VERONESE WEBS AND NONLINEAR PDEs

Joint work with Boris Kruglikov

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Workshop on Integrable Nonlinear Equations

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2 Introduction

$f : \mathbb{R}^3 \to \mathbb{R}$

$Af_x f_{yz} + Bf_y f_{xz} + Cf_z f_{xy} = 0, A + B + C = 0$ – dispersionless Hirota equation (or (A, B, C)-equation)

$f_{xz} - f_{yy} + f_y f_{xx} - f_x f_{xy} = 0$ – hyper-CR equation

Aim of the talk: to discuss underlying geometric structures and introduce three more equations

$(\beta(y) - \gamma(z))f_x f_{yz} + (\gamma(z) - \alpha(x))f_y f_{xz} + (\alpha(x) - \beta(y))f_z f_{xy} = 0$ (Type I)

$f_x f_{zx} - f_z f_{xx} + f_y f_{xy} - f_x f_{yy} = 0$ (Type II)

$f_x f_{zx} - f_z f_{xx} + (\beta(y) - \gamma(z))(f_x f_{yz} - f_y f_{xz}) + \beta'(y) f_x f_z = 0$ (Type III)

3 Plan

1. Veronese webs and Einstein–Weyl structures
2. Dual description and partial Nijenhuis (1,1)-tensors
3. “Usual” Hirota equation
4. “Unusual” Hirota equations I, II, and III
5. Associated Einstein–Weyl structures
6. Contact and Bäcklund transformations

4 Veronese webs

Definition

$$\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{P}^1=\mathbb{K}\cup\{\infty\}};$$

here $\mathcal{F}_\lambda$ is a foliation of codimension 1 on M^{n+1} such that

$$\forall x \in M \ \exists \text{ a local coframe } (\alpha_0,\dots,\alpha_n), \alpha_i \in \Gamma(T^*M)$$

with

$$(T\mathcal{F}_\lambda)^\perp = \langle \alpha_0 + \lambda\alpha_1 + \dots + \lambda^n\alpha_n \rangle.$$

near x.

5 Motivation: bihamiltonian structures and classical webs

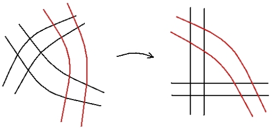
Poisson str. of const rank $\overset{\textit{Darboux th}}{\longrightarrow}$ $\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \cdots + \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial q_k}$

Pair of compatible Poisson structures of const rank $\longrightarrow$???

Idea of Gelfand and Zakharevich:

Pair of compatible Poisson structures of const rank $\longrightarrow$ 1-param. family of foliations

Classical webs



6 Einstein–Weyl structures

Weyl structures: torsion free connections D adapted to conformal classes of metrics $[g]$, given by g and 1-form ω such that $Dg = g \otimes \omega$.

Einstein–Weyl structures: W. str. whose symmetrised Ricci tensor is proportional to some metric $g \in [g]$

Einstein–Weyl structures in (2+1)-dim: $\longleftrightarrow$ existence of 2-dim family $\{\mathcal{G}_\gamma\}_{\gamma \in \Gamma}$ of null totally geodesic hypersurfaces

Einstein–Weyl structures of hyper-CR type: E–W. str. in (2+1)-dim with Γ fibered over P^1

Theorem of Dunajski–Kryński:

Veronese webs in dim 3 $\overset{1:1}{\longleftrightarrow}$ E–W str. of hyper-CR type

7 Partial Nijenhuis operators

Definition

A *PNO* on a manifold M is a pair $(\mathcal{F}, N)$, where $\mathcal{F}$ is a foliation on M and $N : T\mathcal{F} \to TM$ is a partial (1,1)-tensor such that $\forall\, X, Y \in \Gamma(T\mathcal{F})$

- $[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y] \ \in \ \Gamma(T\mathcal{F})$;
- $T_N(X, Y) := [NX, NY] - N[X, Y]_N = 0$.

Example

Let $N : TM \to TM$ be a Nijenhuis (1,1)-tensor, i.e. $T_N \equiv 0$. Then (M, N) is a PNO.

8 Partial Nijenhuis operators: Lemma 1

Lemma

Let $(\mathcal{F}, N)$ be a PNO on M. Then

- *$(\mathcal{F}, N_\lambda)$ is a PNO; here $N_\lambda := N - \lambda I$, $I : T\mathcal{F} \hookrightarrow TM$ a canonical inclusion*
- *$[X, Y]_{N_\lambda}$ is a Lie bracket on $\Gamma(T\mathcal{F})$*
- *$N_\lambda : \Gamma(T\mathcal{F}) \to \Gamma(TM)$ is a homomorphism of Lie algebras.*

In particular, if $N_\lambda(T\mathcal{F}) \subset TM$ is a distribution, it is integrable:

$$N_\lambda(T\mathcal{F}) = T\mathcal{F}_\lambda.$$

9 Partial Nijenhuis operators: Lemma 2

Lemma

Let $N : TM \to TM$ be a Nijenhuis (1,1)-tensor, i.e. $T_N \equiv 0$, and let $\mathcal{F}$ be a foliation. Assume

- *$\forall\, x \in M$: $N_x|_{T_x\mathcal{F}} : T_x\mathcal{F} \to T_xM$ is an isomorphism onto the image*
- *$N(T\mathcal{F}) \subset TM$ is an integrable distribution.*

Then $(\mathcal{F}, N|_{T\mathcal{F}})$ is a PNO.

Remark Given a partial Nijenhuis operator $(\mathcal{F}, N)$, there can exist *different* Nijenhuis (1,1)-tensors $\widetilde{N}$ such that $\widetilde{N}|_{T\mathcal{F}} = N$.

10 Veronese webs: dual description

Theorem

There exists a 1-1-correspondence between Veronese webs $\{\mathcal{F}_\lambda\}$ on M^{n+1} and PNOs $(\mathcal{F}, N)$ such that the pair of operators (N, I), $I : T\mathcal{F} \hookrightarrow TM$, has a unique Kronecker block in the Jordan–Kronecker decomposition, i.e. exist local frames $v_1, \ldots, v_n \in \Gamma(T\mathcal{F}), w_0, \ldots, w_n \in \Gamma(TM)$ in which

$$N = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

11 Proof of the theorem

$(\Longleftarrow)$ $(\mathcal{F}, N) \mapsto N_\lambda(T\mathcal{F}) = T\mathcal{F}_\lambda$ (use Lemma 1)
$(\Longrightarrow)$ *Variation of a construction of F.J. Turiel:*
Let $\{\mathcal{F}_\lambda\}$ be a Veronese web on M^{n+1} ($=M^3$ for simplicity).
Fix $\lambda_1, \lambda_2, \lambda_3$ pairwise distinct nonzero. Then

$$D_1 = T\mathcal{F}_{\lambda_2} \cap T\mathcal{F}_{\lambda_3}, D_2 = T\mathcal{F}_{\lambda_3} \cap T\mathcal{F}_{\lambda_1}, D_3 = T\mathcal{F}_{\lambda_1} \cap T\mathcal{F}_{\lambda_2}$$

are 1-dimensional distributions such that $D_i + D_j$ are integrable 2-dimensional distributions (for instance $D_1 + D_2 = T\mathcal{F}_{\lambda_3}$ etc.). Hence there exists a local coordinate system (x_1, x_2, x_3) such that $D_i = \langle \partial_{x_i} \rangle$. Put

$$N\partial_{x_i} = \lambda_i \partial_{x_i}.$$

Then $T_N \equiv 0$ and $(\mathcal{F}_\infty, N|_{T\mathcal{F}_\infty})$ is a PNO. Indeed $N(T\mathcal{F}_\infty) = T\mathcal{F}_0$ is integrable (use Lemma 2). Finally, $N_{\lambda_i}(T\mathcal{F}_\infty) = T\mathcal{F}_{\lambda_i}$, $i = 1, 2, 3$ and by the uniqueness property of the Veronese curve $N_\lambda(T\mathcal{F}_\infty) = \mathcal{F}_\lambda$.

12 The Hirota equation

Variation of a construction of I. Zakharevich:

Consider $\mathbb{R}^3(x_1, x_2, x_3)$, $\lambda_1, \lambda_2, \lambda_3$ pairwise distinct nonzero numbers. Construct a Nijenhuis (1,1)-tensor $N : T\mathbb{R}^3 \to T\mathbb{R}^3$ by

$$N\partial_{x_i} = \lambda_i \partial_{x_i}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be nondegenerate ($f_{x_i} \neq 0$). Put
$\mathcal{F}_\infty : T\mathcal{F}_\infty := \langle df \rangle^\perp$. Then

$$(N(T\mathcal{F}_\infty))^\perp = \langle \frac{1}{\lambda_1} f_{x_1} dx_1 + \frac{1}{\lambda_2} f_{x_2} dx_2 + \frac{1}{\lambda_3} f_{x_3} dx_3 \rangle =: \langle \omega \rangle.$$

$N(T\mathcal{F}_\infty)$ is integrable $\iff d\omega \wedge \omega = 0 \iff$

$$\frac{1}{\lambda_1}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) f_{x_1} f_{x_2 x_3} + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_1}\right) f_{x_2} f_{x_3 x_1} + \frac{1}{\lambda_3}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) f_{x_3} f_{x_1 x_2} = 0$$

Theorem

There is a 1-1-correspondence between Veronese webs $\{\mathcal{F}_\lambda\}$ with $\mathcal{F}_{\lambda_i} = \{dx_i = 0\}$, $\mathcal{F}_\infty = \{df = 0\}$ and the solutions of the Hirota $(\lambda_2 - \lambda_3, \lambda_3 - \lambda_1, \lambda_1 - \lambda_2)$-equation.

13 Another version of the underlying nonlinear equation (I)

Consider $\mathbb{R}^3(x_1, x_2, x_3)$, fix $p \in \mathbb{R}^3$ and $\phi_1(x_1), \phi_2(x_2), \phi_3(x_3)$ any functions which have pairwise distinct nonzero values at p.
Construct a Nijenhuis (1,1)-tensor $N : T\mathbb{R}^3 \to T\mathbb{R}^3$ by

$$N\partial_{x_i} = \phi_i(x_i)\partial_{x_i}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be nondegenerate ($f_{x_i} \neq 0$) around p. Put $\mathcal{F}_\infty : T\mathcal{F}_\infty := \langle df \rangle^\perp$. Then

$$(N(T\mathcal{F}_\infty))^\perp = \langle \frac{1}{\phi_1} f_{x_1} dx_1 + \frac{1}{\phi_2} f_{x_2} dx_2 + \frac{1}{\phi_3} f_{x_3} dx_3 \rangle =: \langle \omega \rangle.$$

$N(T\mathcal{F}_\infty)$ is integrable $\iff d\omega \wedge \omega = 0 \iff$

$$\frac{1}{\phi_1}(\frac{1}{\phi_2} - \frac{1}{\phi_3}) f_{x_1} f_{x_2 x_3} + \frac{1}{\phi_2}(\frac{1}{\phi_3} - \frac{1}{\phi_1}) f_{x_2} f_{x_3 x_1} + \frac{1}{\phi_3}(\frac{1}{\phi_1} - \frac{1}{\phi_2}) f_{x_3} f_{x_1 x_2} = 0$$

$\iff$

$$(\phi_2 - \phi_3) f_{x_1} f_{x_2 x_3} + (\phi_3 - \phi_1) f_{x_2} f_{x_3 x_1} + (\phi_1 - \phi_2) f_{x_3} f_{x_1 x_2} = 0.$$

14 Another version of the underlying nonlinear equation (II)

Consider $\mathbb{R}^3(x_1, x_2, x_3)$, fix $a \in \mathbb{R}$. Construct a Nijenhuis (1,1)-tensor $N : T\mathbb{R}^3 \to T\mathbb{R}^3$ by

$$N\partial_{x_1} = a\partial_{x_1}, N\partial_{x_2} = \partial_{x_1} + a\partial_{x_2}, N\partial_{x_3} = \partial_{x_2} + a\partial_{x_3} \qquad (1)$$

(the Jordan 3×3-block with a constant eigenvalue a). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be nondegenerate. Put $\mathcal{F}_\infty : (T\mathcal{F}_\infty)^\perp := \langle df \rangle$. Then

$$(N(T\mathcal{F}_\infty))^\perp = \langle (N^*)^{-1} df \rangle = \langle f_{x_1}(dx_1 - \frac{1}{a}dx_2 + \frac{1}{a^2}dx_3) + f_{x_2}(dx_2 - \frac{1}{a}dx_3) + f_{x_3}dx_3 \rangle \quad =: \quad \langle \omega \rangle.$$

$N(T\mathcal{F}_\infty)$ is integrable $\iff d\omega \wedge \omega = 0 \iff$

$$f_{x_1} f_{x_3 x_1} - f_{x_3} f_{x_1 x_1} + f_{x_2} f_{x_1 x_2} - f_{x_1} f_{x_2 x_2} = 0$$

15 Yet another version of the underlying nonlinear equation (III)

Consider $\mathbb{R}^3(x_1, x_2, x_3)$, fix $p \in \mathbb{R}^3$ and $a(x_2), b(x_3)$ any functions which have distinct nonzero values at p. Construct a Nijenhuis (1,1)-tensor $N : T\mathbb{R}^3 \to T\mathbb{R}^3$ by

$$N\partial_{x_1} = a(x_2)\partial_{x_1}, N\partial_{x_2} = a(x_2)\partial_{x_2} + \partial_{x_1}, N\partial_{x_3} = b(x_3)\partial_{x_3}$$

(the Jordan 2×2-block with the eigenvalue a and a 1×1-block with the eigenvalue b). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be nondegenerate. Put $\mathcal{F}_\infty : (T\mathcal{F}_\infty)^\perp := \langle df \rangle$. Then

$$(N(T\mathcal{F}_\infty))^\perp = \langle (N^*)^{-1} df \rangle = \langle f_{x_1}(\frac{1}{a(x_2)} dx_1 - \frac{1}{a(x_2)^2} dx_2) + f_{x_2}(\frac{1}{a(x_2)} dx_2) + f_{x_3} \frac{1}{b(x_3)} dx_3 \rangle \quad =: \quad \langle \omega \rangle.$$

$N(T\mathcal{F}_\infty)$ is integrable $\iff d\omega \wedge \omega = 0 \iff$

$$f_{x_1} f_{x_3 x_1} - f_{x_3} f_{x_1 x_1} + (a(x_2) - b(x_3))(f_{x_1} f_{x_2 x_3} - f_{x_2} f_{x_1 x_3}) + a'(x_2) f_{x_1} f_{x_3} = 0$$

16 Contact symmetries (type I)

Theorem

The contact symmetry algebra of equation of type I with constant ϕ_i is generated by the point symmetries $g_1(x)\partial_{x_1} + g_2(x_2)\partial_{x_2} + g_3(x_3)\partial_{x_3} + g_4(f)\partial_u$ with arbitrary functions g_1, g_2, g_3, g_4 of one argument, i.e. the corresponding Lie pseudogroup is generated by the transformations

$$x_i \mapsto X_i(x_i), \;\; f \mapsto F(f).$$

The contact symmetry algebra of equation of type I with variable ϕ_i is generated by the point symmetries $c_1 \cdot \left(\frac{\partial_{x_1}}{\phi_1'(x_1)} + \frac{\partial_{x_2}}{\phi_2'(x_2)} + \frac{\partial_{x_3}}{\phi_3'(x_3)}\right) + c_2 \cdot \left(\frac{\phi_1(x_1)\partial_{x_1}}{\phi_1'(x_1)} + \frac{\phi_2(x_2)\partial_{x_2}}{\phi_2'(x_2)} + \frac{\phi_3(x_3)\partial_{x_3}}{\phi_3'(x_3)}\right) + f(u)\partial_u$ with arbitrary two constants c_1, c_2 and one function f of one argument.

The structures of these two Lie algebras are quite different: 1) $\bigoplus_{n=1}^{4} \mathrm{Vect}(\mathbb{R})$; 2) the direct sum of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ with a solvable non-Abelian 2D algebra.

17 Einstein–Weyl structures and Lax pair (type I)

Theorem

1. The following Weyl structure on a 3D-space with coordinates (x_1, x_2, x_3), parametrized by one function $f = f(x_1, x_2, x_3)$

$$g = \frac{(\phi_2 - \phi_3)^2 f_{x_1}}{f_{x_2} f_{x_3}} dx_1^2 + \frac{2(\phi_1 - \phi_3)(\phi_2 - \phi_3)}{f_{x_3}} dx_1 dx_2 + c.p.$$

$$\omega = \Big(\Big(\frac{1}{\phi_1 - \phi_2} + \frac{1}{\phi_1 - \phi_3} \Big) \phi_1' - \Big(\frac{1}{\phi_1 - \phi_2} \frac{f_{x_1}}{f_{x_2}} \Big) \phi_2' - \Big(\frac{1}{\phi_1 - \phi_3} \frac{f_{x_1}}{f_{x_3}} \Big) \phi_3' - \frac{f_{x_1 x_1}}{f_{x_1}} \Big) dx_1 + c.p.$$

is Einstein-Weyl iff the function f satisfies equation of type I.

2. Lax pair for equation of type I:
$v^\lambda := f_{x_2}(\phi_1 - \lambda)\frac{\partial}{\partial x_1} - f_{x_1}(\phi_2 - \lambda)\frac{\partial}{\partial x_2}$,
$w^\lambda := f_{x_3}(\phi_2 - \lambda)\frac{\partial}{\partial x_2} - f_{x_2}(\phi_3 - \lambda)\frac{\partial}{\partial x_3}$

18 Realization theorem

Veronese curve for the equation of type I:
$(\phi_2-\lambda)(\phi_3-\lambda)f_{x_1}dx_1+(\phi_3-\lambda)(\phi_1-\lambda)f_{x_2}dx_2+(\phi_1-\lambda)(\phi_2-\lambda)f_{x_3}dx_3$

Definition

Let $\mathcal{F}_\lambda$ be a Veronese web, $T\mathcal{F}_\lambda = \langle \alpha_0 + \lambda\alpha_1 + \cdots + \lambda^n\alpha_n \rangle^\perp$. A smooth function $\phi : M \to \mathbb{R}$ is called *self-propelled* if $\alpha_0 + \phi\alpha_1 + \cdots + \phi^n\alpha_n \sim d\phi$, where $\sim$ means proportionality up to multiplication by a nonvanishing function.

Lemma

Let $\mathcal{F}_\lambda$ be a Veronese web in $\mathbb{R}^3$ defined by $(T\mathcal{F}_\lambda)^\perp = \langle \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2 \rangle$. Then locally there exist three functionally independent self-propelled functions $\phi_1(x), \phi_2(x), \phi_3(x)$.
The relation $\alpha_0 + \phi\alpha_1 + \phi^2\alpha_2 \sim d\phi$ is equivalent to the following system of first order nonlinear PDEs:

$$\phi X_0\phi = X_1\phi, \phi X_1\phi = X_2\phi, \tag{2}$$

where X_0, X_1, X_2 is the frame dual to the coframe $\alpha_0, \alpha_1, \alpha_2$.

19 Realization theorem

Theorem

Let a Veronese web $\mathcal{F}_\lambda$ in $\mathbb{R}^3$ be given, let ϕ_1, ϕ_2, ϕ_3 be independent self-propelled functions for it, and let $f : \mathbb{R}^3 \to \mathbb{R}$ be such that $\mathcal{F}_\infty = \{f = const\}$. Then f is a solution of equation (Type I), where x_i is an invertible function of ϕ_i.

Proof

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & \phi_1\phi_2\phi_3 \\ 1 & 0 & -\phi_1\phi_2 - \phi_1\phi_3 - \phi_2\phi_3 \\ 0 & 1 & \phi_1 + \phi_2 + \phi_3 \end{bmatrix} \mapsto \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{bmatrix}$$

Realization theorem $\mapsto$ Bäcklund transformations

Example $[X_0, X_1] = X_0, [X_1, X_2] = X_2, [X_0, X_2] = 2X_1 \mapsto$ nonflat Veronese web $\alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2$. Realize X_i as left-invariant vector fields on $SL(2)$:

$$X_0 = -y\frac{\partial}{\partial x} - \frac{yz+1}{x}\frac{\partial}{\partial z}, X_1 = \frac{1}{2}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}), X_2 = x\frac{\partial}{\partial y},$$

where $\begin{bmatrix} x & y \\ z & \frac{yz+1}{x} \end{bmatrix} \in SL(2)$. Let $F(x, y, z, \lambda)$ be the function such that $\mathcal{F}_\lambda = \{F(x, y, z, \lambda) = const\}$.
Then the formula $F(x, y, z, \phi(x, y, z)) = c$ gives implicitly a 1-parametric family of self-propelled functions.

Explicitly

$$F(x,y,z,\lambda)=\frac{\lambda yz+xz+\lambda}{(\lambda y+x)x}\mapsto$$

$$\phi_1=\frac{-x}{y},\phi_2=\frac{-xz}{yz+1},\phi_3=\frac{x^2-xz}{yz+1-xy}$$

The function $F(x,y,z,\infty)=(yz+1)/yx$ “cutting” the foliation $\mathcal{F}_\infty$ can be expressed as

$$\frac{\phi_1-\phi_3}{\phi_2-\phi_3},$$

which gives a particular solution of the equation of type I

$$(\phi_2-\phi_3)f_{\phi_1}f_{\phi_2\phi_3}+(\phi_3-\phi_1)f_{\phi_2}f_{\phi_3\phi_1}+(\phi_1-\phi_2)f_{\phi_3}f_{\phi_1\phi_2}=0.$$

22 Finally

For equations of type II and III we also have:

- Contact symmetry algebras
- Formulae for Einstein–Weyl structures
- Realization theorems
- Bäcklund transformations between equations of type I, II, and III

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Lacking (?): Bäcklund transformations between equations of type I, II, III and hyper-CR

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Many thanks for your attention!