# Homogeneous geodesics in sub-Riemannian geometry

Alexey Podobryaev
A. K. Ailamazyan Program Systems Institute of RAS
Pereslavl-Zalesskiy

Alexandre Vinogradov Memorial Conference
Moscow
December 17, 2021

# Plan

1. What sub-Riemannian geometry?
2. Homogeneous geodesics.
3. A criterion for homogeneous geodesics.
4. Examples.
5. Existence of homogeneous geodesics.
6. Geodesic orbit sub-Riemannian structures.
7. Integrability of a geodesic flow.

# A sub-Riemannian structure on a manifold $M$

Let $\Delta \subset TM$ be a distribution of $r$-dimensional subspaces equipped with a scalar product $B(\cdot,\cdot)$.

## Definition

A curve $\gamma : [0, T] \to M$ is called *an admissible curve* if

$$\dot{\gamma}(t) \in \Delta_{\gamma(t)} \qquad \text{a.e.}$$

## Definition

*The sub-Riemannian length* of an admissible curve $\gamma$ is

$$\int_0^T \sqrt{B(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt.$$

Remark. For $r = \dim M$ we get a Riemannian structure.

The goal is to describe the shortest arcs.

# An optimal control problem

Let $X_1, \dots, X_r$ be a orthonormal frame for distribution $\Delta$ with respect to $B(\,\cdot\,,\,\cdot\,)$.
The problem is to find controls $u_1, \dots, u_r \in L^\infty([0,T],\mathbb{R})$ and a curve $\gamma : [0,T] \to M$ such that

$$\dot\gamma(t) = u_1(t)X_1(\gamma(t)) + \cdots + u_r(t)X_r(\gamma(t)) \qquad \text{a.e.},$$

$$\int_0^T \sqrt{u_1^2(t) + \cdots + u_r^2(t)}\, dt \to \min.$$

# The energy functional

$$\int_0^T \sqrt{u_1^2(t) + \cdots + u_r^2(t)}\, dt \to \min \;\Leftrightarrow$$

$$\Leftrightarrow\; J = \frac{1}{2} \int_0^T (u_1^2(t) + \cdots + u_r^2(t))\, dt \to \min\,.$$

# Existence of solutions

**Theorem (Rashevskiy, Chow)**

*If $M$ is connected and* $\operatorname{span}\{\Delta_m^1, \Delta_m^2, \dots\} = T_m M$ *for any* $m \in M$*, where*

$$\Delta^1 = \Delta, \quad \Delta^k = \Delta^{k-1} + [\Delta, \Delta^{k-1}],$$

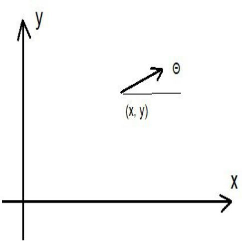
*then there exists an admissible curve connecting any two given points.*

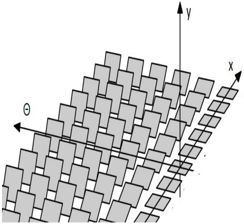
**Theorem (Filippov)**

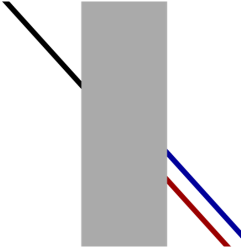
*The reachable set is compact under some broad conditions.*

# Example. Sub-Riemannian structure on the group of isometries of a plane.

The bundle of unit tangent vectors on a plane. A model of a car.



$$\left\{\begin{array}{rcl} \dot{x} & = & u_1 \cos\theta, \\ \dot{y} & = & u_1 \sin\theta, \\ \dot{\theta} & = & u_2, \end{array}\right. \qquad \int_0^T u_1^2(t) + u_2^2(t)\, dt \to \min .$$

## Sub-Riemannian geodesics

Consider functions that are linear on the fibers of the cotangent bundle

$$h_i : T^*M \to \mathbb{R}, \quad h_i(\lambda) = \langle X_i(\pi(\lambda)), \lambda \rangle, \quad \lambda \in T^*M,$$

where $\pi : T^*M \to M$.
Introduce a family of functions on $T^*M$

$$H_u^\nu = u_1(t)h_1 + \dots u_r(t)h_r - \frac{\nu}{2}(u_1^2(t) + \dots + u_r^2(t)).$$

**Theorem (Pontryagin maximum principle)**

*If $\tilde{u} : [0, T] \to U \subset \mathbb{R}^k$ is an optimal control and $\tilde{\gamma} : [0, T] \to M$ is a shortest arc, then there exists a Lipschitz curve $\lambda : [0, T] \to T^*M$ and $\nu \geqslant 0$ such that*
(1) $\pi(\lambda(t)) = \tilde{\gamma}(t), t \in [0, T]$;
(2) $\dot{\lambda}(t) = \vec{H}^\nu_{\tilde{u}(t)}(\lambda(t))$;
(3) $H^\nu_{\tilde{u}(t)}(\lambda(t)) = \max_{u \in U} H^\nu_u(\lambda(t))$ *for a.e.* $t \in [0, T]$;
(4) $(\lambda(t), \nu) \neq 0$.

# Sub-Riemannian geodesics

We will consider the normal case $\nu \neq 0$.
The maximized Hamiltonian is quadratic $H = \frac{1}{2}(h_1^2 + \cdots + h_r^2)$.
The trajectories of the Hamiltonian vector field $\vec{H}$ project to *geodesics* (their small arcs are optimal).
Any normal geodesic is defined by its initial momentum.

# Invariant case

Assume that a Lie group $G$ acts on $M$ transitively and a sub-Riemannian structure is $G$-invariant. Let $K$ be a stabilizer of a point $o \in M$.

Consider a lift of our problem to the group $G$

$$\gamma(0) \in K, \qquad \gamma(T) \in gK.$$

The transversality condition of the Pontryagin maximum principle: $\lambda \in (T_g gK)^\circ$, where $g = \pi(\lambda)$.

A trivialisation via group action: $T^*G = \mathfrak{g}^* \times G \supset \mathfrak{k}^\circ \times G$, where

$$\mathfrak{k}^\circ = (\mathfrak{g}/\mathfrak{k})^* = \mathfrak{m}^*.$$

# Normal case for an invariant sub-Riemannian structure

Normal case ($\nu = 1$). Maximized Hamiltonian
$H = \frac{1}{2}(h_1^2 + \dots h_r^2) \in C^\infty(\mathfrak{m}^*)$.
The Hamiltonian system is

$$\begin{cases} \dot{g} & = & g \circ d_p H, \\ \dot{p} & = & (\mathrm{ad}^* \, d_p H) p, \end{cases}$$

where $g \in G$, $p \in \mathfrak{m}^*$.

The vertical part of the Hamiltonian system is independent.

# Homogeneous geodesics

Let $(M, \Delta, B)$ be a sub-Riemannian manifold.
Let $G \subset \mathrm{Isom}\, M$ be a closed subgroup of isometries. Assume that $G$ acts on $M$ transitively and effectively.
Let $K \subset G$ be an isotropy subgroup for a point $o \in M$. So, $M = G/K$.
Notice that $K$ is compact and there is an $\mathrm{Ad}\, K$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m} = T_o M$.

**Definition**

A geodesic $\gamma : [0, T] \to M$ passing through the point $o$ is called *a homogeneous geodesic* if $\gamma(t) = \exp(tX)o$ for same $X \in \mathfrak{g}$.

# Some properties of homogeneous geodesics

1. A simple parametrization.
2. The cut time (the time of loss of optimality) is independent on a starting point on a geodesic.

**Question.** How many homogeneous geodesics could be there?

# A criterion for homogeneous geodesic in the Riemannian case

**Theorem (Geodesic Lemma (Kowalski, Vanhecke))**

*A curve* $\exp(tX)o$ *is a homogeneous geodesic iff*

$$B(X_{\mathfrak{m}}, [X, \mathfrak{g}]_{\mathfrak{m}}) = 0,$$

*where* $X_{\mathfrak{m}}$ *is a* $\mathfrak{m}$*-component of* $X$.

**Example**

If $M$ is a compact Lie group and a metric $B$ is defined by the Killing form, then any geodesic is homogeneous. The form $B$ is bi-invariant in this case. We have $M = G \times G/G$ as a homogeneous space.

# A criterion for homogeneous geodesic in the sub-Riemannian case

A geodesic is defined by its initial covector (instead of initial vector in the Riemannian case).

**Theorem**

*The following conditions are equivalent:*
(1) *A geodesic with an initial momentum $p$ is homogeneous.*
(2) *There exists $X \in \mathfrak{g}$ such that*

$$p([X, \mathfrak{g}]) = 0 \qquad \textit{and} \qquad X_{\mathfrak{m}} = d_p H.$$

(3) *The trajectory of the vertical part of the Hamiltonian vector field passing through the point $p$ lies in $(\mathrm{Ad}^* K)$-orbit.*

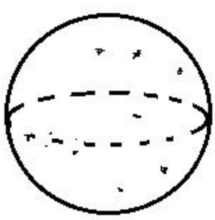
# Examples

**Example**

Consider a Riemannian metric on $\mathrm{SO}_3$ with eigenvalues $I_1, I_2, I_3 > 0$.

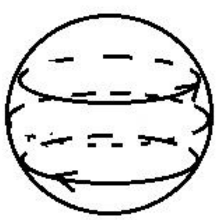
The vertical part of the Hamiltonian vector field on the level surface $H = \frac{1}{2}$.

$I_1 = I_2 = I_3$



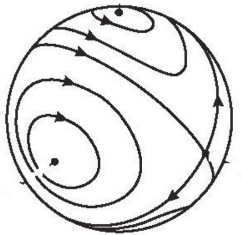
Any geodesic is homogeneous, $M = \mathrm{SO}_3 \times \mathrm{SO}_3/\mathrm{SO}_3$.

$I_1 = I_2 \neq I_3$



Any geodesic is homogeneous, $M = \mathrm{SO}_3 \times \mathrm{SO}_2/\mathrm{SO}_2$.

$I_1 \neq I_2 \neq I_3 \neq I_1$



Six homogeneous geodesics.

# Example

## Example

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. The distribution $\Delta$ is generated by $\mathfrak{p}$ and $B$ is a restriction of the Killing form to $\mathfrak{p}$. The any geodesic is homogeneous (Agrachev, Brockett, Kupka, Jurdjevic).

$$\gamma(t) = \exp\left(t(X+Y)\right)\exp\left(-tY\right),$$

where $X \in \mathfrak{p}$ and $Y \in \mathfrak{k}$.

## Existence of homogeneous geodesics

**Theorem**

*Let $\mathcal{K}$ be the Killing form of the Lie algebra $\mathfrak{g}$. If $\mathrm{Ker}\,\mathcal{K} = \mathfrak{m}$ or $\mathcal{K}|_{\Delta} \neq 0$, then there exists a homogeneous geodesic passing through the point $o \in M$.*

Kowalski, Szenthe: Existence of homogeneous geodesics for Riemannian manifolds.

# Geodesic orbit sub-Riemannian manifolds

**Definition**

A sub-Riemannian manifold is *geodesic orbit* if any geodesic is homogeneous.

**Proposition**

*A sub-Riemannian manifold is geodesic orbit iff*

$$\{H, \mathbb{R}[\mathfrak{m}^*]^K\} = 0,$$

*where* $\{\cdot\,,\,\cdot\}$ *is the Poisson bracket,* $H$ *is the normal Hamiltonian of the Pontryagin maximum principle, and* $\mathbb{R}[\mathfrak{m}^*]^K$ *is an algebra of left-invariant polynomial functions on* $T^*M$ *(i.e., the algebra of* $\mathrm{Ad}^*\,K$*-invariant functions on* $\mathfrak{m}^*$*).*

# Geodesic orbit sub-Riemannian manifolds

## Corollary

*If the algebra of left-invariant polynomial functions is commutative with respect to the Poisson bracket, then a sub-Riemannian structure is geodesic orbit. In particular, sub-Riemannian weakly symmetric spaces are geodesic orbit.*

## Definition

A homogeneous space is called *weakly symmetric* if for any two points there exists an isometry that replace these points one with another.

## Example

Selberg's original example $M = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{SO}_2/\mathrm{SO}_2$. The sub-Riemannian structure models a car on a hyperbolic plane.

## Integrability in non-commutative sense

**Definition**

A Poisson algebra $\mathcal{F}$ on $T^*M$ is called *complete* if

$$\dim \operatorname{span}\{d_x f \mid f \in \mathcal{F}\} + \dim \operatorname{Ker}\{\,\cdot\,,\,\cdot\,\}|_{\mathcal{F}} = \dim T^*M.$$

**Definition**

A Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ is called *integrable in non-commutative sense* if $\{H, \mathcal{F}\} = 0$ for some complete algebra $\mathcal{F}$.

# Integrability in non-commutative sense

A generalization of Jovanovic's result to the sub-Riemannian case.

**Theorem**

*If a sub-Riemannian structure is geodesic orbit, then the corresponding geodesic flow is integrable in non-commutative sense.*

Indeed, take $\mathcal{F} = \mathbb{R}[\mathfrak{m}^*]^K + \mu^*(\mathbb{R}[\mathfrak{g}^*])$, where $\mu : T^*M \to \mathfrak{g}^*$ is the momentum map. This is a complete algebra. The normal sub-Riemannian Hamiltonian $H \in \mathbb{R}[\mathfrak{m}^*]^K$. Since any geodesic is homogeneous, we have $\{H, \mathbb{R}[\mathfrak{m}^*]^K\} = 0$. Notice that $\{\mathbb{R}[\mathfrak{m}^*]^K, \mu^*(\mathbb{R}[\mathfrak{g}^*])\} = 0$. It follows that $\{H, \mathcal{F}\} = 0$.

# Free Carnot groups

Consider a free nilpotent Lie algebra of step $s$ and rank $r$:

$$\mathfrak{g} = \bigoplus_{m=1}^{s} \mathfrak{g}_m, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \qquad \mathfrak{g}_k = 0 \quad \text{for} \quad k > s.$$

Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$ and $\dim g_1 = r$.
The corresponding connected and simply connected Lie group is called *a free Carnot group*. Consider a sub-Riemannian structure with distribution generated by $\mathfrak{g}_1$.

There is a nilpotent approximation of sub-Riemannian problems (Agrachev, Sarychev).

# Any Carnot group of step 2 is geodesic orbit

**Example**

Consider a two step free Carnot group $G = V \times \Lambda^2 V$ of rank $r = \dim V$. Multiplication rule:

$$(x_1, \omega_1) \cdot (x_2, \omega_2) = (x_1 + x_2,\ \omega_1 + \omega_2 + x_1 \wedge x_2),$$

$$\text{where} \quad x_1, x_2 \in V, \quad \omega_1, \omega_2 \in \Lambda^2 V.$$

The tangent algebra $\mathfrak{g} = V \oplus \Lambda^2 V$, the distribution $\Delta = V \oplus 0$.
The vertical part of Hamiltonian vector field (Rizzi-Serres):

$$\dot{p} = \varrho p, \quad \dot{\varrho} = 0, \quad \text{where} \quad (p, \varrho) \in V^* \oplus \Lambda^2 V^* = V^* \oplus \mathfrak{so}(V).$$

$\mathrm{Isom}\, G = G \rtimes \mathrm{SO}(V)$ (Kivioja, Le Donne).

## Carnot groups of step more than 2

**Theorem**

*Carnot groups of step more than 2 could not be geodesic orbit.*

The generalization of C. Gordon's result obtained for Riemannian nilpotent manifolds.

# Integrability of the geodesic flow on Carnot groups

Step 1. Euclidian geometry.

Step 2. Geodesic flow is integrable in elementary functions.

Step 3, rank 2. Geodesic flow is integrable in elliptic functions (Sachkov).

Step 3, rank $\geqslant 3$. Geodesic flow is not Liouville integrable (numerically shown by Bizyaev, Borisov, Kilin, Mamaev).

Step $\geqslant 4$. Geodesic flow is not Liouville integrable (proved by Lokutsievskii, Sachkov).

Thank you!