Integrable discretizations of integrable PDE's

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Integrability of the PDE's is closely related to degeneracy of dispersion laws (E. Zakharov and E. I. Shulman (1980)). Say, integrability of the Davey–Stewartson (DSII) equation

$$\begin{aligned} & i(-1)^j\partial_t q_j + \partial_x^2 q_j - \partial_y^2 q_j + 2q_1q_2q_j + vq_j = 0, \qquad j = 1, 2, \\ & \partial_x^2 v - \partial_y^2 v = -4\partial_x^2(q_1q_2) \end{aligned}$$

follows from identity $2(z^2 + w^2) = (z - w)^2 + (z + w)^2$, where z and w are any commuting variables.

Let A, B_1 and B_2 be some arbitrary operators,

$$B = \begin{pmatrix} 0, & B_1 \\ B_2, & 0 \end{pmatrix} \text{ and } \sigma = \sigma_3 = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}.$$

Then the above identity means that we have commutator identity

$$2\sigma[A^2\sigma, B] = [A, [A, B]] + [A\sigma, [A\sigma, B]]$$

Let now B depends on three "times:" t_1, t_2, t_3 as

$$B_{t_1} = [A, B], \qquad B_{t_2} = i[A\sigma, B], \qquad B_{t_3} = i[A^2\sigma, B]$$

then B obeys: $2i\sigma B_{t_3} + \partial_{t_1}^2 B - \partial_{t_2}^2 B = 0$, i.e. linearized version of the DSII equation.

Operator realization. Let operator B be x-dependent function of operator A and impose condition, that $B_{t_1} = B_x$. Taking $B_{t_1} = [A, B]$ into account we get that

$$A = \partial_x - iz, \text{ where } z \in \mathbb{C}$$

In other words B is pseudo-differential operator

$$(Bf)(x) = \int_{\mathbb{R}} dp\, e^{ipx} B(x, i(p-z))\widetilde{f}(p), \qquad \widetilde{f}(p) = \frac{1}{2\pi}\int dx\, e^{-ipx} f(x)$$

which symbol $B(x, z)$ is defined in the complex plane of the second argument: $x \in \mathbb{R}$, $z \in \mathbb{C}$.

Correspondingly, $B_{t_2} = i[(\partial_x - iz)\sigma, B]$, $B_{t_3} = i[(\partial_x - iz)^2\sigma, B]$, so the symbol $B(t, z)$ has the form

$$B(t, z) = e^{2it_1 z_{\mathrm{Im}} + 2i\sigma t_2 z_{\mathrm{Re}} + 2i\sigma t_3 (z_{\mathrm{Im}}^2 - z_{\mathrm{Re}}^2)} b(z)$$

where $b(z)$ is off-diagonal 2×2 matrix depending on z only.

D-bar problem (dressing) Dependence of the symbol on the complex parameter z enables to associate to any operator F with symbol $F(x,z)$ its d-bar derivative: $\bar\partial F$ with symbol $\dfrac{\partial(F(x,z))}{\partial\bar z}$. Now we introduce new operator as solution of the following problem:

$$\bar\partial\nu = \nu B,$$

normalized by condition that for $z\to\infty$

$$\nu = 1 + uA^{-1} + o(z^{-1}),$$

where u is (2×2 matrix) multiplication operator. In what follows we assume unique solvability of this problem. We introduce dependence of operator ν on times by means of the known time dependence of operator B: $\bar\partial\nu(t) = \nu(t)B(t)$. Thus

$$\bar\partial\big(\nu_{t_1} + \nu A\big) = \big(\nu_{t_1} + \nu A\big)B$$

This means that $\nu_{t_1} + \nu A = (A + X)\nu$, where X is some multiplication operator. Thanks to the asymptotic condition we get $X = 0$, so that

$$\nu_{t_1} = [A, \nu]$$

Analogously we derive that

$$\nu_{t_2} = i[A\sigma, \nu] - i[\sigma, u]\nu$$

that thanks to $\nu_{t_1} = [A, \nu]$ can be written in the form $\nu_{t_2} + i\nu\sigma A = i\sigma\nu_{t_1} + i\sigma\nu A - i[\sigma, u]\nu$, that in terms of the symbols takes the form

$$\nu_{t_2}(t, z) + z\nu(t, z)\sigma = i\sigma\nu_{t_1}(t, z) + z\sigma\nu(t, z) - i[\sigma, u(t)]\nu(t, z)$$

Introducing the Jost solution as $\varphi(t, z) = \nu(t, z)e^{-izt_1 + z\sigma t_2}$ we get the two dimensional Zakharov–Shabat L-operator

$$\varphi_{t_2}(t, z) = i\sigma\, \varphi_{t_1}(t, z) - i[\sigma, u(t)]\, \varphi(t, z)$$

In the same way we get $\varphi_{t_3} = i\sigma\, \varphi_{t_1 t_1} - i\sigma\, \varphi_{t_1} + (i[\sigma, u] - i\sigma u_{t_1} - i[\sigma, w])\, \varphi$

Discretization (Darboux transformation). Let besides t-dependence

$$B_{t_1} = [A, B], \qquad B_{t_2} = i[A\sigma, B], \qquad B_{t_3} = i[A^2\sigma, B]$$

operator B depends on the discrete variable n: $B(t, n)$. Let us denote $B^{(1)}(n) = B(n+1)$ and let this dependence is given by means of the same operator $A = \partial_x - iz$ as

$$B^{(1)} = (A - a)B(A - a)^{-1}, \text{ where } a \text{ is a constant diagonal matrix.}$$

This means that with respect to n, t_1 and t_2 we have difference-differential equation:

$$B^{(1)}_{t_2} - B_{t_2} = i\sigma\big(B^{(1)}_{t_1} + B_{t_1}\big) + 2i\sigma B^{(1)}a - 2i\sigma aB$$

and t_3 is the continuous symmetry of this equation. Again we set: $\bar{\partial}\nu^{(1)} = \nu^{(1)}B^{(1)}$, so that

$$\bar{\partial}\nu^{(1)}(A - a) = \nu^{(1)}(A - a)B$$

Then we derive that $\nu^{(1)}(A - a) = (A - a + u_1)\nu$, where $u_1 = u^{(1)} - u$.

In this way we arrive to the Lax pair:

$$\varphi_{t_2} = i\sigma\, \varphi_{t_1} - i[\sigma, u]\, \varphi$$
$$\varphi^{(1)} = \varphi_{t_1} + (u_1 - a)\, \varphi$$

and equation of compatibility for $u = v^{\text{diag}} + w^{\text{antidiag}}$ is

$$v^{(1)}_{t_2} - v_{t_2} = i\sigma v^{(1)}_{t_1} + i\sigma v_{t_1} + 2i\sigma(w_1 - a)v - 2i\sigma v^{(1)}(w_1 - a)$$
$$w_{t_2} - i\sigma w_{t_1} = -2i\sigma v^2$$

Another realization of operators. It is reasonable to consider another realization of operators. Let us start with Let we have a space of infinite sequences $f = \{f(n)\}$, $n \in \mathbb{Z}$, and let T denotes shift operator: $(Tf)(n) = f(n+1)$. For any operator B in this space we introduce dependence on the discrete variable m and two "times" t_1 and t_2 by means of the same relations as above:

$$B^{(1)} = (A-a)B(A-a)^{-1}, \qquad B_{t_1} = [A, B], \qquad B_{t_2} = i[A\sigma, B], \qquad [a, \sigma] = 0$$

and we impose condition that $B^{(1)} = TBT^{-1}$. This means that now $A = zT + a$ and we consider B to be function of operator A, i.e., B is "pseudo-matrix" operator in this space:

$$(Bf)(n) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i} \zeta^{n-1} B(n, z\zeta) \widetilde{f}(\zeta), \qquad \widetilde{f}(\zeta) = \sum_{n\in\mathbb{Z}} \zeta^{-n} f(n)$$

Again we define operator ν by means of the d-bar problem

$$\bar{\partial}\nu = \nu B, \qquad \nu = 1 + uA^{-1} + o(z^{-1}), \quad z \to \infty$$

The construction analogous to the above gives:

$$\begin{aligned} \varphi_{t_1} &= \varphi^{(1)} - (u_1 - a)\,\varphi \\ \varphi_{t_2} &= i\sigma\,\varphi^{(1)} - i\big(\sigma u^{(1)} + (u+a)\sigma\big)\,\varphi \end{aligned}$$

Further discretization of this equation leads to the special case of the non-Abelian Hirota difference equation:

$$u^{(12)}(u^{(2)} - u^{(1)} + a_{12}) + a_{12}u^{(3)} + \mathrm{cycle}\{1, 2, 3\} = 0$$

Here u denotes operator $u(m_1, m_2, m_3)$ and

$$u^{(1)}(m) = u(m_1 + 1, m_2, m_3), \quad u^{(2)}(m) = u(m_1, m_2 + 1, m_3), \quad \dots,$$
$$u_i(m) = u^{(i)}(m) - u(m).$$

Constant operators a_1, a_2, a_3 mutually commute and $a_{ij} = a_i - a_j$. This equation is condition of compatibility of the system

$$\varphi^{(2)} = \varphi^{(1)} + \left(u^{(2)} - u^{(1)} + a_{12}\right)\varphi,$$
$$\varphi^{(3)} = \varphi^{(2)} + \left(u^{(3)} - u^{(2)} + a_{23}\right)\varphi,$$
$$\varphi^{(1)} = \varphi^{(3)} + \left(u^{(1)} - u^{(3)} + a_{31}\right)\varphi,$$

so the Lax pair is any two of these equations.

In our approach this equation appears as commutator identity

$$(A-a_1)(A-a_2)B(A-a_1)^{-1}(A-a_2)^{-1}a_{12}+a_{12}(A-a_3)B(A-a_3)^{-1}+\text{cycle}=0$$

where we assume that operator B acts in $V\otimes W$, and $A-a_j$ stands for $A\otimes I - I\otimes a_j$, $j=1,2,3$ where A is operator in V, and a_j in W. Introducing now evolution by means of

$$B^{(1)}=(A-a_1)B(A-a_1)^{-1},\quad B^{(2)}=(A-a_2)B(A-a_2)^{-1},$$
$$B^{(3)}=(A-a_3)B(A-a_3)^{-1}$$

we see that $B(m)$ obeys the difference equation

$$B^{(12)}a_{12}+a_{12}B^{(3)}+B^{(23)}a_{23}+a_{23}B^{(1)}+B^{(31)}a_{31}+a_{31}B^{(2)}=0, \quad ,$$

or

$$B_{12}a_{12}+[a_{12},B_3]+B_{23}a_{23}+[a_{23},B_1]+B_{31}a_{31}+[a_{31},B_1]=0.$$

Dressing procedure of the type described above gives the Lax pair, as well as the Hirota difference equation.

Limiting cases. The original linear equation

$$B^{(12)}a_{12}+a_{12}B^{(3)}+B^{(23)}a_{23}+a_{23}B^{(1)}+B^{(31)}a_{31}+a_{31}B^{(2)}=0,\quad a_{ij}=a_i-a_j,$$

and evolutions become trivial in two cases: if $a_k\to\infty$ for some k, or if $a_j=a_i$ for some $j\neq i$. Here we consider the first case: we substitute $a_k\to xa_k$, where x is c-number anad then for $x\to\infty$ we get

$$B^{(k)}=a_k\left[B-\frac{1}{x}\partial_{t_k}B\right]a_k^{-1}+\ldots,\quad x\to\infty$$

where $\partial_{t_k}B=[Aa_k^{-1},B]$.

Limit $a_3 \to \infty$, $a_2 \to \infty$. The identity takes the form:

$$(a_2 B a_2^{-1} - a_3 B a_3^{-1})^{(1)} a_1 + (a_2 B_{t_2} - a_3 B_{t_3})^{(1)} - \\ - a_2 a_3 B_{t_2} a_3^{-1} + a_3 a_2 B_{t_3} a_2^{-1} - a_1 a_2 B a_2^{-1} + a_1 a_3 B a_3^{-1} = 0,$$

that is antisymmetric with respect to indexes 2 and 3. Now we have

$$B^{(1)} = T_1 B T_1^{-1}, \quad B_{t_2} = [(T_1 + a_1) a_2^{-1}, B], \quad B_{t_3} = [(T_1 + a_1) a_3^{-1}, B],$$

Substitution: $v(m_1, t_2, t_3) = u(m_1, t_2, t_3) - m_1 a$, Lax pair

$$\alpha \psi_{t_2} = \psi^{(1)} + [\alpha w \alpha^{-1} - w^{(1)}]\psi, \\ \alpha^{-1} \psi_{t_3} = \psi^{(1)} + [\alpha^{-1} w \alpha - w^{(1)}]\psi,$$

and equation:

$$\left(w\alpha - \alpha w^{(1)}\right)_{t_2} - \left(w\alpha^{-1} - \alpha^{-1} w^{(1)}\right)_{t_3} + [w\alpha - \alpha w^{(1)}, w\alpha^{-1} - \alpha^{-1} w^{(1)}] = 0.$$

where α is a constant operator.

Limit $a_3 \to \infty$. The $1/x$ term gives identity

$$B^{(12)}a_{12} + a_3(B^{(2)} - B^{(1)})_{t_3} + a_2 B^{(1)} - a_1 B^{(2)} + \\ + a_3 B^{(2)} a_3^{-1} a_2 - a_3 B^{(1)} a_3^{-1} a_1 + a_{12} a_3 B a_3^{-1} = 0,$$

where

$$B^{(1)} = T_1 B T_1^{-1}, \quad B^{(2)} = (T_1 + a_{12}) B (T_1 + a_{12})^{-1}, \quad B_{t_3} = [(T_1 + a_1) a_3^{-1}, B],$$

so that $\overline{\partial}(\nu_{t_3} + \nu(T_1 + a_1)a_3^{-1}) = (\nu_{t_3} + \nu(T_1 + a_1)a_3^{-1})B$. Thus again taking asymptotic into account we derive: $\nu_{t_3} + \nu(T_1 + a_1)a_3^{-1} = a_3^{-1}(T_1 + a_3 u a_3^{-1} - u^{(1)} + a_1)\nu$. Finally for $w(m_1, m_2, t_3) = u(m_1, m_2, t_3) - m_1 a_1 - m_2 a_2$ we get Lax pair and evolution equation

$$\begin{aligned} &\psi_{t_3} = \psi^{(1)} - w_1 \psi, \\ &\psi^{(2)} = \psi^{(1)} + (w_2 - w_1)\psi, \\ &(w_2 - w_1)_{t_3} + w_{12}(w_2 - w_1) + [w_1, w_2] = 0. \end{aligned}$$

Highest Hirota difference equations. If we introduce, say, 4 discrete variables by means of

$$B(m_1, m_2, m_3, m_4) = (A - a_1)^{m_1}(A - a_2)^{m_2}(A - a_3)^{m_3}(A - a_4)^{m_4} B(\ldots)^{-1}$$

then this function obeys linearized Hirota equation with respect to any 3 of variables $m_1, \ldots, m_4$. But we can introduce higher evolutions:

$$B(m_1, m_2, m_3) = (A - a_1)^{m_1}(A - a_2)^{m_2}(A^2 - a_3^2)^{m_3} B(\ldots)^{-1}$$

In this way we get linear equation

$$\begin{aligned} &\Delta_3\big((a_1 + a_3)\Delta_1 - (a_2 + a_3)\Delta_2\big)\big((a_1 - a_3)\Delta_1 - (a_2 - a_3)\Delta_2\big) = \\ &= (a_1 - a_2)\Delta_1\Delta_2\big((a_1 - a_2)\Delta_1\Delta_2 + 2a_1\Delta_1 - 2a_2\Delta_2\big), \qquad \Delta_j B = B^{(j)} - B \end{aligned}$$

and Lax pair

$$\begin{aligned} \varphi^{(2)} &= \varphi^{(1)} + (u^{(2)} - u^{(1)})\,\varphi \\ \varphi^{(3)} &= \varphi^{(11)} + (u^{(11)} - u^{(3)})\,\varphi^{(1)} + w\,\varphi \end{aligned}$$

Equation:

$$(u^{(2)} - u^{(1)})^{(3)} w = w^{(2)}(u^{(2)} - u^{(11)})$$

$$w^{(2)} - w^{(1)} = (u^{(11)} - u^{(3)})^{(2)}(u^{(2)} - u^{(1)})^{(1)} - (u^{(2)} - u^{(1)})^{(3)}(u^{(11)} - u^{(3)})$$