New Solution for the DKP

Maxim V. Pavlov

Lebedev Physical Institute

13.03.2019

The DKP

The Dispersionless limit of the Kadomtsev–Petviashvili equation

$$u_{yy} = (u_t - uu_x)_x$$

The DKP

The Dispersionless limit of the Kadomtsev–Petviashvili equation

$$u_{yy} = (u_t - uu_x)_x$$

can be written as a quasilinear system of first order

$$u_y = v_x, \quad u_t = uu_x + v_y$$

The Dispersionless limit of the Kadomtsev–Petviashvili equation

$$u_{yy} = (u_t - uu_x)_x$$

can be written as a quasilinear system of first order

$$u_y = v_x, \quad u_t = uu_x + v_y$$

determined by the Lax pair written in the vector field form

$$\lambda_y = p\lambda_x - \lambda_p u_x, \quad \lambda_t = (p^2 + u)\lambda_x - \lambda_p(pu_x + v_x).$$

Here $\lambda(x, t, y, p)$, while the functions $u(x, t, y)$ and $v(x, t, y)$.

The Manakov–Santini Particular Solution

The DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution

$$x - \frac{y^2}{4t} + 2tu = F(t^{1/2}u),$$

where $F(W)$ is an arbitrary function.

The Manakov–Santini Particular Solution

The DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution

$$x - \frac{y^2}{4t} + 2tu = F(t^{1/2}u),$$

where $F(W)$ is an arbitrary function.
The **Problem**:
How to select this particular solution?

The Manakov–Santini Particular Solution

The DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution

$$x - \frac{y^2}{4t} + 2tu = F(t^{1/2}u),$$

where $F(W)$ is an arbitrary function.
The **Problem**:
How to select this particular solution?
How to generalise this particular solution?

The Standard Approach

The Method of Hydrodynamic reductions means that we are looking for pairs of commuting flows (N is an arbitrary natural number)

$$r^i_t = \eta^i(\mathbf{r}) r^i_x, \quad r^i_y = \mu^i(\mathbf{r}) r^i_x, \quad i = 1, 2, ..., N,$$

which have the pair of common conservation laws (the DKP equation)

$$u_y = v_x, \quad u_t = uu_x + v_y.$$

This means that $u(x, t, y) \to u(\mathbf{r})$ and $\lambda(x, t, y, p) \to \lambda(p, \mathbf{r})$.

The Standard Approach

The Method of Hydrodynamic reductions means that we are looking for pairs of commuting flows (N is an arbitrary natural number)

$$r_t^i = \eta^i(\mathbf{r}) r_x^i, \quad r_y^i = \mu^i(\mathbf{r}) r_x^i, \quad i = 1, 2, ..., N,$$

which have the pair of common conservation laws (the DKP equation)

$$u_y = v_x, \quad u_t = u u_x + v_y.$$

This means that $u(x, t, y) \longrightarrow u(\mathbf{r})$ and $\lambda(x, t, y, p) \longrightarrow \lambda(p, \mathbf{r})$.
Then $\eta^i(\mathbf{r}) = (\mu^i(\mathbf{r}))^2 + u(\mathbf{r})$, where the functions $\mu^i(\mathbf{r})$ and $u(\mathbf{r})$ can be found from the Gibbons–Tsarev system

$$\partial_i \mu^k = \frac{\partial_i u}{\mu^i - \mu^k}, \quad \partial_{ik} u = 2 \frac{\partial_i u \cdot \partial_k u}{(\mu^i - \mu^k)^2}, \quad i \neq k.$$

The Standard Approach

The Gibbons–Tsarev system

$$\partial_i \mu^k = \frac{\partial_i u}{\mu^i - \mu^k}, \quad \partial_{ik} u = 2\frac{\partial_i u \cdot \partial_k u}{(\mu^i - \mu^k)^2}, \quad i \neq k$$

is determined by the Löwner system

$$\partial_i \lambda = \frac{\partial_i u}{p - \mu^i} \partial_p \lambda.$$

The Standard Approach

The Gibbons–Tsarev system

$$\partial_i \mu^k = \frac{\partial_i u}{\mu^i - \mu^k}, \quad \partial_{ik} u = 2\frac{\partial_i u \cdot \partial_k u}{(\mu^i - \mu^k)^2}, \quad i \neq k$$

is determined by the Löwner system

$$\partial_i \lambda = \frac{\partial_i u}{p - \mu^i} \partial_p \lambda.$$

Its simplest solutions are polynomial:

$$\lambda = \frac{p^{N+1}}{N+1} + up^{N-1} + vp^{N-2} + a_1 p^{N-3} \ldots + a_{N-3} p + a_{N-4},$$

where the functions $a_k(x, t, y)$ are $a_k(\mathbf{r})$ in this approach.

Applications

Corresponding equation of the Riemann surface $\lambda(p; \mathbf{r})$ plays an important role in:
hydrodynamics (the Dirichlet problem for Laplace's equation),
plasma physics (the distribution function in the Vlasov kinetic equation associated with the Benney system),
nonlinear optics (dispersionless limit of the Kadomtsev–Petviashvili equation),
nonlinear acoustic (Khokhlov–Zabolotzkaya equation),
aerodynamics (Lin–Reissner–Tsien equation),
the theory of Laplacian Growth (the Hele-Shaw equations),
the Topological Field Theory (the so called superpotential in WDVV associativity equations, Frobenius manifolds),
Classical Mechanics (the first integral for Hamiltonian systems with one-and-a-half degree of freedom).

V.M. Teshukov

The Vlasov (collisionless) kinetic equation associated with the Benney system was investigated by V.M. Teshukov

$$\lambda_y = p\lambda_x - \lambda_p h_x, \quad h = \int\limits_{u_-}^{u_+} \lambda dp,$$

where horisontal velocities $u_\pm$ satisfy the two auxiliary equations

$$u_t + uu_x + h_x = 0.$$

Special solutions selected by polynomials of second and third order ($N = 1, 2$)

$$\lambda = \frac{p^{N+1}}{N+1} + up^{N-1} + vp^{N-2} + a_1 p^{N-3} ... + a_{N-3} p + a_{N-4}$$

were considered by V.M. Teshukov and A.A. Chesnokov.

The Ansatz

The DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

is determined by the Lax pair

$$\lambda_y = p\lambda_x - \lambda_p u_x, \quad \lambda_t = (p^2 + u)\lambda_x - \lambda_p(pu_x + v_x).$$

The Ansatz

The DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

is determined by the Lax pair

$$\lambda_y = p\lambda_x - \lambda_p u_x, \quad \lambda_t = (p^2 + u)\lambda_x - \lambda_p(pu_x + v_x).$$

The Manakov–Santini particular solution (here $F(W)$ is an arbitrary function)

$$x - \frac{y^2}{4t} + 2tu = F(t^{1/2}u)$$

The Ansatz

The DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

is determined by the Lax pair

$$\lambda_y = p\lambda_x - \lambda_p u_x, \quad \lambda_t = (p^2 + u)\lambda_x - \lambda_p(pu_x + v_x).$$

The Manakov–Santini particular solution (here $F(W)$ is an arbitrary function)

$$x - \frac{y^2}{4t} + 2tu = F(t^{1/2}u)$$

is selected by the substitution

$$\lambda = kp^2 + lp + m,$$

where $k(x,t,y), l(x,t,y), m(x,t,y)$ are some functions.

The Ansatz

Theorem: *The DKP equation*

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution

$$u(x,t,y)=\frac{1}{k(t)Q(t)}W\text{-}\left(\frac{Q'(t)}{Q(t)}+\frac{k'(t)}{2k(t)}\right)z\text{-}\frac{k''(t)}{4k(t)}y^2\text{-}\frac{L'(t)}{2k(t)}y+D(t)\text{-}\frac{M(t)}{2k(t)},$$

where $k(t), L(t), C(t), M(t), F(W)$ *are arbitrary functions*;

The Ansatz

Theorem: *The DKP equation*

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution

$$u(x,t,y) = \frac{1}{k(t)Q(t)}W - \left(\frac{Q'(t)}{Q(t)} + \frac{k'(t)}{2k(t)}\right)z - \frac{k''(t)}{4k(t)}y^2 - \frac{L'(t)}{2k(t)}y + D(t) - \frac{M(t)}{2k(t)},$$

where $k(t), L(t), C(t), M(t), F(W)$ *are arbitrary functions;*
Here the unknown function W *is a solution of the algebraic equation*

$$\frac{z}{Q(t)(k(t))^{1/2}} - \frac{W}{Q(t)} = F(W) + N(t),$$

while

$$z(x,t,y) = x - \frac{k'(t)}{4k(t)}y^2 - \frac{L(t)}{2k(t)}y + C(t).$$

The Ansatz

Three functions $D(t), Q(t)$ and $N(t)$ are determined by

$$Q'(t) = (k(t))^{-3/2},$$

$$N'(t) = \left(C'(t) - \frac{L^2(t)}{4k^2(t)} + \frac{M(t)}{2k(t)} - D(t)\right)\frac{1}{Q(t)(k(t))^{1/2}},$$

$$D(t) = \frac{L^2(t)}{8k^2(t)} + \left(\frac{k'(t)}{2k(t)} + \frac{(k(t))^{-3/2}}{Q(t)}\right)C(t)$$

$$+\frac{1}{2k(t)Q(t)}\int\left(\frac{M(t)}{(k(t))^{3/2}} - \frac{3L^2(t)}{4(k(t))^{5/2}}\right)dt,$$

while the functions l and m are determined by

$$l = k'(t)y + L(t),$$

$$m = k'(t)x + \frac{1}{2}k''(t)y^2 + L'(t)y + M(t) + 2k(t)u.$$

The Simple Particular Case

If $k(t) = y^\beta, L(t) = 0, C(t) = 0, M(t) = 0$, then ($\beta$ is an arbitrary constant)

$$Q'(t) = t^{-3\beta/2}, \quad Q(t) = \frac{2}{2-3\beta} t^{1-3\beta/2}.$$

Thus, the DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution

$$u(x,t,y) = \frac{2-3\beta}{2} W t^{\beta/2-1} + (\beta - 1)\frac{x}{t} - \frac{\beta(\beta-1)}{2t^2} y^2,$$

where $F(W)$ is an arbitrary function, and the unknown function W is a solution of the algebraic equation

$$\left(x - \frac{\beta}{4y} y^2\right) t^{-\beta/2} = W + \frac{2}{2-3\beta} t^{1-3\beta/2} F(W).$$

The Simple Particular Case

This solution of the DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

also can be re-written in the equivalent form

$$u(x,t,y) = -Ut^{-\beta} - \frac{\beta}{2t}x + \frac{\beta(2-\beta)}{8t^2}y^2,$$

where $U(x,t,y)$ is a solution of the algebraic equation (here $U = F(W)$ and $W = G(U)$)

$$\frac{2}{2-3\beta}t^{1-3\beta/2}U + G(U) = \left(x - \frac{\beta}{4y}y^2\right)t^{-\beta/2}.$$

The Exceptional Case

If $\beta = 2/3$, then the corresponding particular solution becomes (here $Q(t) = \ln t$)

$$u(x,t,y) = \frac{t^{-2/3}}{\ln t} W - \left(\frac{1}{\ln t} + \frac{1}{3}\right)\frac{x}{t} + \left(\frac{1}{\ln t} + \frac{2}{3}\right)\frac{y^2}{6t^2},$$

where the unknown function W is a solution of the algebraic equation

$$\left(x - \frac{y^2}{6t}\right) t^{-1/3} = F(W)\ln t + W.$$

The Exceptional Case

If $\beta = 2/3$, then the corresponding particular solution becomes (here $Q(t) = \ln t$)

$$u(x,t,y) = \frac{t^{-2/3}}{\ln t} W - \left(\frac{1}{\ln t} + \frac{1}{3} \right) \frac{x}{t} + \left(\frac{1}{\ln t} + \frac{2}{3} \right) \frac{y^2}{6t^2},$$

where the unknown function W is a solution of the algebraic equation

$$\left(x - \frac{y^2}{6t} \right) t^{-1/3} = F(W) \ln t + W.$$

This solution also can be re-written in the equivalent form

$$u(x,t,y) = -Ut^{-2/3} - \frac{x}{3t} + \frac{y^2}{9t^2},$$

where $U(x,t,y)$ is a solution of the algebraic equation (here $U = F(W)$ and $W = G(U)$)

$$U \ln t + G(U) = \left(x - \frac{y^2}{6t} \right) t^{-1/3}.$$

The Manakov–Santini Particular Solution

If $\beta = 1$, then $k(t) = t$, and

$$Q'(t) = t^{-3/2}, \quad Q(t) = -2t^{-1/2}.$$

Thus, the DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution (here again $L(t) = 0$, $C(t) = 0$, $M(t) = 0$)

$$u = \frac{y^2}{8t^2} - \frac{x}{2t} + \frac{1}{t} F(t^{1/2} u), \quad \leftrightarrow \quad u = t^{-1/2} G\left(tu + \frac{x}{2} - \frac{y^2}{8t}\right).$$

Here $F(W) = U$ and $G(U) = W$ are arbitrary functions.

The Manakov–Santini Particular Solution

If $\beta = 1$, then $k(t) = t$, and

$$Q'(t) = t^{-3/2}, \quad Q(t) = -2t^{-1/2}.$$

Thus, the DKP equation

$$u_{yy} = (u_t - uu_x)_x$$

has a particular solution (here again $L(t) = 0, C(t) = 0, M(t) = 0$)

$$u = \frac{y^2}{8t^2} - \frac{x}{2t} + \frac{1}{t}F(t^{1/2}u), \quad \leftrightarrow \quad u = t^{-1/2}G\left(tu + \frac{x}{2} - \frac{y^2}{8t}\right).$$

Here $F(W) = U$ and $G(U) = W$ are arbitrary functions.
In this particular case the reduction is

$$\lambda = p^2 t + py + x + 2tu.$$

The Non-Isospectral DKP equation

The non-isospectral DKP equation

$$u_{xt} = yuu_{xx} + yu_x^2 + 2u_y + yu_{yy} + \frac{2}{3}xu_{xy}$$

is determined by the Lax pair

$$\lambda_y = p\lambda_x - \lambda_p u_x,$$

$$\lambda_t - \left(p^2 y + \frac{2}{3}px + yu\right)\lambda_x + \left(\frac{1}{3}p^2 + pyu_x + u + yu_y + \frac{2}{3}xu_x\right)\lambda_p = 0.$$

Here $\lambda(x, t, y, p)$, while the functions $u(x, t, y)$.

The Non-Isospectral DKP equation

The non-isospectral DKP equation

$$u_{xt} = yuu_{xx} + yu_x^2 + 2u_y + yu_{yy} + \frac{2}{3}xu_{xy}$$

is determined by the Lax pair

$$\lambda_y = p\lambda_x - \lambda_p u_x,$$

$$\lambda_t - \left(p^2 y + \frac{2}{3}px + yu\right)\lambda_x + \left(\frac{1}{3}p^2 + pyu_x + u + yu_y + \frac{2}{3}xu_x\right)\lambda_p = 0.$$

Here $\lambda(x,t,y,p)$, while the functions $u(x,t,y)$.
We again select a new solution determined by the substitution

$$\lambda = kp^2 + lp + m,$$

where $k(x,t,y), l(x,t,y), m(x,t,y)$ are some functions.

The Non-Isospectral DKP equation

Theorem: *The non-isospectral DKP possesses the particular solution*

$$u=\frac{y^{-2/3}}{g(t)}W+\frac{1}{9y^2}x^2-\frac{1}{2y}\frac{g'(t)}{g(t)}x+\frac{9g'^2(t)}{8g^2(t)}-\frac{9g''(t)}{4g(t)}+C(y,t),$$

where W is a solution of the algebraic equation

$$F(W)-W\int\frac{dt}{[g(t)]^{3/2}}+\int\left(\frac{k(t)}{2[g(t)]^{3/2}}-\frac{3h^2(t)}{8[g(t)]^{5/2}}\right)dt$$

$$=xy^{-1/3}(g(t))^{-1/2}-\frac{9}{4}\frac{g'(t)}{(g(t))^{3/2}}y^{2/3}-\frac{3}{2}\frac{h(t)}{(g(t))^{3/2}}y^{1/3},$$

and

$$C(y,t)=\left(\frac{3g'(t)h(t)}{4g^2(t)}-\frac{3h'(t)}{2g(t)}\right)y^{-1/3}+\left(\frac{h^2(t)}{8g^2(t)}-\frac{k(t)}{2g(t)}\right)y^{-2/3}.$$

This particular solution depends on four arbitrary functions of a single variable: $g(t), h(t), k(t)$ and $F(W)$.

The Manakov–Santini Particular Solution

In this particular case the reduction is

$$\lambda = kp^2 + lp + m,$$

where

$$k = g(t)y^{2/3}, \quad l = 3g'(t)y^{2/3} + \frac{2x}{3}g(t)y^{-1/3} + h(t)y^{1/3},$$

$$m = 2xg'(t)y^{-1/3} - \frac{1}{9}x^2 g(t)y^{-4/3} + \frac{1}{3}xh(t)y^{-2/3}$$

$$+2g(t)y^{2/3}u + 3h'(t)y^{1/3} + \frac{9}{2}g''(t)y^{2/3} + k(t).$$

The Non-Hydrodynamic Chains

Instead of the expansion

$$\lambda = p + \frac{A^0}{p} + \frac{A^1}{p^2} + \frac{A^2}{p^3} + \dots$$

one can substitute the extended expansion

$$\lambda = a_{-2}p + a_{-1} + \frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3} + \dots$$

into the Lax pair for the DKP equation

$$\lambda_t = p\lambda_x - \lambda_p u_x,$$

$$\lambda_y = (p^2 + u)\lambda_x - \lambda_p(pu_x + v_x).$$

Then we obtain

$$a_{k,t}=a_{k+1,x}+(k-1)a_{k-1}[\frac{1}{a_{-1}(y)}a_{1,x}-\frac{a'_{-1}(y)}{a_{-1}(y)}],\quad k=1,2,...,$$

$$a_{k,y}=a_{k+2,x}+[\frac{1}{a_{-1}(y)}a_1-x\frac{a'_{-1}(y)}{a_{-1}(y)}-\frac{1}{2}t^2\frac{a''_{-1}(y)}{a_{-1}(y)}-t\frac{b'_0(y)}{a_{-1}(y)}-\frac{g_1(y)}{a_{-1}(y)}]a_{k,x}$$
$$+ka_k[\frac{1}{a_{-1}(y)}a_{1,x}-\frac{a'_{-1}(y)}{a_{-1}(y)}]$$
$$+(k-1)a_{k-1}[\frac{1}{a_{-1}(y)}a_{2,x}-\frac{b'_0(y)}{a_{-1}(y)}-t\frac{a''_{-1}(y)}{a_{-1}(y)}],\quad k=1,2,...$$

The Non-Hydrodynamic Reductions

Under the substitution

$$a_k = \frac{1}{k}\sum_{m=1}^{M}\epsilon_m (c^m)^k, \quad \sum_{m=1}^{M}\epsilon_m = 0,$$

one can obtain non-hydrodynamic type system $(k = 1, 2, ..., N)$

$$c_t^k = \left(\frac{(c^k)^2}{2} + \frac{1}{a_{-1}(y)}\sum_{m=1}^{M}\epsilon_m c^m - x\frac{a'_{-1}(y)}{a_{-1}(y)}\right)_x .$$

References

A new class of exact solutions selected by the substitution

$$\lambda = kp^2 + lp + m$$

is presented.

References

A new class of exact solutions selected by the substitution

$$\lambda = kp^2 + lp + m$$

is presented.
This ansatz can be generalised to an arbitrary polynomials ($N > 0$)

$$\lambda = k_1 p^{N+2} + k_2 p^{N+1} + k_3 p^N + ... + k_N p^3 + k_{N+1} p^2 + k_{N+2} p + k_{N+3}.$$

References

A new class of exact solutions selected by the substitution

$$\lambda = kp^2 + lp + m$$

is presented.
This ansatz can be generalised to an arbitrary polynomials ($N > 0$)

$$\lambda = k_1 p^{N+2} + k_2 p^{N+1} + k_3 p^N + ... + k_N p^3 + k_{N+1} p^2 + k_{N+2} p + k_{N+3}.$$

S.V. Manakov, P.M. Santini,
On the solutions of the dKP equation: the nonlinear Riemann Hilbert problem, longtime behaviour, implicit solutions and wave breaking, J. Phys. A: Math. Theor. 41 (2008) 055204,
https://doi.org/10.1088/1751-8113/41/5/055204

References

S.V. Manakov, P.M. Santini,
On the dispersionless Kadomtsev–Petviashvili equation in n+1 dimensions: exact solutions, the Cauchy problem for small initial data and wave breaking, J. Phys. A: Math. Theor. 44 (2011) 405203, https://doi.org/10.1088/1751-8113/44/40/405203

F. Santucci, P.M. Santini,
On the dispersionless Kadomtsev–Petviashvili equation with arbitrary nonlinearity and dimensionality: exact solutions, longtime asymptotics of the Cauchy problem, wave breaking and shocks, J. Phys. A: Math. Theor. 49 (2016) 405203,
https://doi.org/10.1088/1751-8113/49/40/405203

A.M. Kamchtnov, M.V. Pavlov,
On exact solutions of nonlinear acoustic equations, Wave Motion, 67 (2016) 81-88, https://doi.org/10.1016/j.wavemoti.2016.07.009