

First Order Partial Differential Equations: Symmetries, Equivalence and Decoupling

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Generic first Order PDEs

Consider a nonlinear mathematical model given in terms of a general first order PDEs,

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) = 0,$$

where

- ① $\mathbf{x} \in \mathbb{R}^n$ denotes the set of independent variables,
- ② $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^m$ denotes the set of dependent variables,
- ③ $\mathbf{u}^{(1)}$ denotes the set of first order partial derivatives of \mathbf{u} w.r.t. \mathbf{x} .

Quasilinear systems

In mathematical physics a special role is played by quasilinear first order systems:

$$\sum_{i=1}^n A^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} - \mathbf{B}(\mathbf{x}, \mathbf{u}) = \mathbf{0}.$$

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Plan of the talk

1 Lie group theory framework:

- 1.1 Mapping of a nonhomogeneous and nonautonomous quasilinear first order system to homogeneous and autonomous form;
- 1.2 Mapping of a **nonlinear** first order system to homogeneous and autonomous quasilinear form;

2 Decoupling problem of a general quasilinear first order system in **two** independent variables through (locally) invertible point transformations.

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Lie symmetries of DEs

- 1 Lie symmetries of DEs are important ingredients in the process of finding solutions.
- 2 Lie symmetries of DEs can also be used to algorithmically construct a mapping from a (SOURCE) DE to another (TARGET) suitable DE.

Mappings between different DEs

Such a mapping (if it exists) needs not be a group transformation; moreover, any symmetry admitted by the source DE has to be mapped to a symmetry admitted by the target DE.

If the mapping is 1–1 then the mapping must establish a 1–1 correspondence between symmetries of source and target DEs: the corresponding Lie algebras of infinitesimal generators have to be isomorphic.

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Considered Mappings

- Nonautonomous and Nonhomogeneous First Order Quasilinear PDEs vs. Autonomous and Homogeneous First Order Quasilinear PDEs [F. O., IJNLM, 2012] by means of

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \quad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u});$$

- Nonlinear first order PDEs to autonomous and homogeneous quasilinear first order PDEs [F. O., 2015] by means of

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Lemma

Given the system

$$\sum_{i=1}^n A^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{G}(\mathbf{x}, \mathbf{u}),$$

an invertible mapping of the form

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produces a system still in quasilinear form.

Proof.

Straightforward. □

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Theorem [F. O., IJNLM–2012]

$$\sum_{i=1}^n A^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x}, \mathbf{u}) \quad \Leftrightarrow \quad \sum_{i=1}^n \hat{A}^i(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}$$

through an invertible point transformation,

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \quad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}),$$

if and only if it admits as subalgebra of its Lie point symmetries an $(n + 1)$ -dimensional Lie algebra spanned by

$$\Xi_i = \sum_{j=1}^n \xi_i^j(\mathbf{x}) \frac{\partial}{\partial x_j} + \sum_{A=1}^m \eta_i^A(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_A}, \quad (i, \dots, n + 1),$$

such that

$$[\Xi_i, \Xi_j] = 0, \quad [\Xi_i, \Xi_{n+1}] = \Xi_i, \quad i, j = 1, \dots, n.$$

Furthermore, all minors of order n extracted from the $(n + 1) \times n$ matrix with entries ξ_i^j ($i = 1, \dots, n + 1, j = 1, \dots, n$) must be non-vanishing, and the variables \mathbf{w} , which by construction are invariants of Ξ_1, \dots, Ξ_n , must result invariant with respect to Ξ_{n+1} too.

Proof: Necessary condition

Every system of the form

$$\sum_{i=1}^n \hat{A}^i(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}$$

is invariant w.r.t. the n translations and a uniform scaling of the z_i :

$$\Xi_i = \frac{\partial}{\partial z_i} \quad (i = 1, \dots, n), \quad \Xi_{n+1} = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}.$$

These vector fields span an $(n + 1)$ -dimensional **solvable Lie algebra**,

$$[\Xi_i, \Xi_{n+1}] = \Xi_i, \quad i = 1, \dots, n,$$

and the structure of the Lie algebra of point symmetries is not changed by an invertible point transformation.

Proof: Sufficient condition

Under the hypotheses of the theorem, using the operators Ξ_1, \dots, Ξ_n , we may algorithmically construct a set of **canonical variables** $\mathbf{z} = \mathbf{Z}(\mathbf{x})$, $\mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u})$ such that

$$\Xi_i = \frac{\partial}{\partial z_i}, \quad i = 1, \dots, n,$$

whereupon we get an autonomous system. As a result, since $[\Xi_i, \Xi_{n+1}] = \Xi_i$ ($i = 1, \dots, n$), and \mathbf{w} are invariants for Ξ_{n+1} , it is

$$\Xi_{n+1} = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i},$$

ensuring that the autonomous system is also homogeneous.

Example: Rotating shallow water equations

$$h_t + uh_x + vh_y + h(u_x + v_y) = 0,$$

$$u_t + uu_x + vu_y + gh_x = 2\omega v,$$

$$v_t + uv_x + vv_y + gh_y = -2\omega u,$$

where h is the height of the fluid, (u, v) the components of its velocity, g the gravitational constant, and ω the constant angular velocity of the fluid around the x_3 -axis responsible for the Coriolis force.

9-dimensional Lie algebra of point symmetries

$$\Xi_1 = \partial_t, \quad \Xi_2 = \partial_{x_1}, \quad \Xi_3 = \partial_{x_2},$$

$$\Xi_4 = x_2 \partial_{x_1} - x_1 \partial_{x_2} + v \partial_u - u \partial_v,$$

$$\Xi_5 = \cos(2\omega t) \partial_{x_1} - \sin(2\omega t) \partial_{x_2} - 2\omega \sin(2\omega t) \partial_u - 2\omega \cos(2\omega t) \partial_v,$$

$$\Xi_6 = \sin(2\omega t) \partial_{x_1} \cos(2\omega t) \partial_{x_2} + 2\omega \cos(2\omega t) \partial_u - 2\omega \sin(2\omega t) \partial_v,$$

$$\Xi_7 = x_1 \partial_{x_1} + x_2 \partial_{x_2} + u \partial_u + v \partial_v + 2h \partial_h,$$

$$\begin{aligned} \Xi_8 = & \sin(2\omega t) \partial_t + \omega(x_1 \cos(2\omega t) + x_2 \sin(2\omega t)) \partial_{x_1} \\ & + \omega(x_2 \cos(2\omega t) - x_1 \sin(2\omega t)) \partial_{x_2} \\ & + \omega((2\omega x_2 - u) \cos(2\omega t) + (-2\omega x_1 + v) \sin(2\omega t)) \partial_u \\ & - \omega((2\omega x_1 + v) \cos(2\omega t) + (2\omega x_2 + u) \sin(2\omega t)) \partial_v \\ & - 2\omega h \cos(2\omega t) \partial_h, \end{aligned}$$

$$\begin{aligned} \Xi_9 = & \cos(2\omega t) \partial_t + \omega(x_2 \cos(2\omega t) - x_1 \sin(2\omega t)) \partial_{x_1} \\ & - \omega(x_1 \cos(2\omega t) + x_2 \sin(2\omega t)) \partial_{x_2} \\ & - \omega((2\omega x_1 - v) \cos(2\omega t) + (2\omega x_2 - u) \sin(2\omega t)) \partial_u \\ & - \omega((2\omega x_2 + u) \cos(2\omega t) - (2\omega x_1 + v) \sin(2\omega t)) \partial_v \\ & - 2\omega h \sin(2\omega t) \partial_h. \end{aligned}$$

Thee 4–dimensional subalgebra spanned by the vector fields

$$\begin{aligned}\widehat{\Xi}_1 &= \Xi_1 + \omega \Xi_4 - \Xi_9, & \widehat{\Xi}_2 &= \Xi_3 - \Xi_6, \\ \widehat{\Xi}_3 &= -\Xi_2 + \Xi_5, & \widehat{\Xi}_4 &= \frac{1}{2} \left(\Xi_7 - \frac{1}{\omega} \Xi_8 \right),\end{aligned}$$

allows us to introduce

$$\begin{aligned}\tau &= -\frac{1}{2\omega} \cot(\omega t), & \xi &= \frac{1}{2}(y - x \cot(\omega t)), & \eta &= -\frac{1}{2}(x + y \cot(\omega t)), \\ U &= -\frac{1}{2}(u \sin(2\omega t) - v(1 - \cos(2\omega t)) - 2\omega x), \\ V &= -\frac{1}{2}(u(1 - \cos(2\omega t)) + v \sin(2\omega t) - 2\omega y), \\ H &= \frac{1 - \cos(2\omega t)}{2} h,\end{aligned}$$

whereupon we get

$$\begin{aligned}U_\tau + UU_\xi + VU_\eta + gH_\xi &= 0, \\ V_\tau + UV_\xi + VV_\eta + gH_\eta &= 0, \\ H_\tau + UH_\xi + VH_\eta + H(U_\xi + V_\eta) &= 0.\end{aligned}$$

Example: Monatomic perfect gas in rotation and subject to gravity

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^3 \frac{\partial(\rho v_k)}{\partial x_k} = 0,$$

$$\rho \left(\frac{\partial v_i}{\partial t} + \sum_{k=1}^3 v_k \frac{\partial v_i}{\partial x_k} \right) + \frac{K}{m} \frac{\partial(\rho \theta)}{\partial x_i} = \rho(F_i^{(e)} + F_i^{(i)}), \quad i = 1, 2, 3,$$

$$\frac{\partial \theta}{\partial t} + \sum_{k=1}^3 v_k \frac{\partial \theta}{\partial x_k} + \frac{2}{3} \theta \sum_{k=1}^3 \frac{\partial v_k}{\partial x_k} = 0,$$

where ρ is the density, θ the temperature, \mathbf{v} the velocity, $F_i^{(i)}$ the components of the *specific inertial forces*, $F_i^{(e)}$ the components of the *specific external forces* acting on the gas, K is the Boltzmann constant, m the mass of a single particle.

$$F_i^{(i)} = 2\epsilon_{ijl}\omega_l v_j + \omega^2 x_i - (\omega_r x_r)\omega_i, \quad \boldsymbol{\omega} = (0, 0, \omega), \quad \epsilon_{ijl} \text{ Ricci tensor.}$$

$$\rho(\mathbf{F}^{(e)} + \mathbf{F}^{(i)}) = (\rho(2\omega v_2 + \omega^2 x_1), \rho(-2\omega v_1 + \omega^2 x_2), -\rho g).$$

14 Lie point symmetries admitted

$$\Xi_1 = \partial_t, \quad \Xi_2 = \partial_{x_3}, \quad \Xi_3 = x_2 \partial_{x_1} - x_1 \partial_{x_2} + v_2 \partial_{v_1} - v_1 \partial_{v_2},$$

$$\Xi_4 = t \partial_{x_3} + \partial_{v_3}, \quad \Xi_5 = \rho \partial_\rho,$$

$$\Xi_6 = x_1 \partial_{x_1} + x_2 \partial_{x_2} + \left(x_3 + \frac{1}{2} g t^2 \right) \partial_{x_3} + v_1 \partial_{v_1} + v_2 \partial_{v_2} + (v_3 + g t) \partial_{v_3} + 2\theta \partial_\theta,$$

$$\begin{aligned} \Xi_7 = & t \partial_t + \omega t x_2 \partial_{x_1} - \omega t x_1 \partial_{x_2} - g t^2 \partial_{x_3} + (-v_1 + \omega t v_2 + \omega x_2) \partial_{v_1} \\ & + (-\omega t v_1 - v_2 - \omega x_1) \partial_{v_2} + (-v_3 - 2g t) \partial_{v_3} - 2\theta \partial_\theta, \end{aligned}$$

$$\begin{aligned} \Xi_8 = & t^2 \partial_t + t(x_1 + \omega t x_2) \partial_{x_1} - t(\omega t x_1 - x_2) \partial_{x_2} - \frac{1}{2} t(g t^2 - 2x_3) \partial_{x_3} - 3\rho t \partial_\rho \\ & + (-t v_1 + \omega t^2 v_2 + x_1 + 2\omega t x_2) \partial_{v_1} + (-\omega t^2 v_1 - t v_2 - 2\omega t x_1 + x_2) \partial_{v_2} \\ & + \frac{1}{2} (-3g t^2 - 2t v_3 + 2x_3) \partial_{v_3} - 2t\theta \partial_\theta, \end{aligned}$$

$$\Xi_9 = \cos(\omega t) \partial_{x_1} - \sin(\omega t) \partial_{x_2} - \omega \sin(\omega t) \partial_{v_1} - \omega \cos(\omega t) \partial_{v_2},$$

$$\Xi_{10} = \sin(\omega t) \partial_{x_1} + \cos(\omega t) \partial_{x_2} + \omega \cos(\omega t) \partial_{v_1} - \omega \sin(\omega t) \partial_{v_2},$$

... Symmetries

$$\begin{aligned}\Xi_{11} = & -t \cos(\omega t) \partial_{x_1} + t \sin(\omega t) \partial_{x_2} + (-\cos(\omega t) + \omega t \sin(\omega t)) \partial_{v_1} \\ & + (\sin(\omega t) + \omega t \cos(\omega t)) \partial_{v_2},\end{aligned}$$

$$\begin{aligned}\Xi_{12} = & t \sin(\omega t) \partial_{x_1} + t \cos(\omega t) \partial_{x_2} + (\sin(\omega t) + \omega t \cos(\omega t)) \partial_{v_1} \\ & + (\cos(\omega t) - \omega \sin(\omega t)) \partial_{v_2},\end{aligned}$$

$$\begin{aligned}\Xi_{13} = & (-gt^2 - 2x_3) \cos(\omega t) \partial_{x_1} + (gt^2 + 2x_3) \sin(\omega t) \partial_{x_2} + 2(x_1 \cos(\omega t) - x_2 \sin(\omega t)) \partial_{x_3} \\ & + (-2gt \cos(\omega t) - 2v_3 \cos(\omega t) + g\omega t^2 \sin(\omega t) + 2\omega x_3 \sin(\omega t)) \partial_{v_1} \\ & + (g\omega t^2 \cos(\omega t) + 2\omega x_3 \cos(\omega t) + 2gt \sin(\omega t) + 2v_3 \sin(\omega t)) \partial_{v_2} \\ & + 2(v_1 \cos(\omega t) - \omega x_2 \cos(\omega t) - v_2 \sin(\omega t) - \omega x_1 \sin(\omega t)) \partial_{v_3},\end{aligned}$$

$$\begin{aligned}\Xi_{14} = & (gt^2 + 2x_3) \sin(\omega t) \partial_{x_1} + (gt^2 + 2x_3) \cos(\omega t) \partial_{x_2} - 2(x_2 \cos(\omega t) + x_1 \sin(\omega t)) \partial_{x_3} \\ & + (g\omega t^2 \cos(\omega t) + 2\omega x_3 \cos(\omega t) + 2gt \sin(\omega t) + 2v_3 \sin(\omega t)) \partial_{v_1} \\ & + (2gt \cos(\omega t) + 2v_3 \cos(\omega t) - g\omega t^2 \sin(\omega t) - 2\omega x_3 \sin(\omega t)) \partial_{v_2} \\ & + 2(-v_2 \cos(\omega t) - \omega x_1 \cos(\omega t) - v_1 \sin(\omega t) + \omega x_2 \sin(\omega t)) \partial_{v_3}.\end{aligned}$$

Reduction to homogeneous and autonomous form

By taking the 5–dimensional Lie subalgebra of Lie point symmetries spanned

$$\begin{aligned}\widehat{\Xi}_1 &= \Xi_1 + \omega \Xi_3 - g \Xi_4, & \widehat{\Xi}_2 &= \Xi_9, \\ \widehat{\Xi}_3 &= \Xi_{10}, & \widehat{\Xi}_4 &= \Xi_2, & \widehat{\Xi}_5 &= \Xi_5 + \Xi_7,\end{aligned}$$

the non–zero commutators are $[\widehat{\Xi}_i, \widehat{\Xi}_5] = \widehat{\Xi}_i, \quad i = 1, \dots, 4.$

Construct the new independent $(\tau, \xi_1, \xi_2, \xi_3)$ and dependent (R, V_1, V_2, V_3, T) variables

$$\begin{aligned}\tau &= t, \\ \xi_1 &= x_1 \cos(\omega t) - x_2 \sin(\omega t), \\ \xi_2 &= x_1 \sin(\omega t) + x_2 \cos(\omega t), \\ \xi_3 &= x_3 + \frac{1}{2}gt^2, \\ R &= \rho, \quad T = \theta, \\ V_1 &= ((v_1 - \omega x_2) \cos(\omega t) - (v_2 + \omega x_1) \sin(\omega t)), \\ V_2 &= ((v_2 + \omega x_1) \cos(\omega t) + (v_1 - \omega x_2) \sin(\omega t)), \\ V_3 &= v_3 + gt.\end{aligned}$$

Reduction to homogeneous and autonomous form

Hence, the operators $\hat{\Xi}_i$ ($i = 1, \dots, 5$) write as

$$\partial_\tau, \quad \partial_{\xi_1}, \quad \partial_{\xi_2}, \quad \partial_{\xi_3}, \quad \tau\partial_\tau + \xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2} + \xi_3\partial_{\xi_3},$$

and the system reduces to homogeneous and autonomous form:

$$\frac{\partial R}{\partial \tau} + \sum_{k=1}^3 \frac{\partial(RV_k)}{\partial \xi_k} = 0,$$

$$R \left(\frac{\partial V_i}{\partial \tau} + \sum_{k=1}^3 V_k \frac{\partial V_i}{\partial \xi_k} \right) + \frac{K}{m} \frac{\partial(RT)}{\partial \xi_i} = 0, \quad i = 1, 2, 3,$$

$$\frac{\partial T}{\partial \tau} + \sum_{k=1}^3 V_k \frac{\partial T}{\partial \xi_k} + \frac{2}{3} T \sum_{k=1}^3 \frac{\partial V_k}{\partial \xi_k} = 0.$$

General nonlinear systems

If we have a general first order **nonlinear** system of PDEs,

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) = 0$$

and we want to check if it is equivalent to a quasilinear homogeneous and autonomous system, we may look for an invertible mapping like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \quad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}).$$

If this is possible then the nonlinear system has to possess a suitable $(n + 1)$ -dimensional solvable Lie algebra as subalgebra of the algebra of its Lie point symmetries.

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If we have a general first order **nonlinear** system of PDEs,

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) = 0$$

and we want to check if it is equivalent to a quasilinear homogeneous and autonomous system, we may look for an invertible mapping like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \quad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}).$$

If this is possible then the nonlinear system has to possess a suitable $(n + 1)$ -dimensional solvable Lie algebra as subalgebra of the algebra of its Lie point symmetries.

Theorem [F. O., 2015]

A necessary condition in order the nonlinear system

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) = 0$$

be transformed by the invertible map

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \quad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u})$$

into an autonomous and homogeneous first order quasilinear system is that it admits as subalgebra of its Lie point symmetries an $(n + 1)$ -dimensional Lie algebra spanned by

$$\Xi_i = \sum_{j=1}^n \xi_i^j(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_j} + \sum_{A=1}^m \eta_i^A(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_A}, \quad (i, \dots, n + 1),$$

such that

$$[\Xi_i, \Xi_j] = 0, \quad [\Xi_i, \Xi_{n+1}] = \Xi_i, \quad i, j = 1, \dots, n.$$

Furthermore, all minors of order n extracted from the $(n + 1) \times (n + m)$ matrix with entries ξ_i^j and η_i^A must be non-vanishing, and the variables \mathbf{w} , which by constructions are invariants of Ξ_1, \dots, Ξ_n , must result invariant with respect to Ξ_{n+1} too.

Monge–Ampère equation in $(1 + 1)$ dimensions

Consider the 2nd order Monge–Ampère equation (the most general completely exceptional 2nd order equation [Boillat, 1968],

$$\kappa_1 \left(u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2 \right) + \kappa_2 u_{x_1 x_1} + \kappa_3 u_{x_1 x_2} + \kappa_4 u_{x_2 x_2} + \kappa_5 = 0,$$

where $\kappa_i(x_1, x_2, u, u_{x_1}, u_{x_2})$ ($i = 1, \dots, 5$); hereafter, we assume $\kappa_5 = 0$ and κ_i ($i = 1, \dots, 4$) depending at most on first order derivatives. A nonlinear first order system is obtained through the positions

$$u_{x_1} = u_1, \quad u_{x_2} = u_2 :$$

$$\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0,$$

$$\begin{aligned} \kappa_1(u_1, u_2) \left(\frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} - \left(\frac{\partial u_1}{\partial x_2} \right)^2 \right) \\ + \kappa_2(u_1, u_2) \frac{\partial u_1}{\partial x_1} + \kappa_3(u_1, u_2) \frac{\partial u_1}{\partial x_2} + \kappa_4(u_1, u_2) \frac{\partial u_2}{\partial x_2} = 0. \end{aligned}$$

Monge–Ampère equation

The nonlinear system equivalent to the Monge-Ampère equation admits the Lie symmetries spanned by the operators

$$\begin{aligned}\Xi_1 &= \frac{\partial}{\partial x_1}, & \Xi_2 &= \frac{\partial}{\partial x_2}, \\ \Xi_3 &= \left(x_1 - \frac{\partial f}{\partial u_1} \right) \frac{\partial}{\partial x_1} + \left(x_2 - \frac{\partial f}{\partial u_2} \right) \frac{\partial}{\partial x_2},\end{aligned}$$

where $f(u_1, u_2)$ is such that:

$$\kappa_1 + \kappa_4 \frac{\partial^2 f}{\partial u_1^2} - \kappa_3 \frac{\partial^2 f}{\partial u_1 \partial u_2} + \kappa_2 \frac{\partial^2 f}{\partial u_2^2} = 0.$$

It is:

$$[\Xi_1, \Xi_2] = 0, \quad [\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_2.$$

Mapping to a homogeneous and autonomous quasilinear system

By applying the theorem, we introduce

$$z_1 = x_1 - \frac{\partial f}{\partial u_1}, \quad z_2 = x_2 - \frac{\partial f}{\partial u_2}, \quad \text{new indep. var.,}$$

$$w_1 = u_1, \quad w_2 = u_2 - x_2, \quad \text{new dep. var.,}$$

and the generators of the point symmetries write as

$$\Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2};$$

as a result, the nonlinear system becomes:

$$\frac{\partial w_1}{\partial z_2} - \frac{\partial w_2}{\partial z_1} = 0,$$

$$\kappa_2(w_1, w_2) \frac{\partial w_1}{\partial z_1} + \kappa_3(w_1, w_2) \frac{\partial w_1}{\partial z_2} + \kappa_4(w_1, w_2) \frac{\partial w_2}{\partial z_2} = 0.$$

Monge–Ampère equation in $(2 + 1)$ dimensions

The most general 2nd order (hyperbolic) equation completely exceptional in $(2 + 1)$ dimensions [Ruggeri, 1973] is:

$$H = \det \begin{bmatrix} u_{x_1 x_1} & u_{x_1 x_2} & u_{x_1 x_3} \\ u_{x_1 x_2} & u_{x_2 x_2} & u_{x_2 x_3} \\ u_{x_1 x_3} & u_{x_2 x_3} & u_{x_3 x_3} \end{bmatrix}$$

$$\begin{aligned} & \kappa_1 H + \kappa_2 \frac{\partial H}{\partial u_{x_1 x_1}} + \kappa_3 \frac{\partial H}{\partial u_{x_1 x_2}} + \kappa_4 \frac{\partial H}{\partial u_{x_1 x_3}} + \kappa_5 \frac{\partial H}{\partial u_{x_2 x_2}} + \kappa_6 \frac{\partial H}{\partial u_{x_2 x_3}} \\ & + \kappa_7 \frac{\partial H}{\partial u_{x_3 x_3}} + \kappa_8 u_{x_1 x_1} + \kappa_9 u_{x_1 x_2} + \kappa_{10} u_{x_1 x_3} + \kappa_{11} u_{x_2 x_2} \\ & + \kappa_{12} u_{x_2 x_3} + \kappa_{13} u_{x_3 x_3} + \kappa_{14} = 0, \end{aligned}$$

where $\kappa_i(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3})$, $i=1, \dots, 14$; hereafter, we assume $\kappa_{14} = 0$ and κ_i ($i = 1, \dots, 13$) depending at most on first order derivatives.

Monge–Ampère equation in $(2 + 1)$ dimensions

A nonlinear first order system is obtained through the positions

$$u_1 = u_{x_1}, \quad u_2 = u_{x_2}, \quad u_3 = u_{x_3} :$$

$$\begin{aligned} \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0, \quad \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} = 0, \\ \kappa_1 H + \kappa_2 \frac{\partial H}{\partial(\partial u_1/\partial x_1)} + \kappa_3 \frac{\partial H}{\partial(\partial u_1/\partial x_2)} + \kappa_4 \frac{\partial H}{\partial(\partial u_1/\partial x_3)} \\ + \kappa_5 \frac{\partial H}{\partial(\partial u_2/\partial x_2)} + \kappa_6 \frac{\partial H}{\partial(\partial u_2/\partial x_3)} + \kappa_7 \frac{\partial H}{\partial(\partial u_3/\partial x_3)} \\ + \kappa_8 \frac{\partial u_1}{\partial x_1} + \kappa_9 \frac{\partial u_1}{\partial x_2} + \kappa_{10} \frac{\partial u_1}{\partial x_3} + \kappa_{11} \frac{\partial u_2}{\partial x_2} \\ + \kappa_{12} \frac{\partial u_2}{\partial x_3} + \kappa_{13} \frac{\partial u_3}{\partial x_3} = 0. \end{aligned}$$

Symmetries

The latter nonlinear system admits the Lie symmetries spanned by the operators

$$\begin{aligned}\Xi_1 &= \frac{\partial}{\partial x_1}, & \Xi_2 &= \frac{\partial}{\partial x_2}, & \Xi_3 &= \frac{\partial}{\partial x_3}, \\ \Xi_4 &= \left(x_1 - \frac{\partial f}{\partial u_1} \right) \frac{\partial}{\partial x_1} + \left(x_2 - \frac{\partial f}{\partial u_2} \right) \frac{\partial}{\partial x_2} + \left(x_3 - \frac{\partial f}{\partial u_3} \right) \frac{\partial}{\partial x_3},\end{aligned}$$

where κ_j ($i = 1, \dots, 7$) must be expressed **suitably** in terms of κ_j ($j = 8, \dots, 13$) and $f(u_1, u_2, u_3)$. It is:

$$[\Xi_i, \Xi_j] = 0, \quad [\Xi_i, \Xi_4] = \Xi_i, \quad (i, j = 1, 2, 3).$$

$$\begin{aligned}
 \kappa_1 &= - \left(\frac{\partial^2 f}{\partial u_1 \partial u_2} \right)^2 \kappa_{13} + \frac{\partial^2 f}{\partial u_1 \partial u_2} \frac{\partial^2 f}{\partial u_1 \partial u_3} \kappa_{12} + \frac{\partial^2 f}{\partial u_1 \partial u_2} \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_{10} - \frac{\partial^2 f}{\partial u_1 \partial u_2} \frac{\partial^2 f}{\partial u_3^2} \kappa_9 \\
 &- \left(\frac{\partial^2 f}{\partial u_1 \partial u_3} \right)^2 \kappa_{11} + \frac{\partial^2 f}{\partial u_1 \partial u_3} \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_9 - \frac{\partial^2 f}{\partial u_1 \partial u_3} \frac{\partial^2 f}{\partial u_2^2} \kappa_{10} - \frac{\partial^2 f}{\partial u_1^2} \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_{12} \\
 &+ \frac{\partial^2 f}{\partial u_1^2} \frac{\partial^2 f}{\partial u_2^2} \kappa_{13} + \frac{\partial^2 f}{\partial u_1^2} \frac{\partial^2 f}{\partial u_3^2} \kappa_{11} - \left(\frac{\partial^2 f}{\partial u_2 \partial u_3} \right)^2 \kappa_8 + \frac{\partial^2 f}{\partial u_2^2} \frac{\partial^2 f}{\partial u_3^2} \kappa_8, \\
 \kappa_2 &= \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_{12} - \frac{\partial^2 f}{\partial u_2^2} \kappa_{13} - \frac{\partial^2 f}{\partial u_3^2} \kappa_{11}, \\
 \kappa_3 &= 2 \frac{\partial^2 f}{\partial u_1 \partial u_2} \kappa_{13} - \frac{\partial^2 f}{\partial u_1 \partial u_3} \kappa_{12} - \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_{10} + \frac{\partial^2 f}{\partial u_3^2} \kappa_9, \\
 \kappa_4 &= - \frac{\partial^2 f}{\partial u_1 \partial u_2} \kappa_{12} + 2 \frac{\partial^2 f}{\partial u_1 \partial u_3} \kappa_{11} - \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_9 + \frac{\partial^2 f}{\partial u_2^2} \kappa_{10}, \\
 \kappa_5 &= \frac{\partial^2 f}{\partial u_1 \partial u_3} \kappa_{10} - \frac{\partial^2 f}{\partial u_1^2} \kappa_{13} - \frac{\partial^2 f}{\partial u_3^2} \kappa_8, & \kappa_7 &= \frac{\partial^2 f}{\partial u_1 \partial u_2} \kappa_9 - \frac{\partial^2 f}{\partial u_1^2} \kappa_{11} - \frac{\partial^2 f}{\partial u_2^2} \kappa_8, \\
 \kappa_6 &= - \frac{\partial^2 f}{\partial u_1 \partial u_2} \kappa_{10} - \frac{\partial^2 f}{\partial u_1 \partial u_3} \kappa_9 + \frac{\partial^2 f}{\partial u_1^2} \kappa_{12} + 2 \frac{\partial^2 f}{\partial u_2 \partial u_3} \kappa_8.
 \end{aligned}$$

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By applying the theorem, we introduce

$$z_1 = x_1 - \frac{\partial f}{\partial u_1}, \quad z_2 = x_2 - \frac{\partial f}{\partial u_2}, \quad z_3 = x_3 - \frac{\partial f}{\partial u_3} \quad \text{new indep. var.,}$$

$$w_1 = u_1, \quad w_2 = u_2, \quad w_3 = u_3 \quad \text{new dep. var.,}$$

and the generators of the point symmetries write as

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as a result, the nonlinear system becomes:

$$\frac{\partial w_1}{\partial z_2} - \frac{\partial w_2}{\partial z_1} = 0, \quad \frac{\partial w_1}{\partial z_3} - \frac{\partial w_3}{\partial z_1} = 0, \quad \frac{\partial w_2}{\partial z_3} - \frac{\partial w_3}{\partial z_2} = 0,$$

$$\kappa_8 \frac{\partial w_1}{\partial z_1} + \kappa_9 \frac{\partial w_1}{\partial z_2} + \kappa_{10} \frac{\partial w_1}{\partial z_3} + \kappa_{11} \frac{\partial w_2}{\partial z_2} + \kappa_{12} \frac{\partial w_2}{\partial z_3} + \kappa_{13} \frac{\partial w_3}{\partial z_3} = 0.$$

Decoupling problem of quasilinear first order systems

The decoupling of a quasilinear system of PDEs into subsystems of a simpler form — **when it is possible** — has great effects on the properties of its solutions and on the computer time required for its numerical investigation.

Courant

This problem has been formulated by Courant [[Courant, Hilbert: Methods of Mathematical Physics, II, 1962](#)] as follows:

When can a system like

$$\frac{\partial \mathbf{u}}{\partial t} + a(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}$$

be locally decoupled in some coordinates $v_1(\mathbf{u}), \dots, v_n(\mathbf{u})$ into k non-interacting subsystems?

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Theorem (Nijenhuis, 1951)

The necessary and sufficient condition for the complete decoupling of

$$\partial_t \mathbf{u} + \mathbf{a}(\mathbf{u}) \partial_x \mathbf{u} = \mathbf{0}, \quad \mathbf{u} \in \mathbb{R}^n$$

into n non-interacting one-dimensional subsystems is the vanishing of the corresponding Nijenhuis tensor

$$N_{jik} = a_{\alpha i} \frac{\partial a_{jk}}{\partial u_\alpha} - a_{\alpha k} \frac{\partial a_{ji}}{\partial u_\alpha} + a_{j\alpha} \frac{\partial a_{\alpha i}}{\partial u_k} - a_{j\alpha} \frac{\partial a_{\alpha k}}{\partial u_i}.$$

*provided that all eigenvalues of matrix a are real and distinct (**strict hyperbolicity**).*

Necessary and sufficient conditions for the Courant problem have been provided in a series of papers by Bogoyavlenskij (2007) by using the Nijenhuis tensor.

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Decoupling problem solved with simple tools

A very general result stating the **necessary and sufficient conditions** guaranteeing the **partial** decoupling of quasilinear first order system of PDEs is here presented.

The conditions do not involve the Nijenhuis tensor but simply the eigenvalues and the eigenvectors (generalized, if needed) of the coefficient matrix.

A solution to the Courant problem results as a by-product!

Remarkably, the theorem constructively provides the conditions for the decoupling transformation.

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Partial decoupling in 2 subsystems

Definition

The system

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0},$$

where $\mathbf{U} \in \mathbb{R}^n$ and $a \in L(\mathbb{R}^n)$ is a real $n \times n$ matrix with entries depending on \mathbf{U} , is partially decoupled in two subsystems if, suitably sorting the components of \mathbf{U} , we recognize a subsystem of n_1 ($n_1 < n$) equations involving only (U_1, \dots, U_{n_1}) and a subsystem of $n_2 = n - n_1$ equations involving in principle all components of \mathbf{U} .

It means that matrix A has the following block structure

$$A = \begin{bmatrix} A_{(n_1, n_1)}^1 & 0_{(n_1, n_2)} \\ A_{(n_2, n_1)}^2 & A_{(n_2, n_2)}^2 \end{bmatrix}$$

with $A_{(n_i, n_j)}^i$ ($i, j = 1, 2$) $n_i \times n_j$ matrices with entries depending at most on (U_1, \dots, U_{m_i}) , where $m_1 = n_1$ and $m_2 = n_1 + n_2$, whereas $0_{(n_1, n_2)}$ is a $n_1 \times n_2$ matrix of zeros,

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with $A_{(n_i, n_j)}^i$ ($i, j = 1, 2$) $n_i \times n_j$ matrices with entries depending at most on (U_1, \dots, U_{m_i}) , where $m_1 = n_1$ and $m_2 = n_1 + n_2$, whereas $0_{(n_1, n_2)}$ is a $n_1 \times n_2$ matrix of zeros,

Partial decoupling in k subsystems

Definition

The system

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0},$$

where $\mathbf{U} \in \mathbb{R}^n$ and $A \in L(\mathbb{R}^n)$ is a real $n \times n$ matrix with entries depending on \mathbf{U} , is partially decoupled in $k \leq n$ subsystems of some orders n_1, \dots, n_k ($n_1 + \dots + n_k = n$) if, suitably sorting the components of \mathbf{U} , we recognize k subsystems such that the i -th subsystem ($i = 1, \dots, k$) involves at most (U_1, \dots, U_{m_i}) , where

$$m_1 = n_1, \quad m_i = m_{i-1} + n_i \text{ for } i > 1.$$

It means that matrix A is a lower triangular block matrix, where the blocks of the i -th row depend at most on (U_1, \dots, U_{m_i}) .

$$A = \begin{bmatrix} A^1_{(n_1, n_1)} & 0_{(n_1, n_2)} & \dots & \dots & \dots & 0_{(n_1, n_k)} \\ A^2_{(n_2, n_1)} & A^2_{(n_2, n_2)} & 0_{(n_2, n_3)} & \dots & \dots & 0_{(n_2, n_k)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A^{k-1}_{(n_{k-1}, n_1)} & A^{k-1}_{(n_{k-1}, n_2)} & A^{k-1}_{(n_{k-1}, n_3)} & \dots & A^{k-1}_{(n_{k-1}, n_{k-1})} & 0_{(n_{k-1}, n_k)} \\ A^k_{(n_k, n_1)} & A^k_{(n_k, n_2)} & A^k_{(n_k, n_3)} & \dots & \dots & A^k_{(n_k, n_k)} \end{bmatrix},$$

with $A^i_{(n_i, n_j)}$ ($i, j = 1, \dots, k$) are $n_i \times n_j$ matrices with entries depending at most on $U_\alpha^{(r)}$ ($r = 1, \dots, i$, $\alpha = 1, \dots, n_r$), whereas $0_{(n_i, n_j)}$ are $n_i \times n_j$ matrices of zeros, respectively; $m_1 = n_1$, $m_i = m_{i-1} + n_i$ for $i > 1$.

Lemma

Let A be a $n \times n$ real matrix with entries depending on $\mathbf{U} \equiv (U_1, \dots, U_n)$ and assume that such a matrix has real eigenvalues and a complete set of eigenvectors. Matrix A has the structure

$$A = \begin{bmatrix} A_{(n_1, n_1)}^1 & 0_{(n_1, n_2)} \\ A_{(n_2, n_1)}^2 & A_{(n_2, n_2)}^2 \end{bmatrix}$$

with $A_{(n_i, n_j)}^i$ ($i, j = 1, 2$) $n_i \times n_j$ matrices with entries depending at most on (U_1, \dots, U_{m_i}) , where $m_1 = n_1$ and $m_2 = n_1 + n_2$, whereas $0_{(n_1, n_2)}$ is a $n_1 \times n_2$ matrix of zeros, if and only if by computing the n eigenvalues Λ_i (counted with their multiplicity) and the corresponding left and right eigenvectors,

$$\mathbf{L}^{(i)} \equiv (L_1^{(i)}, \dots, L_n^{(i)}), \quad \mathbf{R}^{(i)} \equiv (R_1^{(i)}, \dots, R_n^{(i)})^T,$$

respectively, and suitably sorting the eigenvalues (and the corresponding eigenvectors), the following conditions are satisfied:

$$\begin{aligned} (\nabla_{\mathbf{U}} \Lambda_i) \cdot \mathbf{R}^{(j)} &= 0, & \mathbf{L}^{(i)} \cdot \left((\nabla_{\mathbf{U}} \mathbf{R}^{(\ell)}) \mathbf{R}^{(j)} - (\nabla_{\mathbf{U}} \mathbf{R}^{(j)}) \mathbf{R}^{(\ell)} \right) &= 0, \\ i, \ell &= 1, \dots, n_1, & i \neq \ell, \quad j &= n_1 + 1, \dots, n. \end{aligned}$$

Proof.

- The set of its n eigenvalues, Λ_i , is the **union** of the set of the n_1 eigenvalues of $A_{(n_1, n_1)}^1$, depending at most on U_1, \dots, U_{n_1} , and the set of the n_2 eigenvalues of $A_{(n_2, n_2)}^2$, depending in principle on all components of \mathbf{U} .
- Let us arrange the Λ_i 's in such a way the first n_1 elements are the eigenvalues of $A_{(n_1, n_1)}^1$, and the remaining ones the eigenvalues of $A_{(n_2, n_2)}^2$.
- The left eigenvectors \mathbf{L}_i corresponding to the eigenvalues Λ_i ($i = 1, \dots, n_1$) of matrix $A_{(n_1, n_1)}^1$ may have only the first n_1 components non-vanishing. Moreover, either Λ_i or \mathbf{L}_i may depend only on U_1, \dots, U_{n_1} .
- On the contrary, the right eigenvectors \mathbf{R}_i corresponding to the eigenvalues Λ_i ($i = n_1 + 1, \dots, n$) of matrix $A_{(n_2, n_2)}^2$ may have non-vanishing only the last n_2 components.

As a consequence, the conditions

$$(\nabla_{\mathbf{U}} \Lambda_i) \cdot \mathbf{R}^{(j)} = 0, \quad \mathbf{L}^{(i)} \cdot \left((\nabla_{\mathbf{U}} \mathbf{R}^{(\ell)}) \mathbf{R}^{(j)} - (\nabla_{\mathbf{U}} \mathbf{R}^{(j)}) \mathbf{R}^{(\ell)} \right) = 0,$$

($i, \ell = 1, \dots, n_1$, $i \neq \ell$, $j = n_1 + 1, \dots, n$) are trivially satisfied. Viceversa, if these conditions are fulfilled then the matrix A has the above structure. □

Proof.

- The set of its n eigenvalues, Λ_i , is the **union** of the set of the n_1 eigenvalues of $A^1_{(n_1, n_1)}$, depending at most on U_1, \dots, U_{n_1} , and the set of the n_2 eigenvalues of $A^2_{(n_2, n_2)}$, depending in principle on all components of \mathbf{U} .
- Let us arrange the Λ_i 's in such a way the first n_1 elements are the eigenvalues of $A^1_{(n_1, n_1)}$, and the remaining ones the eigenvalues of $A^2_{(n_2, n_2)}$.
- The left eigenvectors \mathbf{L}_i corresponding to the eigenvalues Λ_i ($i = 1, \dots, n_1$) of matrix $A^1_{(n_1, n_1)}$ may have only the first n_1 components non-vanishing. Moreover, either Λ_i or \mathbf{L}_i may depend only on U_1, \dots, U_{n_1} .
- On the contrary, the right eigenvectors \mathbf{R}_i corresponding to the eigenvalues Λ_i ($i = n_1 + 1, \dots, n$) of matrix $A^2_{(n_2, n_2)}$ may have non-vanishing only the last n_2 components.

As a consequence, the conditions

$$(\nabla_{\mathbf{U}} \Lambda_i) \cdot \mathbf{R}^{(j)} = 0, \quad \mathbf{L}^{(i)} \cdot \left((\nabla_{\mathbf{U}} \mathbf{R}^{(\ell)}) \mathbf{R}^{(j)} - (\nabla_{\mathbf{U}} \mathbf{R}^{(j)}) \mathbf{R}^{(\ell)} \right) = 0,$$

($i, \ell = 1, \dots, n_1$, $i \neq \ell$, $j = n_1 + 1, \dots, n$) are trivially satisfied. Viceversa, if these conditions are fulfilled then the matrix A has the above structure. □

Proof.

- The set of its n eigenvalues, Λ_i , is the **union** of the set of the n_1 eigenvalues of $A_{(n_1, n_1)}^1$, depending at most on U_1, \dots, U_{n_1} , and the set of the n_2 eigenvalues of $A_{(n_2, n_2)}^2$, depending in principle on all components of \mathbf{U} .
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Comment

$$A = \begin{bmatrix} A_{(n_1, n_1)}^1 & 0_{(n_1, n_2)} \\ A_{(n_2, n_1)}^2 & A_{(n_2, n_2)}^2 \end{bmatrix}$$

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$$(\nabla_{\mathbf{U}} \Lambda_i) \cdot \mathbf{R}^{(j)} = 0, \quad \mathbf{L}^{(i)} \cdot \left((\nabla_{\mathbf{U}} \mathbf{R}^{(\ell)}) \mathbf{R}^{(j)} - (\nabla_{\mathbf{U}} \mathbf{R}^{(j)}) \mathbf{R}^{(\ell)} \right) = 0,$$

$$(i, \ell = 1, \dots, n_1, i \neq \ell, j = n_1 + 1, \dots, n)$$

are $n_1^2 n_2$ constraints stating the independence of the n_1^2 entries of matrix $A_{(n_1, n_1)}^1$ from the n_2 variables (U_{n_1+1}, \dots, U_n) ,

Lemma

Let A be a $n \times n$ real matrix with entries depending on $\mathbf{U} \equiv (U_1, \dots, U_n)$. If the matrix A has not a complete set of eigenvectors and/or has complex-valued eigenvalues, let us associate:

- to each real eigenvalue its (left and right) eigenvectors and, if needed, its generalized (left and right) eigenvectors in such a way we have as many linearly independent vectors as the multiplicity of the eigenvalue;
- to each couple of conjugate complex eigenvalues the real part and the imaginary part of its (left and right) eigenvectors (or generalized eigenvectors, if needed) in such a way we have as many couples of linearly independent vectors as the multiplicity of the complex conjugate eigenvalues.

Let us denote with $\mathbf{L}^{(i)}$ and $\mathbf{R}^{(i)}$ ($i = 1, \dots, n$) such vectors. *The conditions in previous Lemma remain unchanged.* In fact, the vectors $\mathbf{L}^{(i)}$ (real eigenvectors, real generalized eigenvectors, real and imaginary parts of complex eigenvectors and generalized complex eigenvectors) corresponding to the eigenvalues Λ_i ($i = 1, \dots, n_1$) of matrix $A_{(n_1, n_1)}^1$ may have only the first n_1 components non-vanishing, and that the vectors $\mathbf{R}^{(i)}$ corresponding to the eigenvalues Λ_i ($i = n_1 + 1, \dots, n$) of matrix $A_{(n_2, n_2)}$ may have non-vanishing only the last n_2 components.

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Lemma

Let A be a $n \times n$ real matrix with entries depending on

$\mathbf{U} \equiv (U_1^{(1)}, \dots, U_{n_1}^{(1)}, \dots, U_1^{(k)}, \dots, U_{n_k}^{(k)})$ ($n_1 + \dots + n_k = n$). Matrix A is a block lower triangular matrix,

$$A = \begin{bmatrix} A_{(n_1, n_1)}^1 & 0_{(n_1, n_2)} & \dots & \dots & \dots & 0_{(n_1, n_k)} \\ A_{(n_2, n_1)}^2 & A_{(n_2, n_2)}^2 & 0_{(n_2, n_3)} & \dots & \dots & 0_{(n_2, n_k)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{(n_{k-1}, n_1)}^{k-1} & A_{(n_{k-1}, n_2)}^{k-1} & A_{(n_{k-1}, n_3)}^{k-1} & \dots & A_{(n_{k-1}, n_{k-1})}^{k-1} & 0_{(n_{k-1}, n_k)} \\ A_{(n_k, n_1)}^k & A_{(n_k, n_2)}^k & A_{(n_k, n_3)}^k & \dots & \dots & A_{(n_k, n_k)}^k \end{bmatrix},$$

with $A_{(n_i, n_j)}^i$ ($i, j = 1, \dots, k$) are $n_i \times n_j$ matrices with entries depending at most on $U_\alpha^{(r)}$ ($r = 1, \dots, i$, $\alpha = 1, \dots, n_r$), whereas $0_{(n_i, n_j)}$ are $n_i \times n_j$ matrices of zeros, respectively, if and only if the set of eigenvalues can be divided into k subsets each containing n_i ($i = 1, \dots, k$) eigenvalues (counted with their multiplicity) with corresponding left and right **vectors** (in the above sense)

...

... Continued

$$\begin{aligned} & \dots \\ & \left\{ \left\{ \Lambda_1^{(1)}, \dots, \Lambda_{n_1}^{(1)} \right\}, \dots, \left\{ \Lambda_1^{(k)}, \dots, \Lambda_{n_k}^{(k)} \right\} \right\}, \\ & \left\{ \left\{ \mathbf{L}_1^{(1)}, \dots, \mathbf{L}_{n_1}^{(1)} \right\}, \dots, \left\{ \mathbf{L}_1^{(k)}, \dots, \mathbf{L}_{n_k}^{(k)} \right\} \right\}, \\ & \left\{ \left\{ \mathbf{R}_1^{(1)}, \dots, \mathbf{R}_{n_1}^{(1)} \right\}, \dots, \left\{ \mathbf{R}_1^{(k)}, \dots, \mathbf{R}_{n_k}^{(k)} \right\} \right\}, \end{aligned}$$

provided that these structure conditions hold:

$$\begin{aligned} (\nabla_{\mathbf{u}} \Lambda_{\alpha}^{(j)}) \cdot \mathbf{R}_{\gamma}^{(j)} &= 0, & \mathbf{L}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}} \mathbf{R}_{\beta}^{(i)}) \mathbf{R}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}} \mathbf{R}_{\gamma}^{(j)}) \mathbf{R}_{\beta}^{(i)} \right) &= 0, \\ i &= 1, \dots, k-1, & j &= i+1, \dots, k, \quad \alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j. \end{aligned}$$

Proof.

The proof is immediate by using the same arguments as above. □

... Continued

...

$$\left\{ \left\{ \Lambda_1^{(1)}, \dots, \Lambda_{n_1}^{(1)} \right\}, \dots, \left\{ \Lambda_1^{(k)}, \dots, \Lambda_{n_k}^{(k)} \right\} \right\},$$

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$$\left\{ \left\{ \mathbf{R}_1^{(1)}, \dots, \mathbf{R}_{n_1}^{(1)} \right\}, \dots, \left\{ \mathbf{R}_1^{(k)}, \dots, \mathbf{R}_{n_k}^{(k)} \right\} \right\},$$

provided that these structure conditions hold:

$$\left(\nabla_{\mathbf{u}} \Lambda_{\alpha}^{(j)} \right) \cdot \mathbf{R}_{\gamma}^{(j)} = 0, \quad \mathbf{L}_{\alpha}^{(i)} \cdot \left(\left(\nabla_{\mathbf{u}} \mathbf{R}_{\beta}^{(i)} \right) \mathbf{R}_{\gamma}^{(j)} - \left(\nabla_{\mathbf{u}} \mathbf{R}_{\gamma}^{(j)} \right) \mathbf{R}_{\beta}^{(i)} \right) = 0,$$

$$i = 1, \dots, k-1, \quad j = i+1, \dots, k, \quad \alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j.$$

Proof.

The proof is immediate by using the same arguments as above. □

Theorem (Partial decoupling in k subsystems)

The first order quasilinear system

$$\frac{\partial \mathbf{u}}{\partial x_1} + a(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0}, \quad \mathbf{u} \in \mathbb{R}^n, \quad a(\mathbf{u}) \text{ } n \times n \text{ matrix}$$

is mapped by a (locally) invertible transformation $\mathbf{u} = \mathbf{h}(\mathbf{U})$ ($\mathbf{U} = \mathbf{H}(\mathbf{u})$), into

$$\frac{\partial \mathbf{U}}{\partial x_1} + A(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x_2} = \mathbf{0}, \quad A = (\nabla_{\mathbf{u}} \mathbf{h})^{-1} a(\nabla_{\mathbf{u}} \mathbf{h}) = (\nabla_{\mathbf{u}} \mathbf{H}) a(\nabla_{\mathbf{u}} \mathbf{H})^{-1},$$

where A is a lower triangular block matrix (with hierarchical dependence of its entries) **iff**, by computing (and suitably sorting) the n eigenvalues λ_i (counted with their multiplicity) and the associated (left and right) **vectors** (real eigenvectors, and, if needed, generalized real eigenvectors, real and imaginary parts of complex eigenvectors or generalized eigenvectors) of matrix a , $\mathbf{l}^{(i)}$ and $\mathbf{r}^{(i)}$, respectively, it is:

$$\nabla_{\mathbf{u}} \lambda_{\alpha}^{(i)} \cdot \mathbf{r}_{\gamma}^{(j)} = 0, \quad \mathbf{l}_{\alpha}^{(i)} \cdot \left(\nabla_{\mathbf{u}} \mathbf{r}_{\beta}^{(i)} \cdot \mathbf{r}_{\gamma}^{(j)} - \nabla_{\mathbf{u}} \mathbf{r}_{\gamma}^{(j)} \cdot \mathbf{r}_{\beta}^{(i)} \right),$$

$$\forall i = 1, \dots, k-1, j = i+1, \dots, k, \alpha, \beta = 1, \dots, n_i, \alpha \neq \beta, \gamma = 1, \dots, n_j.$$

And the decoupling variables are $U_{\alpha}^{(i)} = H_{\alpha}^{(i)}(\mathbf{u})$ such that $(\nabla_{\mathbf{u}} H_{\alpha}^{(i)}) \cdot \mathbf{r}_{\gamma}^{(j)} = 0$.

Proof.

Due to

$$A = (\nabla_{\mathbf{u}}\mathbf{h})^{-1} a(\nabla_{\mathbf{u}}\mathbf{h}) = (\nabla_{\mathbf{u}}\mathbf{H}) a(\nabla_{\mathbf{u}}\mathbf{H})^{-1},$$

we have

$$\Lambda_{\alpha}^{(i)} = \lambda_{\alpha}^{(i)},$$

$$\mathbf{l}_{\alpha}^{(i)} = \mathbf{L}_{\alpha}^{(i)}(\nabla_{\mathbf{u}}\mathbf{H}),$$

$$\mathbf{r}_{\alpha}^{(i)} = (\nabla_{\mathbf{u}}\mathbf{H})^{-1} \mathbf{R}_{\alpha}^{(i)}.$$

The proof is gained by observing that

$$\left(\nabla_{\mathbf{u}}\lambda_{\alpha}^{(i)}\right) \cdot \mathbf{r}_{\gamma}^{(j)} = 0, \quad \Leftrightarrow \quad \left(\nabla_{\mathbf{u}}\Lambda_{\alpha}^{(i)}\right) \cdot \mathbf{R}_{\gamma}^{(j)} = 0,$$

and

$$\mathbf{l}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}_{\beta}^{(i)})\mathbf{r}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{r}_{\gamma}^{(j)})\mathbf{r}_{\beta}^{(i)} \right) = 0 \quad \Leftrightarrow \quad \mathbf{L}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{R}_{\beta}^{(i)})\mathbf{R}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{R}_{\gamma}^{(j)})\mathbf{R}_{\beta}^{(i)} \right) = 0,$$

$$i = 1, \dots, k-1, \quad j = i+1, \dots, k, \quad \alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j.$$



Proof.

Due to

$$A = (\nabla_{\mathbf{u}}\mathbf{h})^{-1} a(\nabla_{\mathbf{u}}\mathbf{h}) = (\nabla_{\mathbf{u}}\mathbf{H}) a(\nabla_{\mathbf{u}}\mathbf{H})^{-1},$$

we have

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and

$$\mathbf{l}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}_{\beta}^{(i)})\mathbf{r}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{r}_{\gamma}^{(j)})\mathbf{r}_{\beta}^{(i)} \right) = 0 \quad \Leftrightarrow \quad \mathbf{L}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{R}_{\beta}^{(i)})\mathbf{R}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{R}_{\gamma}^{(j)})\mathbf{R}_{\beta}^{(i)} \right) = 0,$$

$$i = 1, \dots, k-1, \quad j = i+1, \dots, k, \quad \alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j.$$



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and

$$\mathbf{l}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}_{\beta}^{(i)})\mathbf{r}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{r}_{\gamma}^{(j)})\mathbf{r}_{\beta}^{(i)} \right) = 0 \quad \Leftrightarrow \quad \mathbf{L}_{\alpha}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{R}_{\beta}^{(i)})\mathbf{R}_{\gamma}^{(j)} - (\nabla_{\mathbf{u}}\mathbf{R}_{\gamma}^{(j)})\mathbf{R}_{\beta}^{(i)} \right) = 0,$$

$$i = 1, \dots, k-1, \quad j = i+1, \dots, k, \quad \alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j.$$



Theorem (Courant problem)

For a system of quasilinear PDEs to be locally reducible into k non-interacting subsystems of some orders n_1, \dots, n_k with $n_1 + \dots + n_k = n$ it is necessary and sufficient that the eigenvalues of the coefficient matrix i can be divided into k subsets each containing n_i ($i = 1, \dots, k$) eigenvalues (counted with their multiplicity) with corresponding left and right **vectors**

$$\begin{aligned} & \{ \{ \lambda_1^{(1)}, \dots, \lambda_{n_1}^{(1)} \}, \dots, \{ \lambda_1^{(k)}, \dots, \lambda_{n_k}^{(k)} \} \}, \\ & \{ \{ \mathbf{l}_1^{(1)}, \dots, \mathbf{l}_{n_1}^{(1)} \}, \dots, \{ \mathbf{l}_1^{(k)}, \dots, \mathbf{l}_{n_k}^{(k)} \} \}, \\ & \{ \{ \mathbf{r}_1^{(1)}, \dots, \mathbf{r}_{n_1}^{(1)} \}, \dots, \{ \mathbf{r}_1^{(k)}, \dots, \mathbf{r}_{n_k}^{(k)} \} \}, \end{aligned}$$

such that:

$$\begin{aligned} \nabla_{\mathbf{u}} \lambda_{\alpha}^{(i)} \cdot \mathbf{r}_{\gamma}^{(j)} &= 0, \\ \mathbf{l}_{\alpha}^{(i)} \cdot \left(\nabla_{\mathbf{u}} \mathbf{r}_{\beta}^{(i)} \cdot \mathbf{r}_{\gamma}^{(j)} - \nabla_{\mathbf{u}} \mathbf{r}_{\gamma}^{(j)} \cdot \mathbf{r}_{\beta}^{(i)} \right), \\ \forall i, j &= 1, \dots, k, \quad i \neq j, \quad \alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j. \end{aligned}$$

Moreover, the decoupling variables $\mathbf{U} = \mathbf{H}(\mathbf{u})$ are found from

$$\left(\nabla_{\mathbf{u}} H_{\alpha}^{(i)} \right) \cdot \mathbf{r}_{\gamma}^{(j)} = 0. \quad i, j = 1, \dots, k, \quad i \neq j, \quad \alpha = 1, \dots, n_i, \quad \gamma = 1, \dots, n_j.$$

Example (1D Euler equation of barotropic fluids)

$$\frac{\partial \rho}{\partial x_1} + u \frac{\partial \rho}{\partial x_2} + \rho \frac{\partial u}{\partial x_2} = 0,$$

$$\frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} + \frac{1}{\rho} \frac{\partial p}{\partial x_2} = 0,$$

where $\rho(x_1, x_2)$ is the mass density, $u(x_1, x_2)$ the velocity and $p(\rho)$ the pressure. The constitutive law

$$p(\rho) = \frac{k^2}{3} \rho^3, \quad k \text{ constant,}$$

allow us to introduce the new dependent variables

$$U_1 = u + k\rho, \quad U_2 = u - k\rho,$$

whereupon the source system is transformed in the following fully decoupled one

$$\frac{\partial U_1}{\partial x_1} + U_1 \frac{\partial U_1}{\partial x_2} = 0,$$

$$\frac{\partial U_2}{\partial x_1} + U_2 \frac{\partial U_2}{\partial x_2} = 0.$$

Example (Moving threadline; Ames, Lee, Zaiser, 1968)

Let us consider the motion equations for a moving threadline, where ρ is the mass density, u and v the components of velocity, ϵ the transverse displacement and $T(m)$ the tension:

$$\left\{ \begin{array}{l} \frac{\partial \rho}{\partial x_1} + \frac{\partial}{\partial x_2} (\rho u) = 0, \\ \frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} - \frac{1}{\rho} \frac{\partial}{\partial x_2} \left(\frac{T}{\sqrt{1 + \epsilon^2}} \right) = 0, \\ \frac{\partial v}{\partial x_1} + 2u \frac{\partial v}{\partial x_2} + \left(u^2 - \frac{T}{\rho \sqrt{1 + \epsilon^2}} \right) \frac{\partial \epsilon}{\partial x_2} = 0, \\ \frac{\partial \epsilon}{\partial x_1} - \frac{\partial v}{\partial x_2} = 0, \end{array} \right.$$

where $\rho = m\sqrt{1 + \epsilon^2}$, $T'(m) < 0$.

Imposing the structure conditions for decoupling, the constitutive law

$$T(m) = \frac{k^2}{m}, \quad k \text{ constant,}$$

arises, and the system is partially decoupled (the resulting system results also completely exceptional!).

Thanks for your attention.