Conservation laws of even-order evolution equations

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An evolution equation in two independent variables,

$$\mathcal{E}\colon\quad u_t = F(t, x, u_0, u_1, \dots, u_n), \quad n \geqslant 2, \quad F_{u_n} \neq 0,$$

where $u_j \equiv \partial^j u/\partial x^j$, $u_0 \equiv u$, and $F_{u_j} = \partial F/\partial u_j$, $u_x = u_1$, $u_{xx} = u_2$, and $u_{xxx} = u_3$.

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots, \quad D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots$$

the total derivatives w.r.t. the variables t and x. The subscripts like t, x, u, u_x, etc. stand for the partial derivatives in the respective variables.

Without loss of generality, for any evolution equation the associated quantities like symmetries, cosymmetries, densities and characteristics of CLs can be assumed independent of the t-derivatives or mixed derivatives of u.

We refer to a (smooth) function of t, x and a finite number of u_j as to a *differential function*.

Given a differential function f, its *order* (denoted by $\mathrm{ord}\, f$) is the greatest integer k such that $f_{u_k} \neq 0$ but $f_{u_j} = 0$ for all $j > k$. For $f = f(t, x)$ we assume that $\mathrm{ord}\, f = -\infty$.

Contact transformations of evolution equations

The contact transformations mapping a (fixed) equation $\mathcal{E}$: $u_t = F$ into another equation $\tilde{\mathcal{E}}$: $\tilde u_{\tilde t} = \tilde F$ are well known [Magadeev,1993] to have the form

$$\tilde t = T(t), \quad \tilde x = X(t,x,u,u_x), \quad \tilde u = U(t,x,u,u_x).$$

The nondegeneracy assumptions: $\quad T_t \neq 0, \quad \operatorname{rank}\begin{pmatrix} X_x & X_u & X_{u_x} \\ U_x & U_u & U_{u_x} \end{pmatrix} = 2$

The contact condition: $\quad (U_x + U_u u_x)X_{u_x} = (X_x + X_u u_x)U_{u_x} \quad \Longrightarrow \quad \tilde u_{\tilde x} = V(t,x,u,u_x)$,

where $\quad V = \dfrac{U_x + U_u u_x}{X_x + X_u u_x}$ or $V = \dfrac{U_{u_x}}{X_{u_x}}$ if $X_x + X_u u_x \neq 0$ or $X_{u_x} \neq 0$, respectively

$$\Longrightarrow \quad \tilde u_k \equiv \frac{\partial^k \tilde u}{\partial \tilde x^k} = \left(\frac{1}{D_x X} D_x\right)^{k-1} V, \quad \tilde F = \frac{U_u - X_u V}{T_t} F + \frac{U_t - X_t V}{T_t}$$

The equiv. group $G^{\sim}_{\rm cont}$ generates the whole equiv. groupoid (the set of admissible contact transformations) of the class, i.e., the class is normalized [ROP & Kunzinger & Eshraghi,2010] w.r.t. contact transformations.

Proposition

The class of evolution equations is contact-normalized.

Point transformations of evolution equations

The class of evolution equations is also point-normalized.

The point equivalence group $G^{\sim}_{\rm point}$:

$$\tilde t = T(t), \quad \tilde x = X(t,x,u), \quad \tilde u = U(t,x,u), \quad \tilde F = \frac{\Delta}{T_t D_x X} F + \frac{U_t D_x X - X_t D_x U}{T_t D_x X},$$

where $T_t \neq 0$ and $\Delta = X_x U_u - X_u U_x \neq 0$.

The point equivalence group of the subclass of quasilinear evolution equations (i.e., $F_{u_n u_n} \neq 0$) is the same, and this subclass is normalized.

A *conserved current* of $\mathcal{E}$ is a pair of differential functions (ρ, σ) satisfying the condition

$$D_t\rho + D_x\sigma = 0 \bmod \check{\mathcal{E}}, \qquad \text{where} \qquad \check{\mathcal{E}} = \mathcal{E} \text{ and all its differential consequences.}$$

Here ρ is the *density* and σ is the *flux* for the conserved current (ρ, σ). Let

$$\frac{\delta}{\delta u} = \sum_{i=0}^{\infty}(-D_x)^i\partial_{u_i}, \quad f_* = \sum_{i=0}^{\infty} f_{u_i} D_x^i, \quad f_*^\dagger = \sum_{i=0}^{\infty}(-D_x)^i \circ f_{u_i}$$

denote the operator of variational derivative, the Fréchet derivative of a differential function f, and its formal adjoint, respectively.

A conserved current (ρ, σ) is called *trivial* if $D_t\rho + D_x\sigma = 0$ on the entire jet space $\Longleftrightarrow$ $\rho \in \operatorname{Im} D_x$, i.e., there exists a differential function ζ: $\rho = D_x\zeta$.

Two conserved currents are *equivalent* if they differ by a trivial conserved current.

A *CL* of $\mathcal{E}$ is an equivalence class of conserved currents of $\mathcal{E}$.

The set $\mathrm{CL}(\mathcal{E})$ of CLs of $\mathcal{E}$ is a vector space, and the zero element of this space is the CL being the equivalence class of trivial conserved currents.

For any $\mathcal{L} \in \mathrm{CL}(\mathcal{E})$ there exists a unique differential function γ called the *characteristic* of $\mathcal{L}$ such that for any conserved current $(\rho, \sigma) \in \mathcal{L}$ there exists a trivial conserved current $(\tilde\rho, \tilde\sigma)$:

$$D_t(\rho + \tilde\rho) + D_x(\sigma + \tilde\sigma) = \gamma(u_t - F).$$

The characteristic γ of any CL is a *cosymmetry*, i.e., it satisfies the equation

$$D_t\gamma + F_*^\dagger\gamma = 0 \bmod \check{\mathcal{E}}, \qquad \text{or equivalently,} \qquad \gamma_t + \gamma_* F + F_*^\dagger\gamma = 0.$$

The characteristic of the CL associated with (ρ, σ) is

$$\gamma = \delta\rho/\delta u.$$

Therefore, a cosymmetry γ to be a characteristic of a CL iff $\gamma_* = \gamma_*^\dagger$.

$$D_t\rho + \gamma_* F = D_x\hat\sigma \qquad \text{for some } \hat\sigma.$$

$\mathrm{ord_d}\,\mathcal{L} = 2\,\mathrm{ord_{char}}\,\mathcal{L}$ if they are both positive. $\mathrm{ord_{char}}\,\mathcal{L} \in \{-\infty, 0\}$ if $\mathrm{ord_d}\,\mathcal{L} = 0$.

$$\mathrm{ord_{char}}\,\mathcal{L} \in \{-\infty, 0\} \cup 2\mathbb{N}$$

Total linear independence of cosymmetries

linear dependence $\leftarrow\text{-}\text{-}\text{-}\text{-}\rightarrow$ total linear independence
(meromorphic functions, local approach)

Definition

Diff. functions $f^1,\dots,f^p\colon \Omega\to\mathbb{R}$, where Ω is an open subset of the jet space, are called *totally linear independent (t.l.i.)* if for any open subset Ω' of Ω the functions $f^1|_{\Omega'},\dots,f^p|_{\Omega'}$ are linearly independent.

Lemma

Diff. functions $f^1,\dots,f^p\colon \Omega\to\mathbb{R}$ are t.l.i. over a ring of smooth functions of t iff

$$\operatorname{supp} W(f^1,\dots,f^p)=\Omega,\qquad \text{where}\qquad W(f^1,\dots,f^p)=\det(D_x^{s'-1}f^s)_{s,s'=1}^p$$

denotes the Wronskian of of these functions w.r.t. the operator of total derivative D_x.

Lemma [Magadeev, 1993, for symmetries]

Symmetries (resp. cosymmetries) $\gamma^1,\dots,\gamma^p\colon \Omega\to\mathbb{R}$ of an evolution equation $\mathcal{E}$ are t.l.i. iff $\operatorname{supp} W(\gamma^1,\dots,\gamma^p)=\Omega$.

Evolution equations having low-order CLs. I

Theorem [Svinolupov, 1985, time-independent case; ROP & Sergyeyev, 2010]

Any pair $(\mathcal{E}, \mathcal{L})$, where $\mathcal{E}$ is an evolution equation and $\mathcal{L}$ is a nonzero CL of $\mathcal{E}$ with $\mathrm{ord_d}\,\mathcal{L} \leqslant 1$ is $G^{\sim}_{\rm cont}$-equivalent to a pair $(\tilde{\mathcal{E}}, \tilde{\mathcal{L}})$, where $\tilde{\mathcal{E}}$ is an equation of the same form and $\tilde{\mathcal{L}}$ is a CL of $\tilde{\mathcal{E}}$ with the characteristic equal to 1. If $\mathrm{ord_d}\,\mathcal{L} = 0$, the pairs are $G^{\sim}_{\rm point}$-equivalent.

Corollary

$\mathcal{E}$ admits a nonzero CL $\mathcal{L}$ with $\mathrm{ord_d}\,\mathcal{L} \leqslant 1$ (resp. $\mathrm{ord_d}\,\mathcal{L} = 0$) iff it is locally reduced by a contact (resp. point) transformation to the form

$$\tilde{u}_{\tilde{t}} = D_{\tilde{x}}G(\tilde{t}, \tilde{x}, \tilde{u}_0, \dots, \tilde{u}_{n-1}),\ \ G_{\tilde{u}_{n-1}} \neq 0.$$

Evolution equations having low-order CLs. II

Theorem [ROP & Sergyeyev, 2010]

Let $\mathcal{L}^{\mathrm{I}}$ and $\mathcal{L}^{\mathrm{II}}$ be linearly independent CLs of $\mathcal{E}$ of density order 0. Any such triple $(\mathcal{E}, \mathcal{L}^{\mathrm{I}}, \mathcal{L}^{\mathrm{II}})$ is $G^{\sim}_{\mathrm{point}}$-equivalent to a triple $(\tilde{\mathcal{E}}, \tilde{\mathcal{L}}^{\mathrm{I}}, \tilde{\mathcal{L}}^{\mathrm{II}})$, where $\tilde{\mathcal{E}}$ is an evolution equation that admits CLs $\tilde{\mathcal{L}}^{\mathrm{I}}$ and $\tilde{\mathcal{L}}^{\mathrm{II}}$ with the characteristics equal to 1 and $\tilde{x}$, respectively.

$$\tilde{t} = t, \quad \tilde{x} = X(t,x,u) = \gamma^{\mathrm{II}}/\gamma^{\mathrm{I}}, \quad \tilde{u} = U(t,x,u)\colon \quad X_x U_u - X_u U_x = \gamma^{\mathrm{I}};$$
$$\gamma^i = \rho^i_u, \quad i = \mathrm{I}, \mathrm{II}$$

Corollary

An evolution equation has (at least) two linearly independent CLs of density order 0 iff it can be locally reduced by a point transformation to the “second normal” form

$$\tilde{u}_{\tilde{t}} = D^2_{\tilde{x}} H(\tilde{t}, \tilde{x}, \tilde{u}_0, \ldots, \tilde{u}_{n-2}), \quad H_{\tilde{u}_{n-2}} \neq 0.$$

Order and linear dependence of characteristics

Theorem

Given an evolution equation $\mathcal{E}$ with CL characteristics γ^s, $s=1,\dots,p$, and $\operatorname{ord}\gamma^s \leqslant r \in 2\mathbb{N}\cup\{0\}$,

$$\operatorname{ord} W(\gamma^1,\dots,\gamma^p) < r+p-1 \qquad (1)$$

iff γ^1, …, γ^p are linearly dependent or $\operatorname{ord}\gamma^s < r$ (if $r\leqslant 2$, $\operatorname{mod} G^{\sim}_{\rm cont}$), $s=1,\dots,p$.

In general, $\operatorname{ord} W(\gamma^1,\dots,\gamma^p) \leqslant r+p-1$. So,

lowering the upper bound for the Wronskian order
$\Longrightarrow$
lowering the upper bound for CL characteristic orders

Sketch of the proof.

$$(1) \quad\iff\quad \partial_{u_{r+p-1}} W(\gamma^1,\dots,\gamma^p)=0 \quad\iff\quad W(\gamma^1,\dots,\gamma^p)\big|_{\gamma^s_{u_r}\rightsquigarrow D_x^{p-1}\gamma^s}=0$$

If $r=2$, then (1) is $G^{\sim}_{\rm cont}$-invariant, and we can set $\gamma^p=1$.

If $r=0$, then (1) is $G^{\sim}_{\rm point}$-invariant, and we can set $\gamma^p=1$ and $\gamma^{p-1}=x$

Then we separately prove the cases $r=0$, $r=2$ and $r>2$ by induction.

Second-order evolution equation

$\mathcal{E}$: $u_t = F(t, x, u, u_x, u_{xx})$

- Density order of CLs $\leqslant 1$.
- Density order of CLs $= 1$ if $F_{u_2 u_2} = 0$.

Theorem [Bryant&Griffiths, 1995, $F_t = 0$; ROP&Samoilenko, 2008, gen. case]

$\dim \mathrm{CL}(\mathcal{E}) \in \{0, 1, 2, \infty\}$ for any second-order $(1+1)$D evolution equation $\mathcal{E}$.
The equation $\mathcal{E}$ is (locally) reduced by a contact transformation

1. to the form $u_t = D_x G(t, x, u, u_x)$, where $G_{u_x} \neq 0$, iff $\dim \mathrm{CL}(\mathcal{E}) \geqslant 1$;
2. to the form $u_t = D_x^2 H(t, x, u)$, where $H_u \neq 0$, iff $\dim \mathrm{CL}(\mathcal{E}) \geqslant 2$;
3. to a linear evolution equation iff $\dim \mathrm{CL}(\mathcal{E}) = \infty$.

If the equation $\mathcal{E}$ is quasi-linear (i.e., $F_{u_{xx} u_{xx}} = 0$) then the contact transformation is a prolongation of a point transformation.

Corollary

The space of CLs of $u_t = D_x^2 H(t, x, u)$ is 2D iff H is not linear in u.
The space of CLs of $u_t = D_x G(t, x, u, u_x)$ is 1D if G is not a fractionally linear in u_x.

CLs of linear evolution equations. Even order

$\alpha = \alpha(t, x)$ is a cosymmetry of a linear evolution equation $\mathcal{L}$ iff it satisfies the *adjoint* equation

$$\mathcal{L}^*: \qquad L^*\alpha = 0, \qquad L^* := -\mathrm{D}_t + \sum_{i=0}^{n} (-\mathrm{D}_x)^i \circ A^i.$$

Any such cosymmetry is a characteristic of a linear CL for $\mathcal{L}$ whose canonical conserved current (ρ, σ) has the components

$$\rho = \alpha(t, x)u, \quad \sigma = \sum_{i=0}^{n-1} \sigma^i(t, x)u_i,$$

$$\sigma^{n-1} = -\alpha A^n,\ \sigma^i = -\alpha A^{i+1} - \sigma^{i+1}_x,\ i = n-2, \dots, 0.$$

Theorem [ROP & Sergyeyev, 2010]

For any linear $(1+1)$D evolution equation of even order, all its cosymmetries depend only on (t, x), and the cosymmetry space is isomorphic to the solution space of the adjoint equation.

$\sim \mathrm{CL}(\mathcal{L})$ is exhausted by linear ones and is isomorphic to the solution space of $\mathcal{L}^*$.

CLs of linear evolution equations. Odd order

Theorem [ROP & Sergyeyev, 2010]

For any linear evolution equation of odd order, all its cosymmetries are affine in the totality of variables $u_0, u_1, u_2, \ldots$:

$$\gamma = \Gamma u + \alpha(t,x), \quad \Gamma = \sum_{k=0}^{r} g^k(t,x) D_x^k.$$

where $\alpha = \alpha(t,x)$ satisfies the *adjoint* equation $\mathcal{L}^*$: $L^*\alpha = 0$.

$\sim$ The space of its CLs is spanned by linear and quadratic ones.

Indeed, let $\gamma = \Gamma u$. γ is a characteristic of a CL iff $\Gamma = \Gamma^*$. (We always can take its formally self-adjoint part $\tilde{\Gamma} = (\Gamma + \Gamma^*)/2$; $\tilde{\gamma} = \tilde{\Gamma} u$ is a cosymmetry if so is γ.)
This CL has the density $F = \frac{1}{2} u \Gamma u$ with a flux G quadratic in u_0, u_1, ... [Olver, 1993, Theorem 5.104].

Γu is a characteristic of a CL for $\mathcal{L}$ iff $\Gamma = \Gamma^*$ satisfies the following equivalent conditions:

- it maps the solutions of $\mathcal{L}$ into solutions of $\mathcal{L}^*$;
- $L^*\Gamma = -\Gamma L$, i.e., ΓL is formally skew-adjoint.

Example. Infinite series of quadratic CLs of arbitrarily high orders.

$$\mathcal{L}\colon\quad u_t = u_{xxx}, \qquad L = \mathrm{D}_t - \mathrm{D}_x^3 = -L^*, \quad \text{i.e.,}\quad \mathcal{L}\sim\mathcal{L}^*$$

the space of cosymmetries $\equiv$ the space of characteristics of generalized symmetries $=$

$$\langle D_x^k \Upsilon^l u,\ k,l=0,1,2,\dots,\ \alpha(t,x)\rangle,$$

where $\Upsilon = x + 3tD_x^2$ and $\alpha=\alpha(t,x)$ runs through the solution set of $\mathcal{L}$.

Each cosymmetry $f(t,x)$ is associated with the CL with the density $\rho=\alpha(t,x)u$.

Cosymmetries $D_x^k\Upsilon^l u$ are characteristics of CLs only with even $k=2m$. The CLs in question are quadratic with the densities $\rho^{lm}=\frac{1}{2}u\,D_x^m\Upsilon^l D_x^m u$ and the density orders $l+m$, where $l,m=0,1,2\dots$.

Example. No quadratic CLs.

$$\mathcal{L}\colon\quad u_t = u_{xxx}+xu, \qquad L = \mathrm{D}_t - \mathrm{D}_x^3 - x \neq -L^*, \quad \text{i.e.,}\quad \mathcal{L}\not\sim\mathcal{L}^*$$

the space of characteristics of gen. symmetries $=$

$$\langle (D_x^3+x)^k(D_x+t)^l u,\ k,l=0,1,2,\dots,\ f(t,x)\rangle,$$

where $f=f(t,x)$ runs through the solution set of $\mathcal{L}$

the space of cosymmetries $=\langle\alpha(t,x)\rangle$,
where $\alpha=\alpha(t,x)$ runs through the solution set of $\mathcal{L}^*$

Standard problem on CLs

Given a system of DEs, find the space of its CLs or at least a subspace of this space with certain additional constraints, e.g., on order of CLs.

Various applications, e.g., the parameterization of DEs need solving the inverse problem.

Inverse problem on CLs

Derive the general form of systems of DEs with a prescribed set of CLs.

More generally, the inverse problem on CLs can be interpreted as the study of properties of DEs for which something is known about their CLs.

? What data on CLs should be given ?

Inverse problem on CLs

- There are two types of triviality of conserved currents.
- The form of conserved currents strongly varies if the form of corresponding systems varies within the class under consideration.

$\Longrightarrow$ **!** **Whole conserved currents are not appropriate initial data for the inverse problem on CLs** **!**

- There i one type of triviality of conserved currents.
- The form of characteristics is more stable under varying the form of corresponding systems varies within the class under consideration.

$\Longrightarrow$ If the form of DE systems under consideration are fixed, then

! **CL characteristics are appropriate initial data for the inverse problem on CLs** **!**

Reformulation of inverse problem on CLs

Derive the general form of systems of DEs with a prescribed set of CL characteristics.

If a systems in the extended Kovalevskaya form, then

$$\textcolor{red}{!} \qquad \textbf{densities} \quad \rightsquigarrow \quad \textbf{characteristics} \qquad \textcolor{red}{!}$$

Theorem

An evolution equation

$$\mathcal{E}: \quad u_t = F(t, x, u_0, u_1, \dots, u_n), \quad n \geqslant 2, \quad F_{u_n} \neq 0,$$

admits p linearly independent CLs with densities ρ^s and characteristics $\gamma^s = \delta\rho^s/\delta u$ iff

$$F = \mathrm{DT}[\gamma^1, \dots, \gamma^p]^\dagger H - \sum_{s=1}^{p} \mathrm{DT}[\gamma^1, \dots, \cancel{\gamma^s}, \dots, \gamma^p]^\dagger \left(\frac{W(\gamma^1, \dots, \cancel{\gamma^s}, \dots, \gamma^p)}{W(\gamma^1, \dots, \gamma^p)} \rho^s_t \right)$$

for some $H = H[u]$.

Here

- $W(\gamma^1, \dots, \gamma^p) = \det(D_x^{s'-1}\gamma^s)_{s,s'=1}^p$ denotes the Wronskian of γ^1, ..., γ^p w.r.t. the operator of total derivative D_x,
- $\mathrm{DT}[\gamma^1, \dots, \gamma^p]$ is the operator in D_x associated with the "Darboux transformation"

$$\mathrm{DT}[\gamma^1, \dots, \gamma^p]H = \frac{W(\gamma^1, \dots, \gamma^p, H)}{W(\gamma^1, \dots, \gamma^p)}.$$

- † denotes "formally adjoint", $Q = Q^j D_x^j$, $Q^\dagger = (-D_x)^j \circ Q^j$.

$$F=\mathrm{DT}[\gamma^1,\dots,\gamma^p]^\dagger H-\sum_{s=1}^{p}\mathrm{DT}[\gamma^1,\dots,\cancel{\gamma^s},\dots,\gamma^p]^\dagger\left(\varphi^s\rho^s_t\right), \tag{2}$$

where $\varphi^s:=(-1)^{p-s}\dfrac{W(\gamma^1,\dots,\cancel{\gamma^s},\dots,\gamma^p)}{W(\gamma^1,\dots,\gamma^p)}$ are adjoint diff. functions to $\gamma^1,\dots,\gamma^p$,

$\mathrm{DT}[\gamma^1,\dots,\gamma^p]^\dagger=(-1)^p\mathrm{DT}[\varphi^1,\dots,\varphi^p]$.

$$F=(-1)^p\frac{W(\hat\varphi^1,\dots,\hat\varphi^p,\hat H)}{\left(W(\gamma^1,\dots,\gamma^p)\right)^p}-\sum_{s=1}^{p}\mathrm{DT}[\gamma^1,\dots,\cancel{\gamma^s},\dots,\gamma^p]^\dagger\frac{\hat\varphi^s\rho^s_t}{W(\gamma^1,\dots,\gamma^p)}, \tag{3}$$

where $\hat\varphi^s:=(-1)^{p-s}W(\gamma^1,\dots,\cancel{\gamma^s},\dots,\gamma^p)$.

- (2) depends rather on $\langle\rho^1,\dots,\rho^p\rangle$ than on $\{\rho^1,\dots,\rho^p\}$.
- For fixed F and $\langle\rho^1,\dots,\rho^p\rangle$, H is defined up to $H\to H+\zeta^s(t)\varphi^s$.
- If $\tilde\rho^s=\rho^s+D_x\theta^s$ for some diff. functions θ^s, then $\tilde H=H+(\zeta^s-\theta^s_t)\varphi^s$ for some $\zeta^s=\zeta^s(t)$.
- Let $q:=\max\limits_s\operatorname{ord}\gamma^s$ ($\in\{-\infty,0\}\cup2\mathbb{N}$) and ρ^s: $\operatorname{ord}\rho^s\leqslant\max(0,q+p-1)$. (For best ρ^s: $\operatorname{ord}\rho^s=\max(0,\frac12\operatorname{ord}\gamma^s)$.) Then $\operatorname{ord}\hat H\leqslant\max(n-p,q+p-2)$.
- If $\max\limits_s\operatorname{ord}\gamma^s\leqslant n-2p+2$, then $\operatorname{ord}\hat H\leqslant n-p$.
- If $\max\limits_s\operatorname{ord}\gamma^s\leqslant n-2p$, then $\operatorname{ord}H=\operatorname{ord}\hat H=n-p$.

CLs of even-order evolution equations

Let $\mathcal{E}$ be a $(1+1)$D even-order ($n\in 2\mathbb{N}$) evolution equation.

Lemma [Ibragimov, 1983; Abellanas & Galindo, 1979; Kaptsov, 1980]

For any CL $\mathcal{F}$ of $\mathcal{E}$ we have $\mathrm{ord}_{\mathrm{char}}\,\mathcal{F}\leqslant n$.

Theorem

$\mathrm{Cosym}_{\mathrm{f}}(\mathcal{E})=\mathrm{Ch}_{\mathrm{f}}(\mathcal{E})$.

Theorem

If $\dim\mathrm{CL}(\mathcal{E})\geqslant p$, then $\mathrm{ord}\,\gamma\leqslant n-2p+2$ for any $\gamma\in\mathrm{Ch}_{\mathrm{f}}(\mathcal{E})$ $(\mathrm{mod}\,G^{\sim}_{\mathrm{cont}}$ if $2p\geqslant n+2)$.

Roughly speaking, the greater dimension of the space of CLs, the lower upper bound for orders of CL characteristics. In particular, $\mathrm{ord}\,\gamma=-\infty$ mod $G^{\sim}_{\mathrm{cont}}$ if $2p>n+2$.

Corollary

$\dim\mathrm{CL}(\mathcal{E})>n$ iff $\mathcal{E}$ is linearizable by a contact transformation and thus $\dim\mathrm{CL}(\mathcal{E})=\infty$.

CLs of fourth-order evolution equations

$\mathcal{E}$: $\quad u_t = F(t,x,u,u_1,u_2,u_3,u_4), \quad F_{u_4} \neq 0$

- $\dim \mathrm{CL}(\mathcal{E}) \geqslant 1$: $\quad F = -\dfrac{D_x\hat\sigma^1 + \rho^1_t}{\gamma^1}$,

  $\mathrm{ord}\,\rho^1 \leqslant 2, \quad \mathrm{ord}\,\hat\sigma^1 \leqslant 3, \quad \gamma^1 = \dfrac{\delta\rho^1}{\delta u} \neq 0.$

- $\dim \mathrm{CL}(\mathcal{E}) \geqslant 2$, mod $G^\sim_{\mathrm{cont}}$: $\quad F = D_xG, \quad G = -\dfrac{D_x\check\sigma^1 + \rho^1_t}{D_x\gamma^1}$,

  $\mathrm{ord}\,\rho^1 \leqslant 1, \quad \mathrm{ord}\,\check\sigma^1 \leqslant 2, \quad \gamma^1 = \dfrac{\delta\rho^1}{\delta u}, \quad D_x\gamma^1 \neq 0 \quad (\rho^2 = u,\ \gamma^2 = 1).$

- $\dim \mathrm{CL}(\mathcal{E}) \geqslant 3$, mod $G^\sim_{\mathrm{cont}}$: $\quad F = D_x^2G, \quad G = -\dfrac{D_x\bar\sigma^1 + \rho^1_t}{D_x^2\rho^1_u}$,

  $\mathrm{ord}\,\rho^1 = 0, \quad \mathrm{ord}\,\bar\sigma^1 \leqslant 1, \quad D_x^2\rho^1_u \neq 0 \quad (\rho^2 = u,\ \gamma^2 = 1,\ \rho^3 = xu,\ \gamma^3 = x).$

- $\dim \mathrm{CL}(\mathcal{E}) \geqslant 4$, mod $G^\sim_{\mathrm{cont}}$:

  $$F = \mathrm{DT}[\gamma^1,\dots,\gamma^4]^\dagger H - \sum_{s=1}^{p} \mathrm{DT}[\gamma^1,\dots,\cancel{\gamma^s},\dots,\gamma^4]^\dagger \left(\frac{W(\gamma^1,\dots,\cancel{\gamma^s},\dots,\gamma^p)}{W(\gamma^1,\dots,\gamma^4)}\gamma^s_t u \right),$$

  $\mathrm{ord}\,\gamma^s = -\infty,\ \gamma^3 = x,\ \gamma^4 = 1,\ \mathrm{ord}\,H = 0.$

- $\dim \mathrm{CL}(\mathcal{E}) > 4$, mod $G^\sim_{\mathrm{cont}}$: $\quad \mathcal{E}$ is a linear equation $\quad\Longrightarrow\quad \dim \mathrm{CL}(\mathcal{E}) = \infty.$

Thank you for your attention!