Lie Remarkable PDEs

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LOCAL AND NONLOCAL GEOMETRY OF PDES AND INTEGRABILITY

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Direct Lie problem

Start with a DE

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0,$$

where $\mathbf{x}\in\mathcal{X}\subseteq\mathbb{R}^n$ and $\mathbf{u}\in\mathcal{U}\subseteq\mathbb{R}^m$ denote the independent and dependent variables, whereas $\mathbf{u}^{(r)}$ the set of all derivatives up to the order r, one is interested to find the admitted group of Lie symmetries.

Let us look for the generators of the admitted symmetries

$$\Xi=\sum_{i=1}^{n}\xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\eta_\alpha(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_\alpha}.$$

This is done, building the rth prolongation $\Xi^{(r)}$, via Lie's algorithm

$$\Xi^{(r)}\Delta\Big|_{\Delta=0}=0.$$

The admitted generators $\Xi_1,\Xi_2,\dots$ span a Lie algebra.

Inverse Lie problem

On the contrary, in the inverse problem one chooses a Lie algebra of symmetries and determine the most general DE (having an assigned structure) admitting it [Bluman, Cole, 1989], or look for the additional constraints to be imposed in order to have the requested invariance. Various examples of inverse problems relevant in mathematical physics include:

- quasilinear hyperbolic systems with scaling invariance in connection to the study of the propagation of weak discontinuity waves into nonconstant states described by self similar solutions [Ames, Donato, IJNLM 1987];
- the determination of the structural form of a quasilinear first order system which results invariant with respect to the Galilean group [Shugrin, 1981; Ruggeri, 1989; Oliveri, 1993].
- ...
- Lie remarkable equations.

Lie remarkable equations [Oliveri, Note Mat., 2005]

Let us suppose we have a DE

$$\widehat{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}\right) = 0,$$

with $\widehat{\Delta}$ assigned function of its arguments, and Ξ_k $(k = 1, \ldots, p)$ the generators of its Lie point symmetries.
If we consider a generic DE

$$\Delta\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)}\right) = 0,$$

with Δ unspecified function of its arguments, and require the invariance w.r.t. Ξ_k $(k = 1, \ldots, p)$, we recover some restrictions on the functional dependence of Δ that rarely led to

$$\Delta \equiv \widehat{\Delta}.$$

When this occurs we call Lie remarkable the differential equation.

Geometric formulation

If E is a manifold then we denote by $\chi(E)$ the Lie algebra of vector fields on E.

Let E be a $(n+m)$-dimensional smooth manifold and L an n-dimensional embedded submanifold of E. Let us consider a local chart dividing the coordinates in two sets, (x_i, u_α), $i = 1 \dots n$ and $\alpha = 1 \dots m$, such that the submanifold L is locally described as the graph of a vector function $u_\alpha = f_\alpha(x_1, \dots, x_n)$.

Jet space

Let $\iota\colon L \hookrightarrow E$ and $\iota'\colon L' \hookrightarrow E$ be two submanifolds, and $p \in L \cap L'$. We say that L and L' have a *contact of order* r at p if ι and ι' have a contact of order r at p.
This equivalence relation allows for the introduction of *rth jet of n-dimensional submanifolds of* E, denoted by $J^r(E,n)$.
The set $J^r(E,n)$ has a natural manifold structure, and it is

$$\dim J^r(E,n) = n + m\binom{n+r}{r}.$$

Within a geometrical framework, a differential equation $\mathcal{E}$ is not more than a submanifold of $J^r(E,n)$, and we can better precise the definition of Lie remarkable equations.

Definition (Weakly Lie remarkable equations)

Let E be a manifold, $\dim(E) = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An ℓ–dimensional equation $\mathcal{E} \subset J^r(E, n)$ is said to be *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) ℓ-dimensional equation in $J^r(E, n)$ passing at any $\theta \in \mathcal{E}$ and admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

Definition (Strongly Lie remarkable equations)

Let E be a manifold, $\dim(E) = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An ℓ-dimensional equation $\mathcal{E} \subset J^r(E, n)$ is said to be *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) ℓ–dimensional equation in $J^r(E, n)$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

[Manno, Oliveri, Vitolo, JMAA, 2007; TMP, 2007].

Remark (trivial)

A strongly Lie remarkable equation is also weakly Lie remarkable.

For each $\theta \in J^r(E,n)$, let us denote by $S_\theta(\mathcal{E}) \subset T_\theta J^r(E,n)$ the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at θ. Let us set

$$S(\mathcal{E}) \stackrel{\text{def}}{=} \bigcup_{\theta \in J^r(E,n)} S_\theta(\mathcal{E}).$$

In general, $\dim S_\theta(\mathcal{E})$ may change with $\theta \in J^r(E,n)$. It is clear that $\dim \mathrm{sym}(\mathcal{E}) \geq S_\theta(\mathcal{E})$, for all $\theta \in J^r(E,n)$. If the rank of $S(\mathcal{E})$ at each $\theta \in J^r(E,n)$ equals $\dim \mathrm{sym}(\mathcal{E})$, then $S(\mathcal{E})$ is an involutive (smooth) distribution. The points of $J^r(E,n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E,n)$. It follows that $\mathcal{E}$ can not coincide with the set of points of maximal rank of $S(\mathcal{E})$. Some results have been proved [Manno, Oliveri, Vitolo, JMAA, 2007].

Theorem

A necessary condition for the differential equation $\mathcal{E}$ to be strongly Lie remarkable is that $\dim(\mathrm{sym}(\mathcal{E})) > \dim(\mathcal{E})$.

Theorem

A necessary condition for the differential equation $\mathcal{E}$ to be weakly Lie remarkable is that $\dim(\mathrm{sym}(\mathcal{E})) \geq \dim(\mathcal{E})$.

Theorem

If $S(\mathcal{E})|_{\mathcal{E}}$ is an ℓ-dimensional distribution on $\mathcal{E} \subset J^r(E, n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.

Theorem

Let $S(\mathcal{E})$ be such that for any $\theta \notin \mathcal{E}$ we have $\dim(S_\theta(\mathcal{E})) > \ell$. Then $\mathcal{E}$ is a strongly Lie remarkable equation.

[Manno, Oliveri, Vitolo, JMAA, 2007; TMP, 2007].

Surfaces in $\mathbb{R}^3$

Let $E = \mathbb{R}^3$ endowed with the standard Euclidean metric. The mean and Gaussian curvatures of a surface are the real functions on $J^2(\mathbb{R}^3, 2)$ defined by

$$H = \frac{1}{2}\frac{(1+u_{x_2}^2)u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + (1+u_{x_1}^2)u_{x_2x_2}}{(1+u_{x_1}^2+u_{x_2}^2)^{3/2}},$$

$$G = \frac{u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2}{(1+u_{x_1}^2+u_{x_2}^2)^2}.$$

Minimal surface equation in $\mathbb{R}^3$

The minimal surface equation $\mathcal{E}$ is given by $H = 0$, *i.e.*,

$$(1+u_{x_2}^2)u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + (1+u_{x_1}^2)u_{x_2x_2} = 0.$$

Minimal surface equation in $\mathbb{R}^3$

A straightforward computation shows that the point symmetries of minimal surface equation are the isometries and the homotheties of $\mathbb{R}^3$:

$$\begin{array}{ll} \partial_{x_1}, \quad \partial_{x_2}, & \partial_u, \\ x_2\partial_{x_1} - x_1\partial_{x_2}, & u\partial_{x_1} - x_1\partial_u, \\ u\partial_{x_2} - x_2\partial_u, & x_1\partial_{x_1} + x_2\partial_{x_2} + u\partial_u. \end{array}$$

Moreover, $S(\mathcal{E})$ has maximal rank on an open subset of $\mathcal{E}$.

Theorem

The equation of minimal surfaces in $\mathbb{R}^3$ *is weakly Lie remarkable.*

Minimal surface equations in $\mathbb{R}^{2+m}$

The mean curvature can be generalized to surfaces in $\mathbb{R}^{2+m}$, with $m \geq 2$, and the DE of minimal surfaces reads

$$(1+\mathbf{u}_y^2)\mathbf{u}_{xx} - 2(\mathbf{u}_x \cdot \mathbf{u}_y)\mathbf{u}_{xy} + (1+\mathbf{u}_x^2)\mathbf{u}_{yy} = \mathbf{0}, \qquad \mathbf{u} \in \mathbb{R}^m.$$

Its dimension as a submanifold of $\mathbb{R}^{2+m}$ is $\ell = 2+5m$. The dimension of the algebra of isometries and omotheties in $\mathbb{R}^{2+m}$ is $d = \binom{m+2}{2} + m + 3$.

$$\begin{aligned} &d < \ell, && m = 2, 3, \\ &d = \ell, && m = 4 \quad \text{(weakly Lie remarkable)}, \\ &d > \ell, && m \geq 5 \quad \text{(strongly Lie remarkable)}. \end{aligned}$$

Relationship between Lie remarkability and differential invariants

Theorem

Let $\mathfrak{s}$ be a Lie subalgebra of $\chi(J^r(E,n))$. Let us suppose that the rth prolongation subalgebra of $\mathfrak{s}$ acts regularly on $J^r(E,n)$ and that the set of rth order functionally independent differential invariants of $\mathfrak{s}$ are

$$\{I_1, I_2, \dots, I_q\} \in C^\infty(J^r(E,n)).$$

Then the submanifold of $J^r(E,n)$ described by

$$I_1 = \kappa_1, \dots, I_q = \kappa_q$$

with $\kappa_1, \dots, \kappa_q \in \mathbb{R}$ is a weakly Lie remarkable equation.

Algorithm

To prove that a PDE is strongly or weakly Lie remarkable the following steps are required:

1. determine its Lie point symmetries;
2. determine the rank k of the distribution generated by its rth order prolongations;
3. determine the submanifolds where the rank of the distribution decreases;
4. compare the rank k of the distribution and the dimension ℓ of the submanifold characterized by the equation.

If $k = \ell$ the equation is weakly Lie remarkable; if $k > \ell$ the equation is strongly Lie remarkable.

Differential invariant

The unique second order scalar differential invariant I of the algebra spanned by

$$\begin{aligned} &\partial_{x_1}, \quad \partial_{x_2}, && \partial_u, \\ &x_2\partial_{x_1} - x_1\partial_{x_2}, && u\partial_{x_1} - x_1\partial_u, \\ &u\partial_{x_2} - x_2\partial_u, && x_1\partial_{x_1} + x_2\partial_{x_2} + u\partial_u. \end{aligned}$$

is

$$I = \frac{H^2}{G}.$$

Then

$$\begin{aligned} &G = \kappa H^2, \qquad 0 \leq \kappa \leq 1, \\ &H = 0 \end{aligned}$$

are weakly Lie remarkable equations.

2nd order Monge-Ampère equation

The 2nd order Monge-Ampère equation in 2 independent variables, introduced by Ampère in 1815, has the form

$$Hu_{tt} + 2Ku_{tx} + Lu_{xx} + M + N(u_{tt}u_{xx} - u_{tx}^2) = 0,$$

where H, K, L, M, N ($N \neq 0$) may depend on t, x, u, u_t, u_x.

In 1968, Boillat discovered that MA equation is the only 2nd order equation being completely exceptional (or linearly degenerate) in the Lax–Boillat sense.

The property of C.E. has been used to derive Monge-Ampère equations involving more than 2 independent variables [Ruggeri, 1973; Donato, Ramgulam & Rogers, 1992; Boillat, 1991].

2nd order Monge-Ampère equation

Proposition

[Boillat, 1992] Given an unknown field

$$u(x_1, \ldots, x_n), \qquad (x_1 \text{ denoting the time}),$$

and its associated Hessian matrix $H = \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|$, *the most general 2nd order PDE being completely exceptional (and called Monge-Ampère equation) is provided by a linear combination of all minors extracted from* H, *with coefficients depending at most on* x_i, u *and first order derivatives of* u.

Monge-Ampère equations may possess the remarkable property of being characterized by their Lie point symmetries.

Monge–Ampère equation

Theorem

Equation

$$\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1 \partial x_2} = \kappa$$

is *weakly Lie remarkable* *if* $\kappa \neq 0$,

Proof.

If $\kappa \neq 0$, we have 9 point symmetries generated by

$$\begin{array}{lll} \Xi_1 = \partial_{x_1}, & \Xi_2 = \partial_{x_2}, & \Xi_3 = \partial_u, \\ \Xi_4 = x_1\partial_{x_2}, & \Xi_5 = x_2\partial_{x_1}, & \Xi_6 = x_1\partial_u, \\ \Xi_7 = x_2\partial_u, & \Xi_8 = x_1\partial_{x_1} + u\partial_u, & \Xi_9 = x_2\partial_{x_2} + u\partial_u. \end{array}$$

The 2nd order prolonged vector fields give rise to a distribution of rank 7 on the whole jet space provided we exclude the 5–dimensional submanifolds locally described by $u_{x_1x_1} = u_{x_1x_2} = u_{x_2x_2} = 0$ where the rank reduces to 5. □

Monge–Ampère equation

Theorem

Equation

$$\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$$

is strongly Lie remarkable.

Proof.

The equation admits a 15–dimensional Lie algebra of point symmetries spanned by

$$\partial_a, \qquad a\partial_b, \qquad a\left(x_1\partial_{x_1} + x_2\partial_{x_2} + u\partial_u\right), \qquad \forall a, b \in \{x_1, x_2, u\}.$$

The 2nd order prolonged vector fields give rise to a distribution of rank 8 on the whole jet space, provided we exclude the submanifold characterized by the equation itself (where the rank reduces to 7) and the submanifold $u_{x_1x_1} = u_{x_1x_2} = u_{x_2x_2} = 0$ (where the rank reduces to 5). □

Higher order Monge-Ampère equations

The property of complete exceptionality has been used by Boillat [Boillat, 1992] to determine higher order Monge-Ampère equations for the unknown $u(x_1, x_2)$.

By considering an equation of order $N > 2$, we need to consider the Hankel matrix

$$H = \begin{bmatrix} X_0 & X_1 & X_2 & \dots & X_{M-1} & X_M \\ X_1 & X_2 & X_3 & \dots & X_M & X_{M+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{M-1} & X_M & X_{M+1} & \dots & X_{2M-2} & X_{2M-1} \\ X_M & X_{M+1} & X_{M+2} & \dots & X_{2M-1} & X_{2M} \end{bmatrix},$$

where $X_k = \dfrac{\partial^N u}{\partial x_1^{N-k} \partial x_2^k}$.

Higher order Monge-Ampère equations

Theorem (Boillat, 1992)

The most general nonlinear C.E. equation is given, if $N = 2M$, by a linear combination of all minors, including the determinant, of the Hankel matrix, whereas in the case where $N = 2M - 1$, we have to consider the linear combination of all minors extracted from the Hankel matrix where the last row has been removed.

In both cases the coefficients of the linear combination are functions of x_1, x_2, u and its derivatives up to the order $N - 1$.
Let us limit ourselves to the case where the coefficients are constant.

3rd order Monge-Ampère equation

Consider the 3rd order Monge-Ampère equation

$$\begin{aligned}
&\widetilde{\kappa}_1(u_{x_1x_1x_2}u_{x_1x_1x_1} - u^2_{x_1x_2x_2}) + \widetilde{\kappa}_2(u_{x_1x_1x_1}u_{x_2x_2x_2} - u_{x_1x_1x_2}u_{x_1x_2x_2}) \\
&\quad + \widetilde{\kappa}_3(u_{x_1x_1x_1}u_{x_1x_2x_2} - u^2_{x_1x_1x_2}) + \widetilde{\kappa}_4 u_{x_1x_1x_1} + \widetilde{\kappa}_5 u_{x_1x_1x_2} + \widetilde{\kappa}_6 u_{x_1x_2x_2} \\
&\quad + \widetilde{\kappa}_7 u_{x_2x_2x_2} + \widetilde{\kappa}_8 = 0.
\end{aligned}$$

The substitution

$$u \rightarrow u + \alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^2 + \alpha_4 x_2^3$$

provides the equation

$$\begin{aligned}
&\kappa_1(u_{x_1x_1x_2}u_{x_1x_1x_1} - u^2_{x_1x_2x_2}) + \kappa_2(u_{x_1x_1x_1}u_{x_2x_2x_2} - u_{x_1x_1x_2}u_{x_1x_2x_2}) \\
&\quad + \kappa_3(u_{x_1x_1x_1}u_{x_1x_2x_2} - u^2_{x_1x_1x_2}) = \kappa.
\end{aligned}$$

It is an 11–dimensional submanifold, but possesses only 10 symmetries. However . . .

3rd order Monge-Ampère equation

Theorem

The equation

$$(u_{x_1x_1x_2}u_{x_1x_1x_1} - u^2_{x_1x_2x_2}) + \lambda(u_{x_1x_1x_1}u_{x_2x_2x_2} - u_{x_1x_1x_2}u_{x_1x_2x_2}) + \lambda^2(u_{x_1x_1x_1}u_{x_1x_2x_2} - u^2_{x_1x_1x_2}) = \mu,$$

where

$$\lambda = \frac{\kappa_3}{\kappa_2}, \qquad \mu = \frac{\kappa\kappa_3}{\kappa_2^2},$$

obtained by choosing $\kappa_1 = \dfrac{\kappa_2^2}{\kappa_3}$ *in*

$$\kappa_1(u_{x_1x_1x_2}u_{x_1x_1x_1} - u^2_{x_1x_2x_2}) + \kappa_2(u_{x_1x_1x_1}u_{x_2x_2x_2} - u_{x_1x_1x_2}u_{x_1x_2x_2}) + \kappa_3(u_{x_1x_1x_1}u_{x_1x_2x_2} - u^2_{x_1x_1x_2}) = \kappa,$$

is weakly Lie remarkable.

3rd order Monge-Ampère equation

Proof.

The Lie algebra of point symmetries admitted is spanned by

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial u}, \qquad \Xi_{12} = F(x_1 - \lambda x_2)\frac{\partial}{\partial u},$$

$$\Xi_4 = (2x_1 - 3\lambda x_2)\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}, \qquad \Xi_5 = \lambda^2 x_2\frac{\partial}{\partial x_1} + (2x_1 - \lambda x_2)\frac{\partial}{\partial x_2},$$

$$\Xi_6 = \lambda x_2\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2u\frac{\partial}{\partial u}, \qquad \Xi_7 = x_1\frac{\partial}{\partial u}, \qquad \Xi_8 = x_2\frac{\partial}{\partial u},$$

$$\Xi_9 = x_1^2\frac{\partial}{\partial u}, \qquad \Xi_{10} = x_1 x_2\frac{\partial}{\partial u}, \qquad \Xi_{11} = x_2^2\frac{\partial}{\partial u},$$

where F is an arbitrary function of $(x_1 - \lambda x_2)$; their 3rd order prolongations give rise, provided $F''' \neq 0$, to a distribution of rank 11. □

In a similar way, it is possible to prove that some Monge–Ampère equations of order higher than the third are weakly Lie remarkable.

4th order Monge-Ampère equation

Theorem

The fourth order Monge-Ampère equation

$$u_{x_1x_1x_1x_1}(u_{x_1x_1x_2x_2}u_{x_2x_2x_2x_2} - u^2_{x_1x_2x_2x_2}) + 2u_{x_1x_1x_1x_2}u_{x_1x_1x_2x_2}u_{x_1x_2x_2x_2} - u^3_{x_1x_1x_2x_2} - u^2_{x_1x_1x_1x_2}u_{x_2x_2x_2x_2} = 0$$

is weakly Lie remarkable.

Proof.

1. The equation characterizes a 16–dimensional submanifold in the 17–dimensional jet space $J^4(\mathbb{R}^3, 2)$;
2. The Lie algebra of point symmetries is 19–dimensional;
3. The 4th order prolongations give rise to a distribution of rank 16 provided we exclude a singular subset.

□

Strongly Lie remarkable equations in $J^2(\mathbb{R}^4, 2)$ with $\mathcal{A}(\mathbb{R}^4)$

$$\dim(\mathcal{A}(\mathbb{R}^4)) = 20, \qquad \dim(J^2(\mathbb{R}^4, 2) = 14.$$

The rank of the distribution prolonged to the second order is 14, and reduces to 12 only on the 12–dimensional manifold of $J^2(\mathbb{R}^4, 2)$ characterized by the system of equations

$$\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_1 x_2} - \frac{\partial^2 u_1}{\partial x_1 x_2}\frac{\partial^2 u_2}{\partial x_1^2} = 0,$$
$$\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_2^2} - \frac{\partial^2 u_1}{\partial x_2^2}\frac{\partial^2 u_2}{\partial x_1^2} = 0,$$

that, consequently, is strongly Lie remarkable.
The system admits also the symmetries generated by

$$a\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right),$$

where $a \in \{x_1, x_2, u_1, u_2\}$.

Strongly Lie remarkable equations in $J^2(\mathbb{R}^5, 2)$ with $\mathcal{A}(\mathbb{R}^5)$

$$\dim(\mathcal{A}(\mathbb{R}^5)) = 30, \qquad \dim(J^2(\mathbb{R}^4, 2) = 23.$$

The rank of the distribution prolonged to the second order is 23, and reduces to 19 only on the 19–dimensional manifold of $J^2(\mathbb{R}^5, 2)$ characterized by the system of equations

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_1 x_2} - \frac{\partial^2 u_1}{\partial x_1 x_2}\frac{\partial^2 u_2}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_2^2} - \frac{\partial^2 u_1}{\partial x_2^2}\frac{\partial^2 u_2}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial^2 u_3}{\partial x_1 x_2} - \frac{\partial^2 u_2}{\partial x_1 x_2}\frac{\partial^2 u_3}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial^2 u_3}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_2^2}\frac{\partial^2 u_3}{\partial x_1^2} = 0.
\end{aligned}$$

Strongly Lie remarkable equations in $J^2(\mathbb{R}^5, 3)$ with $\mathcal{A}(\mathbb{R}^5)$

$$\dim(\mathcal{A}(\mathbb{R}^5)) = 30, \qquad \dim(J^2(\mathbb{R}^4, 2) = 20.$$

The rank of the distribution prolonged to the second order is 20, and reduces to 16 only on the 16–dimensional manifold of $J^2(\mathbb{R}^5, 3)$ characterized by the system of equations

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_1 x_2} - \frac{\partial^2 u_1}{\partial x_1 x_2}\frac{\partial^2 u_2}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_1 x_3} - \frac{\partial^2 u_1}{\partial x_1 x_3}\frac{\partial^2 u_2}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_2^2} - \frac{\partial^2 u_1}{\partial x_2^2}\frac{\partial^2 u_2}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_2 x_3} - \frac{\partial^2 u_1}{\partial x_2 x_3}\frac{\partial^2 u_2}{\partial x_1^2} = 0,\\
&\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_3^2} - \frac{\partial^2 u_1}{\partial x_3^2}\frac{\partial^2 u_2}{\partial x_1^2} = 0.
\end{aligned}$$

Strongly Lie remarkable equations in $J^2(\mathbb{R}^{n+m}, n)$ with $\mathcal{A}(\mathbb{R}^{n+m})$

Theorem

The system of differential equations

$$\boldsymbol{\Delta} = 0,$$

with the components of $\boldsymbol{\Delta}$ *given by*

$$\Delta_{\alpha,\alpha+1;1,1,p,q} \equiv \frac{\partial^2 u_\alpha}{\partial x_1^2}\frac{\partial^2 u_{\alpha+1}}{\partial x_p \partial x_q} - \frac{\partial^2 u_{\alpha+1}}{\partial x_1^2}\frac{\partial^2 u_\alpha}{\partial x_p \partial x_q} = 0,$$

where $\alpha = 1, \ldots, m-1$ *and* $p, q = 1, \ldots, n$, *is strongly Lie remarkable with respect to the affine Lie algebra* $\mathcal{A}(\mathbb{R}^{n+m})$.

Proof.

$$\Delta_{\alpha,\alpha+1;1,1,p,q} \equiv \frac{\partial^2 u_\alpha}{\partial x_1^2}\frac{\partial^2 u_{\alpha+1}}{\partial x_p \partial x_q} - \frac{\partial^2 u_{\alpha+1}}{\partial x_1^2}\frac{\partial^2 u_\alpha}{\partial x_p \partial x_q} = 0$$

is made by $(m-1)(n^2+n-2)/2$ independent differential equations and characterizes in the second order jet space a manifold with dimension

$$\ell = n^2 + 2n + 3m - 2 + \frac{mn(1-n)}{2} < (n+m)(n+m+1), \qquad \forall\, m, n.$$

The rank of the distribution prolonged to the second order does suffice. From the above equations it follows

$$\Delta_{\alpha,\beta;i,j,p,q} \equiv \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\frac{\partial^2 u_\beta}{\partial x_p \partial x_q} - \frac{\partial^2 u_\beta}{\partial x_i \partial x_j}\frac{\partial^2 u_\alpha}{\partial x_p \partial x_q} = 0,$$

for all $\alpha, \beta \in \{1, \dots, m\}$ and all $i, j, p, q \in \{1, \dots, n\}$.
Consider the infinitesimal generators of $\mathcal{A}(\mathbb{R}^{n+m})$, say

$$\partial_{x_i}, \qquad \partial_{u_\beta}, \qquad x_j\partial_{x_i}, \qquad u_\beta\partial_{x_i}, \qquad x_i\partial_{u_\beta}, \qquad u_\gamma\partial_{u_\beta},$$

and prove that each of these vector fields is admitted.

Proof.

Since neither the independent variables nor the dependent ones appear, it is evident the invariance with respect to the $(n+m)$ translations of the independent and dependent variables.

The second order prolongations of the vector fields $x_i \frac{\partial}{\partial u_\beta}$ read

$$\Xi^{(2)} = x_i \frac{\partial}{\partial u_\beta} + \delta_{\alpha\beta}\delta_{ik} \frac{\partial}{\partial(\partial u_\alpha/\partial x_k)},$$

whereupon

$$\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) = 0.$$

□

Proof.

Let us consider the second order prolongations of generators $x_j \frac{\partial}{\partial x_i}$:

$$\Xi^{(2)} = x_j \frac{\partial}{\partial x_i} - \delta_{jk} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial}{\partial(\partial u_\alpha/\partial x_k)} - \left(\delta_{jl} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_k} + \delta_{jk} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_l}\right) \frac{\partial}{\partial(\partial^2 u_\alpha/\partial x_k \partial x_l)},$$

where δ_{ij} is the Kronecker symbol and the Einstein summation convention over repeated indices has been used. It is:

$$\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) = = 2\delta_{j1}\Delta_{\alpha,\alpha+1;1,i,p,q} + \delta_{jp}\Delta_{\alpha,\alpha+1;1,1,i,q} + \delta_{jq}\Delta_{\alpha,\alpha+1;1,1,i,p},$$

that vanishes on $\mathbf{\Delta} = 0$. □

Proof.

The second order prolongations of the vector fields $u_\beta \frac{\partial}{\partial x_i}$ read

$$\Xi^{(2)} = u_\beta \frac{\partial}{\partial x_i} - \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_k}\frac{\partial}{\partial u_{\alpha,k}} - \left(\frac{\partial u_\alpha}{\partial x_i}\frac{\partial^2 u_\beta}{\partial x_k \partial x_l} + \frac{\partial u_\beta}{\partial x_k}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_l} + \frac{\partial u_\beta}{\partial x_l}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_k}\right)\frac{\partial}{\partial(\partial^2 u_\alpha/\partial x_k \partial x_l)},$$

whereupon

$$\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) = \frac{\partial u_\alpha}{\partial x_i}\Delta_{\beta,\alpha+1;1,1,p,q} + \frac{\partial u_{\alpha+1}}{\partial x_i}\Delta_{\alpha,\beta+1;1,1,p,q} + 2\frac{\partial u_\beta}{\partial x_1}\Delta_{\alpha,\alpha+1;1,i,p,q} + \frac{\partial u_\beta}{\partial x_p}\Delta_{\alpha,\alpha+1;1,1,i,q} + \frac{\partial u_\beta}{\partial x_q}\Delta_{\alpha,\alpha+1;1,1,i,p},$$

vanishing on $\boldsymbol{\Delta} = 0$. □

Proof.

Finally, the second order prolongation of the vector fields $u_\gamma \frac{\partial}{\partial u_\beta}$ is

$$\Xi^{(2)} = u_\gamma \frac{\partial}{\partial u_\beta} + \delta_{\alpha\beta} \frac{\partial u_\gamma}{\partial x_k} \frac{\partial}{\partial(\partial u_\alpha/\partial x_k)} + \delta_{\alpha\beta} \frac{\partial^2 u_\gamma}{\partial x_k \partial x_l} \frac{\partial}{\partial(\partial^2 u_\alpha/\partial x_k \partial x_l)},$$

whereupon

$$\begin{aligned}
&\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) = \\
&= \delta_{\alpha\beta}\left(\frac{\partial^2 u_\gamma}{\partial x_1^2}\frac{\partial^2 u_{\alpha+1}}{\partial x_p \partial x_q} - \frac{\partial^2 u_{\alpha+1}}{\partial x_1^2}\frac{\partial^2 u_\gamma}{\partial x_p \partial x_q}\right) \\
&+ \delta_{\alpha+1\beta}\left(\frac{\partial^2 u_\alpha}{\partial x_1^2}\frac{\partial^2 u_\gamma}{\partial x_p \partial x_q} \frac{\partial^2 u_\gamma}{\partial x_1^2}\frac{\partial^2 u_\alpha}{\partial x_p \partial x_q}\right) = \\
&= \delta_{\alpha\beta}\Delta_{\gamma,\alpha+1;1,1,p,q} + \delta_{\alpha+1\beta}\Delta_{\alpha,\gamma;1,1,p,q},
\end{aligned}$$

vanishing on $\mathbf{\Delta} = 0$, so completing the proof. □

Note that the system admits also projective transformations!

Hierarchy of 2nd order Monge–Ampère (?) equations

Strongly Lie remarkable	Algebra of Lie symmetries
$\dfrac{d^2u}{dx^2} = 0,$	$\mathcal{P}(\mathbb{R}^2),$
$\dfrac{\partial^2 u}{\partial x_1^2}\dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial^2 u}{\partial x_1 \partial x_2} = 0,$	$\mathcal{P}(\mathbb{R}^3),$
$\det\left(\left\Vert \dfrac{\partial^2 u}{\partial x_i \partial x_j}\right\Vert\right) = 0,$	$\mathcal{P}(\mathbb{R}^{n+1}),$
$\dfrac{\partial^2 u_\alpha}{\partial x_1^2}\dfrac{\partial^2 u_{\alpha+1}}{\partial x_p \partial x_q} - \dfrac{\partial^2 u_{\alpha+1}}{\partial x_1^2}\dfrac{\partial^2 u_\alpha}{\partial x_p \partial x_q} = 0,$	$\mathcal{P}(\mathbb{R}^{n+m}).$

	$\dim(J^r(\mathbb{R}^{n+m}, n))$	$\dim(\mathcal{A}(\mathbb{R}^{n+m}))$	$\dim(\mathcal{P}(\mathbb{R}^{n+m}))$
	$n + m\binom{n+r}{r}$	$(n+m)(n+m+1)$	$(n+m)(n+m+2)$

Lie remarkability in $J^3(\mathbb{R}^3, 2)$ corresponding to $\mathcal{P}(\mathbb{R}^3)$

A third order strongly Lie remarkable equation with respect to the Lie algebra $\mathcal{P}(\mathbb{R}^3)$ does exist [Manno, Oliveri, Vitolo, TMP, 2007].
Its local form reads:

$$
\begin{aligned}
& u_{xx}^3 u_{yyy}^2 + u_{xxx}^2 u_{yy}^3 + 6u_{xx}u_{xxx}u_{xy}u_{yy}u_{yyy} - 6u_{xxx}u_{xxy}u_{xy}u_{yy}^2 \\
& \quad - 6u_{xx}u_{xxx}u_{xyy}u_{yy}^2 - 6u_{xx}^2 u_{xy}u_{xyy}u_{yyy} - 6u_{xx}^2 u_{xxy}u_{yy}u_{yyy} \\
& \quad - 8u_{xxx}u_{xy}^3 u_{yyy} + 9u_{xx}u_{xxy}^2 u_{yy}^2 + 9u_{xx}^2 u_{xyy}^2 u_{yy} \\
& \quad + 12u_{xxx}u_{xy}^2 u_{xyy}u_{yy} + 12u_{xx}u_{xxy}u_{xy}^2 u_{yyy} - 18u_{xx}u_{xxy}u_{xy}u_{xyy}u_{yy} = 0.
\end{aligned}
$$

Lie remarkability in $J^3(\mathbb{R}^4,2)$ corresponding to $\mathcal{P}(\mathbb{R}^4)$

$$
\begin{aligned}
&((u_{yy}v_{xy}-u_{xy}v_{yy})(u_{yy}(u_{yy}v_{xx}^2+v_{xy}(u_{xx}v_{xy}-2u_{xy}v_{xx}))+(u_{xy}^2-u_{xx}u_{yy})v_{xx}v_{yy}))\,u_{xxx}\\
&+3\,((u_{xx}v_{xy}-u_{xy}v_{xx})(u_{xy}v_{xy}-u_{yy}v_{xx})(u_{xy}v_{yy}-u_{yy}v_{xy}))\,u_{xxy}\\
&+3\,((u_{xy}v_{xx}-u_{xx}v_{xy})^2(u_{yy}v_{xy}-u_{xy}v_{yy}))\,u_{xyy}\\
&+(((u_{xy}v_{xx}-u_{xx}v_{xy})^2(u_{xx}v_{yy}-u_{xy}v_{xy})))\,u_{yyy}\\
&+((u_{xy}^2-u_{xx}u_{yy})(u_{yy}v_{xx}-u_{xx}v_{yy})(u_{yy}v_{xy}-u_{xy}v_{yy}))\,v_{xxx}\\
&+3\,((u_{xy}^2-u_{xx}u_{yy})(u_{xy}v_{xx}-u_{xx}v_{xy})(u_{xy}v_{yy}-u_{yy}v_{xy}))\,v_{xxy}\\
&+((u_{xy}^2-u_{xx}u_{yy})(u_{xy}v_{xx}-u_{xx}v_{xy})^2)\,v_{yyy}=0,
\end{aligned}
$$

$$
\begin{aligned}
&3\,((u_{yy}v_{xy}-u_{xy}v_{yy})^2(v_{xx}v_{yy}-v_{xy}^2))\,u_{xxy}\\
&+3\,((u_{yy}v_{xx}-u_{xx}v_{yy})(u_{yy}v_{xy}-u_{xy}v_{yy})(v_{xx}v_{yy}-v_{xy}^2))\,u_{xyy}\\
&+((v_{xx}v_{yy}-v_{xy}^2)(u_{yy}^2v_{xx}^2-u_{xy}u_{yy}v_{xx}v_{xy}+v_{yy}(u_{xy}^2v_{xx}-u_{xx}u_{xy}v_{xy}+u_{xx}^2v_{yy})\\
&+u_{xx}u_{yy}(v_{xy}^2-2v_{xx}v_{yy})))\,u_{yyy}+((u_{yy}v_{xy}-u_{xy}v_{yy})^3)\,v_{xxx}\\
&+3\,((u_{xy}v_{xy}-u_{yy}v_{xx})(u_{yy}v_{xy}-u_{xy}v_{yy})^2)\,v_{xxy}\\
&+3\,((u_{yy}v_{xy}-u_{xy}v_{yy})(u_{yy}(u_{yy}v_{xx}^2+v_{xy}(u_{xx}v_{xy}-2u_{xy}v_{xx}))+(u_{xy}^2-u_{xx}u_{yy})v_{xx}v_{yy}))\,v_{xyy}\\
&+(u_{yy}(u_{yy}v_{xx}v_{xy}(3u_{xy}v_{xx}-2u_{xx}v_{xy})+u_{xy}v_{xy}^2(u_{xx}v_{xy}-u_{xy}v_{xx})-u_{yy}^2v_{xx}^3)\\
&+(u_{xy}^2-u_{xx}u_{yy})(-2u_{yy}v_{xx}^2+v_{xy}(u_{xy}v_{xx}-u_{xx}v_{xy}))v_{yy}+u_{xx}(u_{xy}^2-u_{xx}u_{yy})v_{xx}v_{yy}^2)\,v_{yyy}=0.
\end{aligned}
$$

Dimensionality is relevant.

	$\dim\left(J^r(\mathbb{R}^{n+m}, n)\right)$	$\dim\left(\mathcal{A}(\mathbb{R}^{n+m})\right)$	$\dim\left(\mathcal{P}(\mathbb{R}^{n+m})\right)$
	$n + m\binom{n+r}{r}$	$(n+m)(n+m+1)$	$(n+m)(n+m+2)$

3rd order DEs with 2 independent variables

m	$\dim(J^3(\mathbb{R}^{2+m}, n))$	$\dim(\mathcal{A}(\mathbb{R}^{2+m}))$	$\dim(\mathcal{P}(\mathbb{R}^{2+m}))$
1	12	12	15
2	22	20	24
3	32	30	35
4	42	42	48
5	52	56	63
6	62	72	80
7	72	90	99
8	82	110	120
9	92	132	143
10	102	156	168
...	...	...	...

Conjecture

For all $m \in \mathbb{N}$ *there exists a third order system of PDEs that is strongly Lie remarkable with respect to the Lie algebra of projective transformations in* $\mathbb{R}^{2+m}$.

The comparison between the dimension of the equation and the dimension of the algebra indicates that such Lie remarkable systems do exist.

Current work is devoted to compute the general form of such systems.

Conformal Lie Algebra $\mathcal{C}(\mathbb{R}^{n+1})$

The algebra $\mathcal{C}(\mathbb{R}^{n+1})$ has dimension equal to $(n+2)(n+3)/2$.
For $n = 2$, the infinitesimal generators of this algebra are the generators of isometries together with

$$\begin{cases} \dfrac{x_1^2 - x_2^2 - u^2}{2}\dfrac{\partial}{\partial x_1} + x_1 x_2 \dfrac{\partial}{\partial x_2} + x_1 u \dfrac{\partial}{\partial u}, \\ x_1 x_2 \dfrac{\partial}{\partial x_1} + \dfrac{x_2^2 - x_1^2 - u^2}{2}\dfrac{\partial}{\partial x_2} + x_2 u \dfrac{\partial}{\partial u}, \\ x_1 u \dfrac{\partial}{\partial x_1} + x_2 u \dfrac{\partial}{\partial x_2} + \dfrac{u^2 - x_1^2 - x_2^2}{2}\dfrac{\partial}{\partial u}, \\ x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2} + u \dfrac{\partial}{\partial u}. \end{cases}$$

Strongly Lie remarkable equation corresponding to $\mathcal{C}(\mathbb{R}^3)$

The rank of the matrix of 2-prolongations of the vector fields of conformal algebra (dimension 10) is 8 except on the manifold below where it reduces to 7. Thus, the second order equation which is strongly Lie remarkable with respect to the conformal algebra is

$$G = H^2,$$

characterizing an umbilic surface.

Conformal Lie Algebra $\mathcal{C}(\mathbb{R}^4)$

A couple of equations in $J^3(\mathbb{R}^4, 2)$ that are strongly Lie remarkable can be also characterized (15–dimensional Lie algebra of symmetries whose second order prolongation generates distribution or rank 13); the jet space has dimension 14, and the dimension of the equation is 12.

Conjecture

There exist strongly Lie remarkable equations corresponding to $\mathcal{C}(\mathbb{R}^{n+m})$ in $J^2(\mathbb{R}^{n+m}, 2)$.

All the best, Iosif.

Thank you.