Higher symmetries and conservation laws of (1+1)-dimensional Klein–Gordon equation

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Motivation

Consider the $(1+1)$-dimensional Klein–Gordon equation
$(\partial^2/\partial x_0^2 - \partial^2/\partial x_1^2)u + m^2 u = 0$
in the characteristic, or light-cone, variables

$$\mathcal{K}\colon \quad u_{xy} = u, \quad \text{or} \quad \mathfrak{K}u = 0 \quad \text{with} \quad \mathfrak{K} = \mathrm{D}_x\mathrm{D}_y - 1,$$

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Lie symmetries:

$$\mathfrak{g} = (\langle \partial_x,\ \partial_y,\ x\partial_x - y\partial_y \rangle \oplus \langle u\partial_u \rangle) \Subset \langle f(x,y)\partial_u \rangle = \mathfrak{g}^{\rm ess} \Subset \tilde{\Sigma}^{-\infty} \simeq (\mathfrak{e}(1,1) \oplus \mathfrak{a}_1) \Subset \tilde{\Sigma}^{-\infty},$$

where $\mathfrak{e}(1,1)$ is the pseudo-Euclidean Lie algebra, $\mathfrak{a}_1$ is the one-dimensional Lie algebra and f runs through the solution set of $\mathcal{K}$.

Let Σ be the algebra of higher and $\Sigma^{\rm triv}$ the algebra of trivial higher symmetries of $\mathcal{K}$.

$$\Sigma^{\rm q}=\Sigma/\Sigma^{\rm triv}\cong\hat\Sigma^{\rm q}=\{Q=\eta\partial_u\in\Sigma\mid\eta\in\mathcal{F}(\mathcal{K})\},$$
$$\Sigma^n\cong\hat\Sigma^n=\{\eta\partial_u\in\hat\Sigma^{\rm q}\mid\eta\in\mathcal{F}_n(\mathcal{K})\},\quad n\in\mathbb{N}_0\cup\{-\infty\},$$
$$\hat\Sigma^{-\infty}=\{f(x,y)\partial_u\mid f\in\mathcal{F}_{-\infty}(\mathcal{K}),\quad f_{xy}=f\},$$
$$\Sigma^{[n]}=\Sigma^n/\Sigma^{n-1}\text{ for }n\in\mathbb{N},\quad\Sigma^{[0]}=\Sigma^0/\Sigma^{-\infty}\text{ and denote }\Sigma^{[-\infty]}:=\Sigma^{-\infty}.$$

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$$\Lambda=\Big\{\eta\partial_u\in\Sigma\;\Big|\;\eta=\mathfrak{D}u\ \text{ for some }\ \mathfrak{D}=\sum_{|\alpha|\leqslant n}\zeta^\alpha(x,y)\mathrm{D}_x^{\alpha_1}\mathrm{D}_y^{\alpha_2},\ n\in\mathbb{N}_0\Big\},$$
$$\Lambda^{\rm triv}=\Lambda\cap\Sigma^{\rm triv},\quad\Lambda^{\rm q}=\Lambda/\Lambda^{\rm triv},\quad\Lambda^n=\Lambda^{\rm q}\cap\Sigma^n\cong\hat\Lambda^n,\quad n\in\mathbb{N}_0.$$

Denote $u_0:=u$, $u_k:=\mathrm{D}_x^ku$ and $\bar u_k:=\mathrm{D}_y^ku$, $k\in\mathbb{N}$. Let $\eta\in\mathcal{F}_n(\mathcal{K})$, then

$$\eta\partial_u\in\hat\Lambda^n\text{ iff }\eta=\sum_{k=1}^n\eta^k(x,y)u_k+\eta^0(x,y)u+\sum_{k=1}^n\eta^{-k}(x,y)\bar u_k.$$

$$\Lambda^{[n]}=\Lambda^n/\Lambda^{n-1},\ n\in\mathbb{N},\quad\Lambda^{[0]}=\Lambda^0.$$

Lemma

$\dim \Lambda^{[n]} = 2n+1$.

$$\ell_{\mathfrak{K}u}(\eta) = 0 \Rightarrow \Delta_k\colon\ \eta^k_{xy} + \eta^{k-1}_y + \eta^{k+1}_x = 0, \quad k = -n-1, -n, \dots, n, n+1,$$

where $\eta^{-n-2} = \eta^{-n-1} = \eta^{n+1} = \eta^{n+2} = 0$.

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Symmetries are of order n if and only if at least one of η^{-n} and η^{n} does not vanish.

Suppose $\eta^{-n} \neq 0$. Then Δ_{-n-1}: $\eta^{-n}_x = 0 \Rightarrow \eta^{-n} = \theta(y)$; $\quad \Delta_{-n}:\; \eta^{-n+1}_x = 0$.

$\Delta_k:\; \eta^{k+1}_x = -\eta^k_{xy} - \eta^{k-1}_y$, where $k = -n+1, -n+2, \dots, n-1$

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Successively integrate with respect to x,

$$\eta^n = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n\theta}{\mathrm{d}y^n}x^n + R, \text{ where } R \text{ is a polynomial in } x \text{ with } \deg_x R < n\ .$$

Δ_n: $\eta^n_y = 0 \Rightarrow \mathrm{d}^{n+1}\theta/\mathrm{d}y^{n+1} = 0$.

Corollary

$$\dim \Lambda^n = \sum_{k=0}^{n} \dim \Lambda^{[k]} = (n+1)^2 < +\infty.$$

Lemma

The space $\Sigma^{[n]}$ *with* $n \in \mathbb{N}_0$ *is isomorphic to the subspace*

$$\tilde{\Sigma}^{[n]} = \langle\, (\mathrm{J}^n u)\partial_u,\, (\mathrm{J}^k \mathrm{D}_x^{n-k} u)\partial_u,\, (\mathrm{J}^k \mathrm{D}_y^{n-k} u)\partial_u,\, k = 0, \dots, n-1 \rangle$$

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Shapovalov–Shirokov theorem + Corollary $\Rightarrow \Sigma^{[n]} = \Lambda^{[n]}$ for $n \in \mathbb{N}_0$.

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Shapovalov–Shirokov theorem + Corollary $\Rightarrow \Sigma^{[n]} = \Lambda^{[n]}$ for $n \in \mathbb{N}_0$.

D_x, D_y and J are recursion operators of $\mathcal{K}$. So $\mathfrak{D}$ in the universal enveloping algebra generated by these operators is a symmetry operator of $\mathcal{K}$. Thus, $\tilde{\Sigma}^{[n]} \subset \Lambda \subset \Sigma$.

The space $\tilde{\Sigma}^{[n]}$ contains no nonzero trivial higher symmetries of $\mathcal{K}$.
Indeed, let $Q \in \tilde{\Sigma}^{[n]}$ with the generating function

$$\mathcal{F}_n(\mathcal{K}) \ni Q[u] = a\mathrm{J}^n u + \sum_{k=0}^{n-1} \left(b_k \mathrm{J}^k \mathrm{D}_x^{n-k} u + c_k \mathrm{J}^k \mathrm{D}_y^{n-k} u \right)$$

be a trivial symmetry. Here a, b's and c's are constants.

$u^\lambda = e^{\lambda x + \lambda^{-1} y}$ is a family of solution of $\mathcal{K}$, $\lambda \in \mathbb{R}/\{0\}$.

$e^{-\lambda x - \lambda^{-1} y} Q[u^\lambda]$ is a polynomial in $\lambda x - \lambda^{-1} y$, $\lambda x + \lambda^{-1} y$, λ and λ^{-1}, whose collection of terms of maximal total degree, which equals n, coincides with

$$a \left(\lambda x - \lambda^{-1} y \right)^n + \sum_{k=0}^{n-1} \left(\lambda x - \lambda^{-1} y \right)^k \left(b_k \lambda^{n-k} + c_k \lambda^{k-n} \right).$$

$Q[u^\lambda] = 0 \Rightarrow a = 0,\ b_k = c_k = 0,\ k = 0, \ldots, n-1.$

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$Q[u^\lambda] = 0 \Rightarrow a = 0,\ b_k = c_k = 0,\ k = 0, \ldots, n-1.$

Theorem

The quotient algebra Σ^{q} of higher symmetries of the Klein–Gordon equation $\mathcal{K}$ is naturally isomorphic to the algebra $\tilde{\Sigma}^{\mathrm{q}}$, which is the semidirect sum of the algebra

$$\tilde{\Lambda}^{\mathrm{q}} = \langle\, (\mathrm{J}^k u)\partial_u,\, (\mathrm{J}^k \mathrm{D}_x^l u)\partial_u,\, (\mathrm{J}^k \mathrm{D}_y^l u)\partial_u,\, k \in \mathbb{N}_0,\, l \in \mathbb{N} \rangle \simeq \mathfrak{U}(\mathfrak{e}(1,1) \oplus \mathfrak{a}_1)/I$$

with the abelian algebra $\tilde{\Sigma}^{-\infty} = \{ f(x,y)\partial_u \mid f \text{ satisfies } \mathcal{K} \}$. Here I is the two-sided ideal of the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}(1,1) \oplus \mathfrak{a}_1)$ that is generated by the equivalence classes of $e_1 \otimes e_2 - e_0$ and $e_0 \otimes e_k - e_k$, $k = 0, 1, 2, 3$.

Noether symmetries

For $Q \in \tilde{\Lambda}^{\rm q}$ let $\mathfrak{Q}u = \sum_{|\alpha|} \zeta^\alpha(x,y)\mathrm{D}_x^{\alpha_1}\mathrm{D}_y^{\alpha_2}u$ be its generating function, $\mathfrak{K}\mathfrak{Q} = \mathfrak{Q}\mathfrak{K}$.

$\mathcal{K}$ is the Euler–Lagrange equation for the Lagrangian $K = -(u_xu_y + u^2)/2$.

Υ is the algebra of Noether symmetries of K, $\Upsilon^{\rm triv} := \Upsilon \cap \Sigma^{\rm triv}$ and $\Upsilon^{\rm q} := \Upsilon/\Upsilon^{\rm triv}$,

$$\Upsilon^n = \{[Q] \in \Upsilon^{\rm q} \mid \exists\, \eta\partial_u \in [Q]\colon \eta \in \mathcal{F}_n(\mathcal{K})\}, \quad n \in \mathbb{N}_0 \cup \{-\infty\},$$

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Lemma

A linear higher symmetry $Q\in\Lambda$ of $\mathcal{K}$ is a Noether symmetry of K if and only if the corresponding operator $\mathfrak{Q}$ is formally skew-adjoint, $\mathfrak{Q}^*=-\mathfrak{Q}$.

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$$0 = \ell^*_{\mathfrak{Q}u}(\mathfrak{K}u) + \ell^*_{\mathfrak{K}u}(\mathfrak{Q}u) = (\mathfrak{Q}^*\mathfrak{K} + \mathfrak{K}^*\mathfrak{Q})u \quad \Leftrightarrow$$
$$0 = \mathfrak{Q}^*\mathfrak{K} + \mathfrak{K}^*\mathfrak{Q} = \mathfrak{Q}^*\mathfrak{K} + \mathfrak{K}\mathfrak{Q} = \mathfrak{Q}^*\mathfrak{K} + \mathfrak{Q}\mathfrak{K} = (\mathfrak{Q}^* + \mathfrak{Q})\mathfrak{K} \quad \Leftrightarrow$$
$$\mathfrak{Q}^* + \mathfrak{Q} = 0.$$

$$\tilde{\Lambda}^{\rm q} = \langle((\mathfrak{Q}_{kl}u)\partial_u,\ k,l\in\mathbb{N}_0,\ (\bar{\mathfrak{Q}}_{kl}u)\partial_u,\ k\in\mathbb{N}_0,\ l\in\mathbb{N}\rangle,$$

$$\mathfrak{Q}_{kl} = \left(\mathrm{J}+\frac{l}{2}\right)^k \mathrm{D}_x^l,\quad k,l\in\mathbb{N}_0,\qquad \bar{\mathfrak{Q}}_{kl} = \left(\mathrm{J}-\frac{l}{2}\right)^k \mathrm{D}_y^l,\quad k\in\mathbb{N}_0,\quad l\in\mathbb{N}.$$

The algebra $\tilde{\Lambda}^{\rm q}$ is decomposed into the direct sum of two subspaces, $\tilde{\Lambda}^{\rm q} = \tilde{\Lambda}^{\rm q}_- \dotplus \tilde{\Lambda}^{\rm q}_+$, where $\tilde{\Lambda}^{\rm q}_-$ (resp. $\tilde{\Lambda}^{\rm q}_+$) is the subspace of elements in $\tilde{\Lambda}^{\rm q}$ associated with formally skew-adjoint (resp. self-adjoint) operators.

$$\mathrm{D}_x^* = -\mathrm{D}_x,\quad \mathrm{D}_y^* = -\mathrm{D}_y,\quad \mathrm{J}^* = -\mathrm{J},\quad \mathrm{D}_x\mathrm{J} = (\mathrm{J}+1)\mathrm{D}_x,\quad \mathrm{D}_y\mathrm{J} = (\mathrm{J}-1)\mathrm{D}_y,$$

$$\Rightarrow\quad \mathfrak{Q}_{kl}^* = (-1)^{k+l}\mathrm{D}_x^l\left(\mathrm{J}-\frac{l}{2}\right)^k = (-1)^{k+l}\mathfrak{Q}_{kl}\quad\text{and}\quad \bar{\mathfrak{Q}}_{kl}^* = (-1)^{k+l}\bar{\mathfrak{Q}}_{kl}.$$

$$\tilde{\Lambda}^{\rm q}_- = \langle\,(\mathfrak{Q}_{k'0}u)\partial_u,\ k'\in 2\mathbb{N}_0+1,\ (\mathfrak{Q}_{kl}u)\partial_u,\ (\bar{\mathfrak{Q}}_{kl}u)\partial_u,\ k\in\mathbb{N}_0,\ l\in\mathbb{N},\ k+l\in 2\mathbb{N}_0+1\rangle,$$

$$\tilde{\Lambda}^{\rm q}_+ = \langle\,(\mathfrak{Q}_{k'0}u)\partial_u,\ k'\in 2\mathbb{N}_0,\ (\mathfrak{Q}_{kl}u)\partial_u,\ (\bar{\mathfrak{Q}}_{kl}u)\partial_u, k\in\mathbb{N}_0,\ l\in\mathbb{N},\ k+l\in 2\mathbb{N}_0\rangle.$$

Theorem

The quotient algebra $\Upsilon^{\rm q}$ of Noether symmetries of the $(1+1)$-dimensional Klein–Gordon equation $\mathcal{K}$ is naturally isomorphic to the algebra $\tilde{\Upsilon}^{\rm q} = \tilde{\Lambda}^{\rm q}_- \Subset \tilde{\Sigma}^{-\infty}$.

Noether theorem: generating functions of Noether symmetries of $K \leftrightarrow$ generating functions of conservations laws.

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1) $f(x,y)\partial_u \in \tilde\Sigma^{-\infty}$: $f\mathfrak{K}u = \mathrm{D}_x(fu_y) + \mathrm{D}_y(-f_x u) \Rightarrow \mathrm{C}^0_f = (fu_y, -f_x u)$.

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2) $(\mathfrak{Q}_{kl}u)\partial_u \in \tilde{\Lambda}^{\mathrm{q}}_-$, $k+l$ is odd, $\mathrm{J} = \mathrm{D}_x \circ x - \mathrm{D}_y \circ y$.

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a) k is odd, $k' = (k-1)/2$, $l' = l/2$, $\mathfrak{Q}_{kl} = \mathrm{D}_x^{l'}\mathrm{J}^k\mathrm{D}_x^{l'}$. Then

$$
\begin{aligned}
(\mathfrak{Q}_{kl}u)\mathfrak{K}u = \mathrm{D}_x \sum_{l''=0}^{l'-1} (-1)^{l''} \left(\mathrm{D}_x^{l'-l''-1}\mathrm{J}^k\mathrm{D}_x^{l'}u\right)\mathrm{D}_x^{l''}\mathfrak{K}u \\
+ \mathrm{J}\sum_{k''=0}^{k'-1}(-1)^{l'+k''}\left(\mathrm{J}^{2k'-k''}\mathrm{D}_x^{l'}u\right)\mathrm{J}^{k''}\mathrm{D}_x^{l'}\mathfrak{K}u \\
+ \frac{(-1)^{l'+k'}}{2}\left(x\mathrm{D}_y(\mathrm{D}_x\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2 - y\mathrm{D}_x(\mathrm{D}_y\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2 - \mathrm{J}(\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2\right),
\end{aligned}
$$

$$
\Rightarrow \quad \mathrm{C}^1_{k'l'} = \left(-y(\mathrm{D}_y\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2 - x(\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2,\ x(\mathrm{D}_x\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2 + y(\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2\right)
$$

is of order $k' + l' + 1 = (k+l+1)/2$, which is minimal for the conserved currents related to $\mathfrak{Q}_{kl}u$.

b) k is even, $k' = k/2$ and $l' = (l-1)/2$. Hence $\mathfrak{Q}_{kl} = \mathrm{D}_x^{l'}(\mathrm{J}+1/2)^{k'}\mathrm{D}_x(\mathrm{J}-1/2)^{k'}\mathrm{D}_x^{l'} = \mathrm{D}_x^{l'+1}(\mathrm{J}-1/2)^k\mathrm{D}_x^{l'} = \mathrm{D}_x^{l'}(\mathrm{J}+1/2)^k\mathrm{D}_x^{l'+1}$ and

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& + \mathrm{J}\sum_{k''=0}^{k'-1}(-1)^{l'+k''}\left(\left(\mathrm{J}+\frac{1}{2}\right)^{k-k''-1}\mathrm{D}_x^{l'+1}u\right)\left(\mathrm{J}-\frac{1}{2}\right)^{k''}\mathrm{D}_x^{l'}\mathfrak{K}u \\
& + \frac{(-1)^{l'+k'}}{2}\left(\mathrm{D}_y\left(\mathrm{D}_x\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2 - \mathrm{D}_x\left(\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2\right),
\end{aligned}
$$

$$
\Rightarrow \quad \mathrm{C}^2_{k'l'} = \left(-\left(\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2,\ \left(\mathrm{D}_x\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2\right)
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$$\Rightarrow \quad \mathrm{C}^2_{k'l'} = \left(-\left(\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2,\ \left(\mathrm{D}_x\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2\right)$$

is of order $k'+l'+1 = (k+l+1)/2$, which is minimal for the conserved currents related to $\mathfrak{Q}_{kl}u$.

Corollary

Up to adding low-order conservation laws, the Klein–Gordon equation $\mathcal{K}$ possesses $4n-1$ linearly independent conservation laws of order n if $n \geqslant 2$, and an infinite number of linearly independent first-order conservation laws.

Thank you for your attention!