t - u_{t,xx} &= \left(-2u^2 + uu_{xx} + u_x^2\right)_x \\ m_t &= -3u_x m - m_x u, \quad m = u - u_{xx}. \end{aligned}$$

The scalar spectral problem is

$$\Psi_{xxx} = \Psi_x - \lambda m \Psi.$$

The matrix spectral problem

$$\Phi_x = \begin{pmatrix} 0 & 0 & 1 \\ -\lambda m & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Multipeakon solution.

$$u(x,t) = \sum_{j-1}^{N} p_j(t) e^{-|x-q_j(t)|}$$

$$\begin{aligned} \mathring{p}_j &= 2\sum_{k=1}^{N} p_j p_k sgn(q_j - q_k) e^{-|q_j - q_k|} \\ \mathring{q}_j &= \sum_{k=1}^{N} p_k e^{-|q_j - q_k|} \end{aligned}$$

4 types of generalizations

Scalar,Vector, Hamilton, Lax pair

1.) Scalar

$$(1-\partial_{xx})u_t = W(u, u_x, u_{xx}, u_{xxx})$$

How to find the polynomial W?

Hint: Higher symmetries Novikov

For square and cubic we have 9 equations.

Only 2 cubic equations are interesting.

$$\begin{aligned} m_t &+ (m(u^2 - u_x^2))_x = 0 \\ m_t &+ u^2 m_x + 3uu_x m = 0 \end{aligned}$$

3.) Generalization of Lax Pair

A.) Two-component C-H

$$
\begin{aligned}
\Psi_{xx} &= \left(\frac{1}{4} - \lambda m + \lambda^2 \rho^2\right) \Psi \\
\Psi_t &= -\left(\frac{1}{2\lambda} + u\right) \Psi_x + \frac{1}{2} u_x \Psi, \\
m_t &= -2mu_x - m_x u + \rho\rho_x, \qquad \rho_t = -(u\rho)_x
\end{aligned}
$$

Ivanov and Holm generalization CH(N,K)

$$
\begin{aligned}
\Psi_{xx} &= \Big(\sum_{i=1}^{N} q_i(x,t)\lambda^i + \frac{1}{4}\Big)\Psi \\
\Psi_t &= \sum_{j=0}^{K} \big(-u_j(x,t)/\lambda^j \, \partial_x + u_j(x,t)_x/2\big)\Psi
\end{aligned}
$$

B.) First cubic peakon equation

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_x = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

$$A = \lambda^{-2} + \frac{1}{2}(uv - u_x v_x) + \frac{1}{2}(uv_x - u_x v)$$

$$B = -\lambda^{-1}(u - u_x) - \frac{1}{2}\lambda m(uv - u_x v_x)$$

$$C = \lambda^{-1}(v + v_x) + \frac{1}{2}\lambda n(uv - u_x v_x)$$

Qiao equation

$$
\begin{aligned}
m_t &= \frac{1}{2}\left[m(uv - u_x v_x)\right]_x - \frac{1}{2} m\left(uv_x - u_x v\right) \\
n_t &= \frac{1}{2}\left[n(uv - u_x v_x)\right]_x + \frac{1}{2} n\left(uv_x - u_x v\right)
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix} \\
J_1 &= \begin{pmatrix} \partial m \partial^{-1} m \partial - m \partial^{-1} m & \partial m \partial^{-1} n \partial + m \partial^{-1} n \\ \partial n \partial^{-1} m \partial + n \partial^{-1} m & \partial n \partial^{-1} n \partial - n \partial^{-1} n \end{pmatrix} \\
H_1 &= \int dx \, (uv + u_x v_x) \\
H_2 &= \int \, dx \; n(u^2 v_x + u_x^2 v_x - 2 u u_x v)
\end{aligned}
$$

Second cubic equation (Song,Qu,Qiao equation)

$$
\begin{aligned}
m_t &= [m(u_x v_x - uv + uv_x - u_x v)]_x, \\
n_t &= [n(u_x v_x - uv + uv_x - u_x v)]_x,
\end{aligned}
$$

where

$$
m = u - u_{xx}, \qquad n = v - v_{xx}
$$

The matrix spectral problem

$$
\Phi_x = \begin{pmatrix} \frac{1}{2} & \lambda m \\ \lambda n & -\frac{1}{2} \end{pmatrix}
$$

Generalizations of previous cubic equation

B.Xia,Z.Qiao,R.Zho

$$
\begin{aligned}
m_t &= F + F_x - \frac{1}{2}m(u_x v_x - uv + uv_x - u_x v)], \\
n_t &= -G + G_x + \frac{1}{2}n(u_x v_x - uv + uv_x - u_x v),
\end{aligned}
$$

where F, G are an arbitrary function

$$
m = u - u_{xx}, \qquad n = v - v_{xx}
$$

The matrix spectral problem is the same as in the previous case.

c.) Third cubic equation $m_t = -u^2 m_x - 3uu_x m$

$$\Psi_x = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix} \Psi$$

$$\Psi_t = \begin{pmatrix} \frac{1}{3\lambda^2} - uu_x & \frac{u_x}{\lambda} - \lambda u^2 m & u_x^2 \\ \frac{u}{\lambda} & -\frac{2}{3\lambda^2} - \frac{u_x}{\lambda} & -\lambda u^2 m \\ -u^2 & \frac{u}{\lambda} & \frac{1}{3\lambda^2} + uu_x \end{pmatrix} \Psi$$

Three component Generalization, (Geng, Xu)

Matrix spectral problem

$$\Psi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1+\lambda^2 v & 0 & u \\ \lambda^2 w & 0 & 0 \end{pmatrix} \Psi$$

The scalar form is

$$\Phi_{xx} = (1+\lambda^2 v)\Phi + \lambda^2 u \partial^{-1}(w\Phi)$$

$$
\begin{aligned}
u_t &= -vp_x + u_x q + \frac{3}{2}u(q_x - p_x r_x + pr) \\
v_t &= 2vq_x + v_x q \\
w_t &= vr_x + w_x q + \frac{3}{2}w(q_x + p_x r_x - pr)
\end{aligned}
$$

where

$$
\begin{aligned}
u &= p - p_{xx}, \quad w = r_{xx} - r, \\
v &= \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x r - 3pr_x)
\end{aligned}
$$

Generalization to 4-component case

$$\Psi_x = U\Psi = \begin{pmatrix} 0 & \lambda m_1 & 1 \\ \\ \lambda n_1 & 0 & \lambda m_2 \\ \\ 1 & \lambda n_2 & 0 \end{pmatrix} \Psi$$

All mentioned equationa are in this spectral problem

Bi-Hamiltonian structure of a 4 component case.

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi$$

The compatibility condition

$$\begin{aligned}
\lambda m_{1,t} &= V_{1,2,x} - V_{3,2} + \lambda(m_1 V_{1,1} + n_2 V_{1,3} - m_1 V_{2,2}) \\
\lambda m_{2,t} &= V_{2,3,x} + V_{2,1} + \lambda(m_2 V_{2,2} - m_2 V_{3,3} - n_1 V_{1,3}) \\
\lambda n_{1,t} &= V_{2,1,x} + V_{2,3} + \lambda(n_1 V_{2,2} - m_2 V_{3,1} - n_1 V_{1,1}) \\
\lambda n_{2,t} &= V_{3,2,x} - V_{1,2} + \lambda(n_2 V_{3,3} + m_1 V_{3,1} - n_2 V_{2,2})
\end{aligned}$$

and also

$$
\begin{aligned}
V_{1,1} &= V_{3,1,x} + V_{3,3} - \lambda(n_2 V_{2,1} - n_1 V_{3,2}), \\
V_{1,3} &= V_{3,3,x} + V_{3,1} + \lambda(m_2 V_{3,2} - n_2 V_{2,3}), \\
V_{2,2,x} &= \lambda(n_1 V_{1,2} + m_2 V_{3,2} - m_1 V_{2,1} - n_2 V_{2,3}) \\
2V_{3,1,x} + V_{3,3,x,x} &= \lambda\Big((\partial n_2 + m_1)V_{2,3} - (\partial m_2 + n_1)V_{3,2} - \\
&\qquad m_2 V_{1,2} + n_2 V_{2,1}\Big) \\
2V_{3,3,x} + V_{3,1,x,x} &= \lambda\Big((\partial n_2 + m_1)V_{2,1} - (\partial n_1 + m_2)V_{3,2} - \\
&\qquad n_1 V_{1,2} + n_2 V_{2,3}\Big)
\end{aligned}
$$

substituting $V_{1,1}, V_{1,3}, V_{2,2}, V_{3,1}, V_{3,3}$ to the first equation we obtain

$$\begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix}_t = (\lambda^{-1}\mathcal{K} + \lambda\mathcal{L}) \begin{pmatrix} V_{21} \\ V_{32} \\ V_{12} \\ V_{23} \end{pmatrix}$$

where

$$\mathcal{K} = \begin{pmatrix} 0 & -1 & \partial & 0 \\ 1 & 0 & 0 & \partial \\ \partial & 0 & 0 & 1 \\ 0 & \partial & -1 & 0 \end{pmatrix}, \quad \mathcal{L} = \mathcal{J} + \mathcal{F}.$$

$$\mathcal{J}_{13} = -2m_1\partial^{-1}n_1 - n_2\partial^{-1}m_2, \quad \mathcal{J}_{14} = m_1\partial^{-1}n_2 + n_2\partial^{-1}m_1,$$
$$\mathcal{J}_{23} = m_2\partial^{-1}n_1 + n_1\partial^{-1}m_2, \quad \mathcal{J}_{24} = -2m_2\partial^{-1}n_2 - n_1\partial^{-1}m_1,$$

$$\mathcal{J} = \begin{pmatrix} 2m_1\partial^{-1}m_1 & -m_1\partial^{-1}m_2 & \mathcal{J}_{13} & \mathcal{J}_{14} \\ -m_2\partial^{-1}m_1 & 2m_2\partial^{-1}m_2 & \mathcal{J}_{23} & \mathcal{J}_{24} \\ -\mathcal{J}_{13}^* & -\mathcal{J}_{23}^* & 2n_1\partial^{-1}n_1 & -n_1\partial^{-1}n_2 \\ -\mathcal{J}_{14}^* & -\mathcal{J}_{24}^* & -n_2\partial^{-1}n_1 & 2n_2\partial^{-1}n_2 \end{pmatrix},$$

$$\mathcal{F} = (2P + S\partial)(\partial^3 - 4\partial)^{-1}P^T - (2S + P\partial)(\partial^3 - 4\partial)^{-1}S^T.$$

where $P = (m_1, m_2, -n_1, -n_2)^T, S = (-n_2, n_1, -m_2, m_1)^T$.

Now expanding V as

$$V = V_{-2}/\lambda^2 + V_{-1}/\lambda + V_0 = \begin{pmatrix} -f_1g_1 & \frac{g_1}{\lambda} & -g_1g_2 \\ \frac{f_1}{\lambda} & -\frac{1}{\lambda^2} + f_1g_1 + f_2g_2 & \frac{g_2}{\lambda} \\ -f_1f_2 & \frac{f_2}{\lambda} & -f_2g_2 \end{pmatrix}.$$

where

$$\begin{aligned} f_1 &= u_2 - v_{1,x}, \qquad f_2 = u_1 + v_{2,x} \\ g_1 &= v_2 + u_{1,x}, \qquad g_2 = v_1 - u_{2,x} \end{aligned}$$

The four component system follows from the zero-curvature $U_t - V_x + [U, V] = 0$ condition and reads

$$\begin{aligned}
m_{1t} &+ n_2 g_1 g_2 + m_1(f_2 g_2 + 2f_1 g_1) = 0,\\
m_{2t} &- n_1 g_1 g_2 - m_2(f_1 g_1 + 2f_2 g_2) = 0,\\
n_{1t} &- m_2 f_1 f_2 - n_1(f_2 g_2 + 2f_1 g_1) = 0,\\
n_{2t} &+ m_1 f_1 f_2 + n_2(f_1 g_1 + 2f_2 g_2) = 0,
\end{aligned}$$

where

$$\begin{aligned}
m_i &= u_i - u_{ixx}, \qquad n_i = v_i - v_{ixx}, \qquad i = 1, 2.\\
f_1 &= u_2 - v_{1,x}, \qquad f_2 = u_1 + v_{2,x}\\
g_1 &= v_2 + u_{1,x}, \qquad g_2 = v_1 - u_{2,x}
\end{aligned}$$

The four-component system is a bi-Hamiltonian system, which can be written as

$$\begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix}_t = \mathcal{K} \begin{pmatrix} \frac{\delta H_0}{\delta m_1} \\ \frac{\delta H_0}{\delta m_2} \\ \frac{\delta H_0}{\delta n_1} \\ \frac{\delta H_0}{\delta n_2} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \frac{\delta H_1}{\delta m_1} \\ \frac{\delta H_1}{\delta m_2} \\ \frac{\delta H_1}{\delta n_1} \\ \frac{\delta H_1}{\delta n_2} \end{pmatrix}$$

where

$$\begin{aligned} H_0 &= \int (f_1 g_1 + f_2 g_2)(m_2 f_2 + n_1 g_1) dx, \\ H_1 &= \int (m_2 f_2 + n_1 g_1) dx. \end{aligned}$$

We would like to find hierrarchy of the equations connetced with U .
The kernal of $\mathcal{L}$, $\mathcal{L}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})^{\mathcal{T}}$ is

$$A = -n_1\Gamma \ + \frac{n_1}{m_1 m_2} K_3 + \frac{1}{m_1} K_1, \qquad D = -m_2\Gamma,$$

$$B = -n_2\Gamma \ + \frac{1}{m_2} K_2, \qquad C = -m_1\Gamma \ + \frac{1}{m_2} K_3,$$

$$\begin{aligned}
K_1 &= (m_2 n_2 \Lambda)_x + (n_1 n_2 - m_1 m_2)\Lambda, \\
K_2 &= -(m_1 n_1 \Lambda)_x + (n_1 n_2 - m_1 m_2)\Lambda, \\
K_3 &= (m_1 m_2 \Lambda)_x + (m_1 n_1 - m_2 n_2)\Lambda, \\
K_4 &= -(n_1 n_2 \Lambda)_x + (m_1 n_1 - m_2 n_2)\Lambda,
\end{aligned}$$

where $\Lambda = \frac{k}{m_1 n_1 + m_2 n_2}$ and Γ is an arbitrary function and k is an arbitrary number.

For $k = 0$ we have

$$(A, B, C, D) = -\lambda(n_1, n_2, m_1, m_2)\Gamma$$

For special case $\Gamma = m_1 n_1 + m_2 n_2$ we have the Casimir of $\mathcal{L}$ as

$$H_c = -\frac{\lambda}{2}\Gamma^2.$$

On the other side assuming that $(V_{2,1}, V_{3,2}, V_{1,2}, V_{2,3}) = (A, B, C, D)$ we have the time part of the Lax pair

$$\tilde{V} = -\lambda \begin{pmatrix} 0 & m_1\Gamma & 0 \\ n_1\Gamma & 0 & m_2\Gamma \\ 0 & n_2\Gamma & 0 \end{pmatrix}.$$

A first positive flow $U_t - \tilde{V}_x + [U, \tilde{V}] = 0$

$$\begin{aligned} m_{1t} + (\Gamma m_1)_x - n_2\Gamma &= 0, \\ m_{2t} + (\Gamma m_2)_x + n_1\Gamma &= 0, \\ n_{1t} + (\Gamma n_1)_x + m_2\Gamma &= 0, \\ n_{2t} + (\Gamma n_2)_x - m_1\Gamma &= 0. \end{aligned}$$

when $\Gamma = m_1 n_1 + m_2 n_2$ we have $H_c = \frac{1}{2}\int dx\Gamma^2$

$$\begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix}_t = \begin{pmatrix} 0 & -1 & \partial & 0 \\ 1 & 0 & 0 & \partial \\ \partial & 0 & 0 & 1 \\ 0 & \partial & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_c}{\delta m_1} \\ \frac{\delta H_c}{\delta m_2} \\ \frac{\delta H_c}{\delta n_1} \\ \frac{\delta H_c}{\delta n_2} \end{pmatrix}$$

Let us consider the combination of $W = V + \tilde{V}$ and the following Lax representation,

$$\Psi_x = U\Psi, \qquad \Psi_t = W\Psi$$

from which follows

$$\begin{aligned}
m_{1t} + (\Gamma m_1)_x + n_2(g_1 g_2 - \Gamma) + m_1(f_2 g_2 + 2f_1 g_1) = 0, \\
m_{2t} + (\Gamma m_2)_x - n_1(g_1 g_2 - \Gamma) - m_2(f_1 g_1 + 2f_2 g_2) = 0, \\
n_{1t} + (\Gamma n_1)_x - m_2(f_1 f_2 - \Gamma) - n_1(f_2 g_2 + 2f_1 g_1) = 0, \\
n_{2t} + (\Gamma n_2)_x + m_1(f_1 f_2 - \Gamma) + n_2(f_1 g_1 + 2f_2 g_2) = 0,
\end{aligned}$$

where $m_i = u_i - u_{ixx}, \quad n_i = v_i - v_{ixx}, \quad i = 1, 2$

The Hamiltonian structure ?

R E D U C T I O N

Recent generalization of Xia,Qiao

$$m_t = (mH)_x + mH + \frac{1}{(N+1)^2}\Big(m(v+v_x)^T(u-u_x) + (u-u_x)(v+v_x)^T m\Big)$$

$$n_t = (nH)_x + nH + \frac{1}{(N+1)^2}\Big(n(u-u_x)^T(v+v_x) + (v+v_x)(u-u_x)^T n\Big)$$

$$m = u - u_{xx},\ \ n = v - v_{xx},\ \ u = (u_1, u_2, \dots),\ \ v = (v_1, v_2, \dots)$$

The spectral problem is

$$\Psi_x = \frac{1}{N+1}\begin{pmatrix} -N & \lambda m \\ \lambda n^T & I_N \end{pmatrix}\Psi$$

For $N = 2$ we have

$$m_{1,t} = (m_1 H)_x + m_1 H + \frac{1}{9}\Big(m_1(2\tilde{f}_1\tilde{g}_1 + \tilde{f}_2\tilde{g}_2) + m_2\tilde{f}_1\tilde{g}_2\Big)$$

$$m_{2,t} = (m_2 H)_x + m_2 H + \frac{1}{9}\Big(m_1\tilde{f}_2\tilde{g}_1 + m_2(\tilde{f}_1\tilde{g}_1 + 2\tilde{f}_2\tilde{g}_2)\Big)$$

$$n_{1,t} = (n_1 H)_x - n_1 H - \frac{1}{9}\Big(n_1(2\tilde{f}_1\tilde{g}_1 + \tilde{f}_2\tilde{g}_2) + n_2\tilde{f}_2\tilde{g}_1\Big)$$

$$n_{2,t} = (n_2 H)_x - n_2 H + \frac{1}{9}\Big(n_1\tilde{f}_1\tilde{g}_2 + n_2(\tilde{f}_1\tilde{g}_1 + 2\tilde{f}_2\tilde{g}_2)\Big)$$

where

$$\tilde{f}_i = u_i - u_{i,x}, \quad \tilde{g}_i = v_i + v_{i,x}, \quad i = 1, 2$$

Reduction to the 3 component case

A.) $m_1 = u_1 = 0 \Rightarrow \Gamma = -u_{2,x}v_2 + v_2 v_1$ and

$$
\begin{aligned}
m_{2,t} &= -(\Gamma m_2)_x + m_2\left[2v_{2,x}(v_1 - u_{2,x}) + v_2(u_2 - v_{1,x})\right] \\
n_{1,t} &= -(\Gamma n_1)_x + n_1\left[2v_2(u_2 - v_{1,x}) + v_{2,x}(v_1 - u_{2,x})\right] \\
&\quad + m_2\left[\Gamma - v_{2,x}(u_2 - v_{1,x})\right] \\
n_{2,t} &= -(\Gamma n_2)_x + n_2\left[2v_{2,x}(u_{2,x} - v_1) + v_2(v_{1,x} - u_2)\right]
\end{aligned}
$$

B.) $m_2 = u_2 = 0 \Rightarrow \Gamma = v_1(u_{1,x} + v_2)$

$$
\begin{aligned}
m_{1,t} &= -(\Gamma m_1)_x + m_1\left[2v_{1,x}(u_{1,x} + v_2) - v_1(v_{2,x} + u_1)\right] \\
n_{1,t} &= -(\Gamma n_1)_x - n_1\left[2v_{1,x}(u_{1,x} + v_2) - v_1(v_{2,x} + u_1)\right] \\
n_{2,t} &= -(\Gamma n_2)_x - n_2\left[2v_1(v_{2,x} + u_1) - v_{1,x}(u_{1,x} + v_2)\right] \\
&\quad + m_1\left[\Gamma + v_{1,x}(v_{2,x} + u_1)\right]
\end{aligned}
$$

symmetry

$m_1 \Rightarrow -m_2,\ n_1 \Rightarrow -n_2,\ n_2 \Rightarrow n_1,\ u_1 \Rightarrow -u_2,\ v_2 \Rightarrow v_1,\ \Gamma \Rightarrow -\Gamma$

Reduction to the 2-component system

AA.) $m_2 = u_2 = 0$ for A case in 3-component case $\Rightarrow \Gamma = v_2 v_1$

$$\begin{aligned} m_t &= -v(m_x u + 3mu_x), \qquad m = u - u_{xx} \\ n_t &= -u(n_x v + 3nv_x), \qquad n = v - v_{xx} \end{aligned}$$

When $u = v$ we have Novikov cubic equation

$$m_t = -u(m_x u + 3mu_x), \qquad m = u - u_{xx}$$

When $u = 1$ or $v = 1$ we have Degasperis-Procesi equation

$$m_t = -(m_x u + 3mu_x), \qquad m = u - u_{xx}$$

BB.) $n_1 = m_2, \quad n_2 = m_1, \quad v_1 = u_2, \quad v_2 = u_1$

$$\begin{aligned} m_{1,t} &= -(\Gamma m_1)_x + m_1(\Gamma + 4(u_{2,x} - u_2)(u_{1,x} + u_1)) \\ m_{2,t} &= -(\Gamma m_2)_x - m_2(\Gamma + 4(u_{2,x} - u_2)(u_{1,x} + u_1)) \end{aligned}$$

If $m_2 = u_2 = 1$ or $m_1 = u_1 = 1$ and $\Gamma = 4u_1u_2$ then our equation reduces to the Camassa-Holm equation.
On the other side when $\Gamma = 4(u_1u_2 - u_{1,x}u_{2,x})$ we obtain Qiao system

$$\begin{aligned} m_{1,t} &= \left(m_1(u_1u_2 - u_{1,x}u_{2,x})\right)_x - m_1(u_1u_{2,x} - u_{1,x}u_2) \\ m_{2,t} &= \left(m_2(u_1u_2 - u_{1,x}u_{2,x})\right)_x + m_2(u_1u_{2,x} - u_{1,x}u_2) \end{aligned}$$

When $u_2 = -1$ this system reduces to the Camassa-Holm equation.

Reduction of Spectral problem

A.) Let $m_1 = u_1 = 0$ then

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 1 \\ \lambda n_1 & 0 & \lambda m_2 \\ 1 & \lambda n_2 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

assuming

$$n_2 = u, \quad m_2 = \frac{v}{u}, \quad n_1 = w + \left(\frac{v}{u}\right)_x,$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda^2 v & 0 & u \\ \lambda^2 w & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

This spectral problem was considered by Geng and Xue.

B.) Let $n_1 = m_2, \quad n_2 = m_1$ then

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} 0 & \lambda m_1 & 1 \\ \lambda m_2 & 0 & \lambda m_2 \\ 1 & \lambda m_1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

assuming that $\Psi_1 + \Psi_3 = \Phi_1, \Psi_1 - \Psi_3 = \Phi_2$ and eliminate of Ψ_2

$$\Phi_{1,xx} = ((\ln m_1)_x + 1)\Phi_{1,x} + (2\lambda^2 m_2 m_1 - (\ln m_1)_x)\Phi_1$$

Song-Qu-Qiao system

$$\varphi_x = \begin{pmatrix} \frac{1}{2} & \lambda m \\ \lambda n & -\frac{1}{2} \end{pmatrix} \varphi.$$

after rescaling $\partial_x \Rightarrow \frac{1}{2}\partial_x$

$$\Phi_{1,xx} = (\ln m)_x \Phi_{1,x} + [-(\ln m)_x + 4\lambda^2 mn + 1]\Phi_1$$

therefore $m = e^x m_1, \quad n = \frac{1}{2}e^{-x} m_2$

Conserved quantities.

These follows from the spectral problem.

These quantities are the conserved for whole hierarchy.

$$\begin{aligned} \Psi_{1,x} &= \lambda m_1 \Psi_2 + \Psi_3 \\ \Psi_{2,x} &= \lambda n_1 \Psi_1 + \lambda m_2 \Psi_3 \\ \Psi_{3,x} &= \Psi_1 + \lambda \Psi_2 \end{aligned}$$

We have 3 different series of conserved quantities because we have three projective coordinates

$$\textbf{I.)}\ \ a = \frac{\varphi_1}{\varphi_2}, \quad b = \frac{\varphi_3}{\varphi_2}, \qquad \textbf{II.)}\ \ \sigma = \frac{\varphi_2}{\varphi_1},\ \tau = \frac{\varphi_3}{\varphi_1},$$

$$\textbf{III.)}\ \ \alpha = \frac{\varphi_1}{\varphi_3},\ \beta = \frac{\varphi_2}{\varphi_3} \tag{1}$$

I.) The spectral problem implies

$$a_x = \lambda m_1 + b - a\rho, \quad b_x = a + \lambda n_2 - b\rho,$$
$$\rho = (\log \Psi_2)_x = \lambda(n_1 a + m_2 b)$$

ρ is a conserved laws.Next a, b, ρ expand in power of λ.

positive

$$a_0 = b_0 = a_2 = b_2 = 0$$
$$a_1 = -v_2 - u_{1,x}, \quad b_1 = -u_1 - v_{2,x}$$

For $k \geq 3$ we have

$$a_{k,x} = b_k - \sum_{i+j=k-1} (n_1 a_i a_j + m_2 a_i b_j)$$
$$b_{k,x} = a_k - \sum_{i+j=k-1} (n_1 a_i b_j + m_2 b_i b_j)$$

$$\rho_1 = -\int (n_1 g_1 + m_2 f_2) dx.$$
$$\rho_3 = \int (n_1 g_1 + m_2 f_2)(f_1 g_1 + f_2 g_2) dx.$$

negative

$$a=\Sigma_{i\geq 0}^{\infty}\tilde{a}_i\lambda^{-i}, \quad b=\Sigma_{j\geq 0}^{\infty}\tilde{b}_j\lambda^{-j}.$$

$$\begin{aligned}\rho_0 &= \int\sqrt{m_1n_1+m_2n_2}dx,\\ \rho_{-1} &= \int\frac{2m_1m_2+2n_1n_2+m_1n_{1x}-m_{1x}n_1+m_{2x}n_2-m_2n_{2x}}{4(m_1n_1+m_2n_2)}dx.\end{aligned}$$

Case 2:

$$\bar{\rho}=(\ln\varphi_1)_x=\lambda m_1\sigma+\tau,$$

where σ,τ satisfy

$$\sigma_x=\lambda n_1+\lambda m_2\tau-\sigma\bar{\rho}, \quad \tau_x=1+\lambda n_2\sigma-\tau\bar{\rho}.$$

$$\bar{\rho}_2 = \frac{1}{2}\int (m_1 + n_2)(f_1 + g_2)dx,$$

$$\bar{\rho}_{-1} = \int \frac{2m_2 n_2^2 + 2m_1 n_1 n_2 - m_{2x} m_1 n_2 - 3n_{2x} m_1 m_2}{4m_1(m_1 n_1 + m_2 n_2)} dx + \int \frac{4m_{1x} m_2 n_2 + m_{1x} m_1 n_1 - n_{1x} m_1^2}{4m_1(m_1 n_1 + m_2 n_2)} dx.$$

Case 3:

$$\hat{\rho} = (\ln \varphi_3)_x = \alpha + \lambda n_2 \beta,$$

where α, β satisfy

$$\alpha_x = \lambda m_1 \beta + 1 - \alpha\hat{\rho}, \quad \beta_x = \lambda n_1 \alpha + \lambda m_2 - \beta\hat{\rho}.$$

$$\hat{\rho}_{-1} = \int \frac{2n_1n_2^2 + 2m_1m_2n_2 - m_{2x}n_2^2 + 4n_{2x}m_1n_1}{4n_2(m_1n_1 + m_2n_2)}dx + \int \frac{-3m_{1x}n_1n_2 + n_{2x}m_2n_2 - n_{1x}m_1n_2}{4n_2(m_1n_1 + m_2n_2)}dx.$$

Thanks

very much

for your attention