Quotients of Differential Equations

Valentin Lychagin

Lab 6, IPU, RAS, Moscow

November 25, 2020

Let Ω be a G-set, i.e a set with an action of a group G :

$$G \times \Omega \to \Omega, g \times \omega \mapsto g\omega.$$

Then the quotient Ω/G is the set of all G-orbits

$$\Omega/G = \bigcup_{\omega \in \Omega} \{G\omega\}.$$

Remark. The projection $\pi : \Omega \to \Omega/G$ allow us to identify functions on the quotient set Ω/G with functions on Ω that are G-invariants

$$f \circ g = f.$$

Survey of Quotients- Topological Spaces

Let Ω be a topological space, G be a topological group and let G-action $G \times \Omega \rightarrow \Omega$ be continuous.
Then the quotient Ω / G carries the natural topological structure, where a subset $U \subset \Omega / G$ is said to be open if and only if the preimage $\pi^{-1}(U) \subset \Omega$ is open.

Remark. We could not guarantee that the quotient Ω / G shall inherit topological properties (like Hausdorff, etc.) of Ω.

Quotients of Topological Spaces-Examples

(A) Let $\Omega = \mathbb{R}^2$, $G = GL_2(\mathbb{R})$, and $GL_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$ be the natural action. Then

$$\mathbb{R}^2 / GL_2(\mathbb{R}) = \mathbf{0} \cup \bigstar,$$

where $\mathbf{0} = GL_2(\mathbb{R})\,(0)$ is the orbit of the origin $0 \in \mathbb{R}^2$, and $\bigstar$ is the orbit of any non zero point.
This is the famous *Serpinski topological space*. It consists of two points and one of them $\mathbf{0}$ is closed but the another one $\bigstar$ is open.
(B) Let $\Omega = \mathbb{R}^2$, $G = \mathbb{R}^* = \mathbb{R} \backslash 0$, and $\mathbb{R}^* \times \mathbb{R}^2 \to \mathbb{R}^2$ be the natural action. Then

$$\mathbb{R}^2 / \mathbb{R}^* = \mathbf{0} \cup \mathbb{R}\mathbf{P}^1,$$

where as above $\mathbf{0}$ is the fat point (any neighborhood coincide with the entire space), and $\mathbb{R}\mathbf{P}^1$ is the projective $1-$ dimensional space.

Survey of Quotients-Smooth Manifolds

Bad news: Practically we do not know when the quotient Ω/G of a smooth manifold Ω by a Lie group G is also smooth manifold.
Except maybe the case: Quotients by free and proper G-actions are smooth manifolds.

Good news. Rosenlicht Theorem (M. Rosenlicht, Some basic theorems on algebraic groups, Amer. J. Math. 78 (1956), 401–443.)
Maxwell Alexander Rosenlicht (15.04.1924, Brooklyn - 22.01.1999, Hawaii)
Harvard University, student of Ockar Zariski.

Rosenlicht Theorem

Let Ω be an algebraic manifold (i.e. an irreducible variety without singularities over a field of zero characteristic), G be an algebraic group and $G \times \Omega \to \Omega$ be an algebraic action.
By $\mathcal{F}(\Omega)$ we denote the field of rational functions on Ω and by $\mathcal{F}(\Omega)^G \subset \mathcal{F}(\Omega)$ the field of rational G-invariants.
We say that an orbit $G\omega \subset \Omega$ is *regular* (as well as the point ω) if there are $m = \operatorname{codim} G\omega$ invariants $x_1, .., x_m$ such that their differentials are linear independent at the points of the orbit.
Let $\Omega_0 = \Omega \smallsetminus \operatorname{Sing}$ be the set of all regular points and $Q(\Omega) = \Omega_0 / G$ the set of all regular orbits. The Rosenlicht theorem states that Ω_0 is open and dense in Ω.
The above invariants $x_1, .., x_m$ we'll consider as local coordinates on the quotient $Q(\Omega)$ at the point $G\omega \in Q(\Omega)$. Moreover on intersections of coordinate charts these coordinates are connected by rational functions. That means that $Q(\Omega)$ is an algebraic manifold of the dimension $m = \operatorname{codim} G\omega$. Thus we have the rational map $\pi : \Omega_0 \to Q(\Omega)$ of algebraic manifolds and $\mathcal{F}(\Omega)^G = \pi^*(\mathcal{F}(Q(\Omega)))$.

Smooth version of the Rosenlicht Theorem

Let G be a Lie group and Ω be a G-manifold.
Follow to the algebraic case, we say that an orbit $G\omega \subset \Omega$ is *regular* (as well as the point ω itself) if there are $m = \operatorname{codim} G\omega$ smooth invariants with differentials linear independent at the points of the orbit.
Let $\Omega_{reg} \subset \Omega$ be the set of regular points then the quotient Ω_{reg} / G is a smooth manifold and the projection $\pi : \Omega_{reg} \to \Omega_{reg} / G$ gives us the isomorphism between algebra of G-invariants $C^{\infty}\left(\Omega_{reg}\right)^{G}$ and $C^{\infty}\left(\Omega_{reg} / G\right)$.
In the contrast to the algebraic case we could not say that Ω_{reg} is dense in Ω.

Algebraic Lie Algebras

Let Ω be an algebraic manifold and let $\mathfrak{g}$ be a Lie subalgebra of the Lie algebra of the vector fields on Ω.
We say that $\mathfrak{g}$ is an *algebraic Lie algebra* if there is an algebraic action of an algebraic group G on Ω such that $\mathfrak{g}$ coincide with the image of Lie algebra $\mathrm{Lie}\,(G)$ under this action.
By *algebraic closer of Lie algebra* $\mathfrak{g}$ we mean the intersection of all algebraic Lie algebras, containing $\mathfrak{g}$.

Examples

(1) $\Omega = \mathbb{R}$,Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_2 = \left\langle \partial_x, x\partial_x, x^2\partial_x \right\rangle$$

is algebraic.
(2) $\Omega = S^1 \times S^1$ -torus

$$\mathfrak{g} = \left\langle \partial_\phi + \lambda\partial_\psi \right\rangle$$

is algebraic iff $\lambda \in \mathbb{Q}$, and the similar true for the case: $\Omega = \mathbb{R}^2$, and

$$\mathfrak{g} = \langle \partial_x + \lambda\partial_y \rangle .$$

In the case when $\lambda \notin \mathbb{Q}$, $\widetilde{\mathfrak{g}} = \langle \partial_x, \partial_y \rangle$

Lie Algebra actions-Rosenlicht Theorem

The Rosenlicht theorem also valids for algebraic Lie algebras, or for their algebraic closer in the case of general Lie algebras.
Let's given a Lie algebra $\mathfrak{g}$ of vector fields on an algebraic manifold Ω and let $\widetilde{\mathfrak{g}} \supset \mathfrak{g}$ be its algebraic closer.
Than the field $\mathcal{F}(\Omega)^{\mathfrak{g}}$ of rational $\mathfrak{g}-$invariants has transcendence degree equal to the codimension of regular $\widetilde{\mathfrak{g}}-$orbits
It also equals to the dimension of the quotient manifold $Q(\Omega)$.

Algebraicity in Jet Geometry

Let $\pi : E(\pi) \to M$ be a smooth bundle over manifold M and let $\pi_k : J^k(\pi) \to M$ be the k-jet bundles. To simplify notations we'll use $\mathbf{J}^k$ instead of $J^k(\pi)$.

The jet geometry depends on $\dim \pi$ and defines by the following pseudo groups:

1. In the case $\dim \pi = 1$,it defines by the pseudo group $\mathrm{Cont}(\pi)$ of the local contact transformations of the manifold $\mathbf{J}^1$, and
2. In the case $\dim \pi \geq 2$,it defines by the pseudogroup $\mathrm{Point}(\pi)$ of the local point transformations, i.e. local diffeomorphisms of the manifold $\mathbf{J}^0$.

It is known that the prolongations of these pseudogroups exhaust all Lie transformations i.e. local diffeomorphisms of jet spaces that preserve the Cartan distributions.

Algebraicity in Jet Geometry-2

it is also known that

1. All bundles $\pi_{k,k-1} : \mathbf{J}^k \to \mathbf{J}^{k-1}$ are affine bundles for $k \geq 2$, when $\dim \pi \geq 2$, and for $k \geq 3$, when $\dim \pi = 1$.
2. Prolongations of the pseudo groups in the standard jet-coordinates $\left(x, u^i_\sigma\right)$ are given by rational in u^i_σ functions.

Therefore,

1. In the case $\dim \pi \geq 2$ the fibres $\mathbf{J}^{k,0}_\theta$ of the projections $\pi_{k,0} : \mathbf{J}^k \to \mathbf{J}^0$ at points $\theta \in \mathbf{J}^0$ are algebraic manifolds and the stationary subgroup $\mathrm{Point}_\theta\left(\pi\right) \subset \mathrm{Point}\left(\pi\right)$ gives us birational isomorphisms of the manifold.
2. In the case $\dim \pi = 1$ the fibres $\mathbf{J}^{k,1}_\theta$ of the projections $\pi_{k,0} : \mathbf{J}^k \to \mathbf{J}^1$, at point $\theta \in \mathbf{J}^1$, *are algebraic manifolds* and the stationary subgroup $\mathrm{Cont}_\theta\left(\pi\right) \subset \mathrm{Cont}\left(\pi\right)$ gives us birational isomorphisms of the manifold.

Algebraic differential equations

We say that a differential equation $\mathcal{E}_k \subset \mathbf{J}^k$ is *algebraic* if fibres $\mathcal{E}_{k,\theta}$ of the projections $\pi_{k,0} : \mathcal{E}_k \to \mathbf{J}^0$, when $\dim \pi \geq 2$, or $\pi_{k,1} : \mathcal{E}_k \to \mathbf{J}^1$, when $\dim \pi = 1$, are algebraic manifolds.

Remark that in the last cases the prolongations $\mathcal{E}_k^{(l)} = \mathcal{E}_{k+l} \subset \mathbf{J}^k$ are algebraic too if $\mathcal{E}_k$ is a formally integrable equation.

By symmetry algebra of the algebraic differential equations we'll mean

1. Point symmetries: $\mathrm{sym}\,(\mathcal{E}_k)$ -the Lie algebra of point vector fields which is transitive on $\mathbf{J}^0$ and the stationary subalgebras $\mathrm{sym}_\theta\,(\mathcal{E}_k)$, $\theta \in \mathbf{J}^0$, produce actions of algebraic Lie algebras on algebraic manifolds $\mathcal{E}_{l,\theta}$, for all $l \geq k$.
2. The similar for contact symmetries: $\mathrm{sym}\,(\mathcal{E}_k)$ of the contact vector fields which is transitive on $\mathbf{J}^1$ and the stationary subalgebras $\mathrm{sym}_\theta\,(\mathcal{E}_k)$, $\theta \in \mathbf{J}^1$, produces actions of algebraic Lie algebras on algebraic manifolds $\mathcal{E}_{l,\theta}$, for all $l \geq k$.

Quotients of Algebraic Differential Equations

Let $\mathfrak{g}$ be an algebraic symmetry Lie algebra of algebraic differential equation $\mathcal{E}_k$, and let $\mathcal{E}_l$ be $(l-k)$-prolongations of $\mathcal{E}_k$. We assume that differential equation $\mathcal{E}_k$ is formally integrable. Then all equations $\mathcal{E}_l$ are algebraic manifolds and we have the tower of algebraic bundles:

$$\mathcal{E}_k \longleftarrow \mathcal{E}_{k+1} \longleftarrow \cdots \longleftarrow \mathcal{E}_l \longleftarrow \mathcal{E}_{l+1} \longleftarrow \cdots .$$

Let $\mathcal{E}_l^0 \subset \mathcal{E}_l$ be the set of *strongly regular points* and $Q_l\left(\mathcal{E}\right)$ be the set of all strongly regular $\mathfrak{g}-$orbits.
Here by strongly regular point (and orbit) we mean such points on $\mathcal{E}_l$ that are regular with respect to $\mathfrak{g}$ -action and their projections on $\mathcal{E}_{l-1}$ are regular too.
Then, as we have seen, $Q_l\left(\mathcal{E}\right)$ are algebraic manifolds and the projections $\varkappa_l : \mathcal{E}_l^0 \to Q_l\left(\mathcal{E}\right)$,are rational maps such that the fields $\mathcal{F}\left(Q_l\left(\mathcal{E}\right)\right)$ - the field of rational functions on $Q_l\left(\mathcal{E}\right)$, and $\mathcal{F}\left(\mathcal{E}_l^0\right)^{\mathfrak{g}}$ - the field of rational functions on $\mathcal{E}_l^0$ that are $\mathfrak{g}$ -invariants (*rational differential invariants*) coincide: $\varkappa_l^*\left(\mathcal{F}\left(Q_l\left(\mathcal{E}\right)\right)\right) = \mathcal{F}\left(\mathcal{E}_l^0\right)^{\mathfrak{g}}$.

Cartan distributions

The $\mathfrak{g}$ -action preserves the Cartan distributions $C(\mathcal{E}_l)$ on the equations and therefore projections $\varkappa_l$ define distributions $C(Q_l)$ on the quotients $Q_l(\mathcal{E})$.
Finally, we get the tower of algebraic bundles of the quotients

$$Q_k(\mathcal{E}) \xleftarrow{\pi_{k+1,k}} Q_{k+1}(\mathcal{E}) \longleftarrow \cdots \longleftarrow Q_l(\mathcal{E}) \xleftarrow{\pi_{l+1,l}} Q_{l+1}(\mathcal{E}) \longleftarrow \cdots ,$$

such that the projection of the distribution $C(Q_l)$ belongs to $C(Q_{l-1}(\mathcal{E}))$.

(1) Let $\omega \in \Omega^1\left(\mathbf{J}^k\right)$ be a differential 1-form on the k-jet manifold. Then the class

$$\omega^h = \pi^*_{k+1,k}(\omega) \,\mathrm{mod}\, \mathrm{Ann}\, C_{k+1},$$

we'll call *horizontal part* of ω.

In the standard jet coordinate it has the following representation

$$\omega = \sum_i a_i dx_i + \sum_{|\sigma|\leq k, j\leq m} a^j_\sigma du^j_\sigma \Longrightarrow \omega^h = \sum_i a_i dx_i + \sum_{|\sigma|\leq k, j\leq m, i\leq n} a^j_\sigma u^j_{\sigma+1_i} dx_i,$$

where $n = \dim M$, $m = \dim \pi$.

As a particular case of this construction we get the *total differential* $f \in C^\infty\left(\mathbf{J}^k\right) \Longrightarrow \widehat{d} f = (df)^h$,or in the standard jet coordinates

$$\widehat{d} f = \sum_{i\leq n} \frac{df}{dx_i} dx_i,$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma}$$

Tresse Derivatives-2

The important for us observation: the operation of taking horizontal part as well as the total differential are invariants of the point and contact transformations.

We say that functions $f_1, ..., f_n \in C^\infty\left(\mathbf{J}^k\right)$ are in *general position* (in a some domain D) if

$$\widehat{d}f_1 \wedge \cdots \wedge \widehat{d}f_n \neq 0$$

in this domain.

Let f be a smooth function in the domain, then we get decomposition in D :

$$\widehat{d}f = \sum_{i \leq n} F_i \widehat{d}f_i,$$

where F_i are smooth functions in the domain $\pi_{k+1,k}^{-1}(D) \subset \mathbf{J}^{k+1}$.

We call them *Tresse derivatives* and denote by

$$\frac{df}{df_i}.$$

Tresse Derivatives of Invariants

The important observation: the operation of taking horizontal part as well as the total differential are invariants of the point and contact transformations.

Therefore we have the following statement:

Let $f_1, .., f_n$ be $\mathfrak{g}$ -invariants of order $\leq k$ in general position. Then for any $\mathfrak{g}$ -invariant f of order $\leq k$ the Tresse derivatives $\frac{df}{df_i}$ are $\mathfrak{g}$ -invariants of order $\leq k+1$.

Biographical information

Artur Marie Leopold Tresse (24.04.1868-5.02.1958)
École normale supérieure (1891) (together with Elie Cartan)
Leipzig university - student of Sophus Lie
Thesis and the book: Sur les invariants différentiels des groupes continus de transformations
Book: Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre y''=w(x,y,y'). Leipzig, S. Hirzel, 1896

Lie - Tresse Theorem

The local version of the following result goes back to Sophus Lie and Artur Tresse and the global version belongs to
Boris Kruglikov & Valentin Lychagin (Global Lie-Tresse theorem, Selecta Math.(2016),22, pp.1357-1411).
Let $\mathcal{E}_k \subset \mathbf{J}^k$ be an algebraic formally integrable differential equation and let $\mathfrak{g}$ be an algebraic symmetry Lie algebra.
Then there are rational differential $\mathfrak{g}$ -invariants $a_1, ..., a_n, b^1, ..., b^N$, of an order $\leq l$, such that the field of all rational differential $\mathfrak{g}$ -invariants is generated by rational functions of these invariants and their Tresse derivatives $\frac{d^{|\alpha|} b^j}{da^\alpha}$.

Comments

In contrast to algebraic invariants, where we have the algebraic only operations, in the case of differential invariants we have more operations-Tresse derivations that allow us to get really new invariants.
Syzygies, in the case of differential invariants, provide us with new differential equations that we'll call *quotient equations.*
From the geometrical point of view the above theorem states that there is a level l and a domain $D \subset Q(\mathcal{E})$ where the invariants a_i, b^j could be considered as local coordinates and the preimage of D in the tower

$$Q_l(\mathcal{E}) \stackrel{\pi_{l+1,l}}{\longleftarrow} Q_{l+1}(\mathcal{E}) \longleftarrow \cdots \longleftarrow Q_r(\mathcal{E}) \stackrel{\pi r}{\longleftarrow} Q_{r+1}(\mathcal{E}) \longleftarrow \cdots$$

is just an infitely prolonged differential equation given by the syzygy.

For this reason we'll call the quotient tower

$$Q_k(\mathcal{E}) \stackrel{\pi_{k+1,k}}{\longleftarrow} Q_{k+1}(\mathcal{E}) \longleftarrow \cdots \longleftarrow Q_r(\mathcal{E}) \stackrel{\pi_{r+1,r}}{\longleftarrow} Q_{r+1}(\mathcal{E}) \longleftarrow \cdots$$

algebraic diffiety.

Relations between differential equations and their quotients

- Let $u = f(x)$ be a solution of differential equation $\mathcal{E}$ and let $a_i(f), b^j(f)$ be values of invariants on the section f. Then $b^j(f) = B^j(a(f))$, and $b^j = B^j(a)$ is the solution of the quotient equation.
- In general, the correspondence between solutions valids on the level of generalized solutions, i.e. on the level of integral manifolds of the Cartan distributions. Also the correspondce will lead us to manifolds with singularities.
- Let $b^j = B^j(a)$ be a solution of the quotient equation. Then considering equations $b^j - B^j(a) = 0$ as differential constrains for the equation $\mathcal{E}$ we get finite type equation $\mathcal{E} \cap \left\{ b^j - B^j(a) = 0 \right\}$ with solution to be an $\mathfrak{g}$-orbit of a solution of $\mathcal{E}$.
- Symmetries of the quotient equation are Bäcklund type transformation for equation $\mathcal{E}$.

Examples ODEs, Metric geomerty.

$\Omega = \mathbb{R}$, $\mathfrak{g} = \langle \partial_x \rangle$.
Invariants=$\langle u, u_1, u_2, \ldots. \rangle$.
Case1. $a = u_0, b = u_1$.
Tresse derivative

$$\frac{d}{du_1} = u_1^{-1} \frac{d}{dx},$$

Then for Tresse derivatives $b_i = \frac{d^i b}{du_0^i}$,we have

$$b_1 = \frac{u_2}{u_1}, b_2 = \frac{u_3}{u_1^2} - \frac{u_2^2}{u_1^3}, b_3 = \frac{u_4}{u_1^3} - 4\frac{u_2 u_3}{u_1^4} + 3\frac{u_2^3}{u_1^5}.$$

Thus, the quotient of ODE equation $F(u_0, u_1, u_2, u_3) = 0$ has the form

$$F\left(a, b_0, b_0 b_1, b_0^2 b_2 + b_0 b_1^2\right) = 0.$$

It is the standard reduction of order for ODEs of the form $F(u_0, u_1, u_2, u_3) = 0$.

Case2:

$a = u_2, b^1 = u_0, b^2 = u_1$
We have

$$\frac{d}{da} = u_3^{-1}\frac{d}{dx},$$

and

$$b_1^1 = u_3^{-1}u_1, b_1^2 = u_3^{-1}u_2, b_2^1 = u_3^{-2}u_2 - u_3^{-3}u_1u_4, b_2^2 = u_3^{-1} - u_3^{-2}u_2u_4.$$

Therefore, the quotient of ODE equation $F(u_0, u_1, u_2, u_3) = 0$ is the ODE system

$$\begin{aligned} F\left(b^1, b^2, a, \frac{a}{b_1^2}\right) &= 0, \\ ab_1^1 - b^2 b_1^2 &= 0. \end{aligned}$$

$\Omega = \mathbb{R}$, $\mathfrak{g} = \langle \partial_x, x\partial_x \rangle$.
Invariants$=\left\langle u_0, \frac{u_2}{u_1^2}, \frac{u_3}{u_1^3}, \frac{u_4}{u_1^4}, ... \right\rangle$
Let's take $a = u_0$, $b = \frac{u_2}{u_1^2}$.
Then

$$b_1 = \frac{u_3}{u_1^3} - 2\frac{u_2^2}{u_1^4}, b_2 = \frac{u_4}{u_1^4} - 7\frac{u_2 u_3}{u_1^5}.$$

Therefore quotient of the equation $F\left(u_0, \frac{u_2}{u_1^2}, \frac{u_3}{u_1^3}, \frac{u_4}{u_1^4}\right) = 0$ is the following

$$F\left(a, b, b_1 + 2b^2, b_2 + 7bb_1 + 6b^3\right) = 0.$$

$\Omega = \mathbb{R}$, $\mathfrak{g} = \mathfrak{sl}_2 = \left\langle \partial_x, x\partial_x, x^2\partial_x \right\rangle$

Invariants $= \left\langle u_0, \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4}, \frac{u_4}{u_1^4} - 6\frac{u_2 u_3}{u_1^5} + 6\frac{u_2^3}{u_1^6}, \ldots \right\rangle$.

Taking $a = u_0$, $b_0 = \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4}$ we get that the quotient of differential equation

$$F\left(u_0, \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4}, \frac{u_4}{u_1^4} - 6\frac{u_2 u_3}{u_1^5} + 6\frac{u_2^3}{u_1^6}\right) = 0$$

is the following

$$F\left(a, b_0, b_1\right) = 0.$$

Integrability ODEs

Assume that we have an ODEs $\mathcal{E}$ of order k. The Lie-Bianchi theorem states that this equation could be integrated by quadratures if we have solvable symmetry Lie algebra $\mathfrak{g}$ and $\dim \mathfrak{g} = k$. If $\mathfrak{g}$ is not solvable but still $\dim \mathfrak{g} = k$ then the integration could be done by using a number of model equations (depending on simple subalgebras in $\mathfrak{g}$).
The above examples show us that, in the case algebraic ODEs and symmetry algebras, condition $\dim \mathfrak{g} = k - 1$ reduces the integrability to the 1-st order quotient equation and integration of a $(k-1)$ order equation having the same symmetry $\mathfrak{g}$. Continue in this way we splits the integration procedure to integration of a series of quotients.

Constant coefficients,hyperbolic case

$\Omega = \mathbb{R}^2$, $\mathfrak{g} = \langle \partial_x, \partial_y \rangle$.
Invariants$=\langle u_{0,0}, u_{1,0}, u_{0,1}, u_{2,0}, u_{1,1}, u_{0,2}, \ldots.. \rangle$
Lie-Tresse coordinates: $a_1 = u_{1,0}, a_2 = u_{0,1}, b = u_{0,0}, c = u_{1,1}$.
Then

$$\begin{aligned} b_{1,0} &= \delta^{-1}\left(u_{1,0}u_{0,2} - u_{0,1}u_{1,1}\right), \; b_{0,1} = \delta^{-1}\left(u_{0,1}u_{2,0} - u_{1,0}u_{1,1}\right), \\ c_{1,0} &= \delta^{-1}\left(u_{0,2}u_{2,1} - u_{1,1}u_{1,2}\right), \; c_{0,1} = \delta^{-1}\left(u_{2,0}u_{1,2} - u_{1,1}u_{2,1}\right), \end{aligned}$$

where $\delta = u_{2,0}u_2 - u_{1,1}^2$ is the Hessian.
The syzygy

$$\begin{aligned} & c^2\left(b_{1,0}^2 b_{02} - 2b_{1,0}b_{0,1}b_{1,1} + b_{0,1}^2 b_{2,0}\right) \\ +c & \left(-a_1 b_{0,1} b_{2,0} - a_2 b_{1,0} b_{0,2} + \left(a_1 b_{1,0} + a_2 b_{0,1}\right) b_{1,1} + b_{1,0} b_{0,1}\right) \\ & -a_1 a_2 b_{1,1} - b_{1,0} b_{0,1}\left(a_1 c_{1,0} + a_2 c_{0,1}\right) = 0. \end{aligned}$$

Thus the quotient of an equation $u_{1,1} = C\left(u_{1,0}, u_{0,1}\right)$ is the above equation for $b\left(u_{1,0}, u_{0,1}\right)$, where capital C stands instead of c.

Examples

1) The wave equation $u_{1,1} = 0$ is 'self dual' it coincides with the quotient.
2) The quotient of the equation

$$u_{1,1} = u_{1,0}\, u_{0,1}$$

is the following

$$a_1 a_2 \left(b_{1,0}^2 b_{02} - 2 b_{1,0} b_{0,1} b_{1,1} + b_{0,1}^2 b_{2,0}\right) = \\ a_1 b_{0,1} b_{2,0} + a_2 b_{1,0} b_{0,2} - \left(a_1 b_{1,0} + a_2 b_{0,1} - 1\right) b_{1,1} - b_{1,0} b_{0,1}$$