AUTOMATIC DETERMINATION OF OPTIMAL SYSTEMS OF LIE SUBALGEBRAS: THE PROGRAM SYMBOLIE

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The problem

- Lie groups of symmetries of PDEs can be used to construct group-invariant solutions. Different Lie subgroups lead to different invariant solutions.
- The number of subgroups is potentially infinite and so the number of group invariant solutions!
- We need to classify these solutions in order to have an *optimal system* of inequivalent group invariant solutions from which all other solutions can be derived by action of the group.
- Lie groups are intimately connected to Lie algebras: a classification of inequivalent subgroups induces a classification of inequivalent Lie subalgebras (easier!), and vice versa.
- It is given an effective algorithm to automatically determine inequivalent Lie subalgebras of a finite-dimensional Lie algebra. The algorithm is implemented in Wolfram Mathematica (package SymboLie).

Definition

If $\mathcal{L}$ is a Lie algebra and V is a vector space, a *linear representation* of $\mathcal{L}$ in V is a homomorphism

$$\rho : \mathcal{L} \to \mathrm{gl}(V).$$

Definition (Adjoint representation)

Every Lie algebra defines, by its own structure, one linear representation, called the *adjoint representation*:

$$\mathrm{ad} : \mathcal{L} \to \mathrm{gl}(\mathcal{L}), \qquad x \mapsto \mathrm{ad}_x(y) := [x, y] \qquad \text{for} \quad x, y \in \mathcal{L}.$$

The map ad_x is linear for each $x \in \mathcal{L}$ and the map $x \to \mathrm{ad}_x$ is itself linear; it is a homomorphism, since

$$\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x \qquad x, y \in \mathcal{L},$$

because of the Jacobi identity. The kernel of ad is $Z(\mathcal{L})$, the centre of $\mathcal{L}$.

Definition (Derivations)

A *derivation* of $\mathcal{L}$ is a linear map $\gamma : \mathcal{L} \to \mathcal{L}$ such that

$$\gamma([x, y]) = [\gamma(x), y] + [x, \gamma(y)] \quad \text{for all} \quad x, y \in \mathcal{L}.$$

The derivations of $\mathcal{L}$ form a Lie algebra ($\mathrm{Der}(\mathcal{L})$) with the bracket

$$[\gamma_1, \gamma_2] = \gamma_1 \circ \gamma_2 - \gamma_2 \circ \gamma_1.$$

The elements of $\mathrm{ad}(\mathcal{L})$ are derivations (called *inner derivations*). In other words, an inner derivation is a map

$$\gamma : \mathcal{L} \to \mathcal{L}, \qquad y \mapsto \gamma(y) = [x, y] \quad \text{for some} \quad x \in \mathcal{L}.$$

Definition

Each derivation $\gamma \in \mathrm{Der}(\mathcal{L})$ determines an automorphism, $\exp(\gamma)$, defined as

$$\exp(\gamma)(y) = y + \gamma(y) + \frac{\gamma^2(y)}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{\gamma^k(y)}{k!}.$$

For an inner derivation $\gamma = \mathrm{ad}_x$,

$$\exp(\mathrm{ad}_x(y)) = y + [x, y] + \frac{1}{2}[x, [x, y]] + \cdots \qquad (1)$$

known as the *Baker-Campbell-Hausdorff formula*.
The *inner automorphisms* of $\mathcal{L}$ consist of the smallest subgroup of $\mathrm{Aut}(\mathcal{L})$, containing all automorphisms of form (1), where x runs through all elements of $\mathcal{L}$. This group is denoted by $\mathrm{Int}(\mathcal{L})$.

Inner Automorphisms of a Group

Let us consider a group $(G, \cdot)$. Among all automorphisms of the group G, there are the automorphisms

$$\phi_a : G \to G, \qquad \phi_a(b) = a^{-1} \cdot b \cdot a,$$

called *inner automorphisms* of the group G.
Since $a^{-1}xa = x \;\Leftrightarrow\; xa = ax$, the existence and number of inner automorphisms gives a measure of the failure of the commutative law.

Definition

A subgroup $H_1 \subseteq G$ is *similar* to a subgroup $H_2 \subseteq G$ if there exists $a \in G$ such that $H_2 = a^{-1}H_1a$, *i.e.*, the subgroups H_1 and H_2 are connected by inner automorphisms of the group.

This relation of similarity is a relation of equivalence and the corresponding equivalence classes are said *conjugacy classes*.

In applications one usually constructs the optimal system of subalgebras, from which the optimal system system of subgroups is reconstructed. This method seems more algorithmic because the group $\mathrm{Int}(\mathcal{L}_r)$ is always a group of linear transformation of the main space, differing from the group $\mathrm{Int}(G_r)$.

Definition

Two Lie subalgebras $\mathcal{L}'$ and $\mathcal{L}''$ of a Lie algebra $\mathcal{L}$ are *similar* if there exists an inner automorphism $\phi \in \mathrm{Int}(\mathcal{L})$ such that $\phi(\mathcal{L}') = \mathcal{L}''$.

Remark

In what follows we shall restrict to consider real Lie algebras.

How to do

Let $\mathcal{L}_r$ be an r–dimensional Lie algebra with a basis $\{\Xi_1, \ldots, \Xi_r\}$. By taking two elements X and Y of $\mathcal{L}_r$,

$$X = \sum_{\alpha=1}^{r} f_\alpha \Xi_\alpha, \qquad Y = \sum_{\alpha=1}^{r} g_\alpha \Xi_\alpha,$$

where f_α and g_α are suitable real constants, it results

$$[X, Y] = \sum_{\alpha,\beta=1}^{r} f_\alpha g_\beta \, [\Xi_\alpha, \Xi_\beta] = \sum_{\alpha,\beta,\gamma=1}^{r} f_\alpha g_\beta C^\gamma_{\alpha\beta} \Xi_\gamma,$$

where $C^\gamma_{\alpha\beta}$ are the structure constants of the Lie algebra. This implies that for the coordinates $\mathbf{f} = (f_1, \ldots, f_r)$ and $\mathbf{g} = (g_1, \ldots, g_r)$ of the generators X and Y with respect to the basis $\{\Xi_1, \ldots, \Xi_r\}$ we may introduce an operation of commutation,

$$[\mathbf{f}, \mathbf{g}]_\gamma = \sum_{\alpha,\beta=1}^{r} f_\alpha g_\beta C^\gamma_{\alpha\beta}, \qquad (\gamma = 1, \ldots, r).$$

With this operation the vector space $\mathbb{R}^r$ gains the structure of a Lie algebra.

Now let us introduce the Lie algebra $\mathcal{L}_r^A$ spanned by the following operators:

$$E_\nu = C^\lambda_{\mu\nu} f_\mu \frac{\partial}{\partial f_\lambda}, \qquad (\lambda, \mu, \nu = 1, 2, \ldots, r).$$

The algebra $\mathcal{L}_r^A$ generates, through the integration of the Lie's equations

$$\frac{df^\star_\lambda}{dt} = C^\lambda_{\mu\nu} f^\star_\mu, \qquad (\lambda, \mu, \nu = 1, \ldots, r)$$
$$f^\star_\lambda(0) = f_\lambda,$$

the group of *inner automorphisms* of the Lie algebra $\mathcal{L}_r$.

All subalgebras of the given Lie algebra $\mathcal{L}$ are decomposed into classes of similar algebras.The set of the representatives of each class is called an *optimal system of subalgebras* [Ovsiannikov, Olver].

Optimal system of subalgebras

The optimal system of subalgebras of a Lie algebra $\mathcal{L}$ with inner automorphisms $A = \mathrm{Int}(\mathcal{L})$ is a set of subalgebras $\Theta(\mathcal{L})$ such that:

1. there are no two elements of this set which can be transformed into each other by inner automorphisms of the Lie algebra $\mathcal{L}$;
2. any subalgebra of the Lie algebra $\mathcal{L}$ can be transformed into one of subalgebras of the set $\Theta(\mathcal{L})$.

The union of the elements of the optimal system of given dimensionality s is called *optimal system of order s* and denoted by the symbol Θ_s. The solution of the classification problem for a finite-dimensional Lie algebra $\mathcal{L}_r$ must be tables of optimal systems for every $s = 1, \dots, r-1$.

Top-down approach

Consider two 1D Lie subalgebras of $\mathcal{L}_r$:

$$X = f_1\Xi_1 + f_2\Xi_2 + \cdots + f_r\Xi_r,$$
$$Y = g_1\Xi_1 + g_2\Xi_2 + \cdots + g_r\Xi_r.$$

They are equivalent if some inner automorphism maps

$$(f_1, f_2, \ldots, f_r) \quad \text{to} \quad (g_1, g_2, \ldots, g_r).$$

The algorithm for finding the optimal system of 1D Lie subalgebras usually takes a t-uple

$$(f_1, f_2, \ldots, f_r)$$

and, through *judicious* applications of inner automorphisms, simplify as many of the coefficients f_α.

This approach is difficult to be implemented in a computer, since we need to solve algebraic equations and make some choices during the process.

The determination of optimal systems is relatively easy only for small dimensionality, but also in this cases the solution is accomplished by a method which requires choices and distinguishing cases from a few possibilities at certain stages of the work, and the "simplicity" of the results obtained is not clear.

Bottom-up approach

Take all possible coordinates of the $(2^r - 1)$ 1D Lie subalgebras, *i.e.*, all possible t-uples with r components:

$$(f_1, 0, \ldots, 0),\ (0, f_2, 0, \ldots, 0),\ \ldots,\ (f_1, f_2, 0, \ldots, 0),\ \ldots,\ (f_1, f_2, \ldots, f_r)$$

They can be labeled with integers from 1 to $(2^r - 1)$ with the correspondence

$$(f_1, f_2, \ldots, f_r) \mapsto \sum_{\alpha=0}^{r-1} \tilde{f}_\alpha 2^\alpha, \qquad \tilde{f}_\alpha = \operatorname{sgn}(|f_\alpha|).$$

Given a t-uple with k non-zero components, if the application of a whatever inner automorphism produces a t-uple with more than k components which are functionally independent (this is ascertained by computing the rank of the Jacobian matrix of the components w.r.t. the coefficients f_α and the parameters involved in automorphisms!), then the two t-uples are equivalent. With this approach, all possible 1D subalgebras can be partitioned, through the construction of a suitable multigraph, $\mathcal{G}(\mathcal{L}_r)$ (the vertices are the subalgebras and the edges the automorphisms), in equivalence classes!

The same approach can be applied to higher dimensional subalgebras.

SYMBOLIE: the game of the name

The name merges the word *Symbol* with *Lie*: in fact, Sophus Lie denoted the infinitesimal generator of a Lie group of transformations as the symbol. The name is due to Lucia Margheriti, a PhD student of the University of Messina who in 2008 faced this problem.

SYMBOLIE

SYMBOLIE is a program written in the Wolfram Language and runs in Mathematica. It has a set of functions able to:

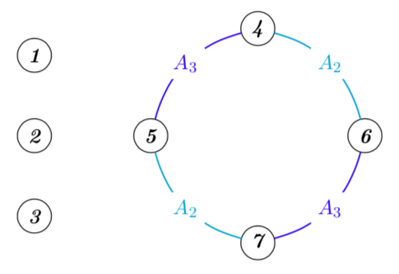
- compute the structure constants and commutator table of a Lie algebra;
- compute the inner automorphisms of a Lie algebra;
- identify inequivalent Lie subalgebras;
- ...

Remarkably, the Lie algebra may be defined either realizing it in terms of vector fields or matrices, or as an abstract object by assigning the structure constants.

A 3D solvable Lie algebra

Let $\mathcal{L}_3$ be the 3D solvable Lie algebra spanned by

$$\{\Xi_1 = \partial_x, \quad \Xi_2 = \partial_y, \quad \Xi_3 = x\partial_x + y\partial_y\}.$$



Non-zero commutators: $[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_2$. The (multi)graph has 4 connected components giving an optimal system Θ_1 of $\mathcal{L}_3$:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \Xi_1 & \Xi_2 & \Xi_1 + a_1\Xi_2 & \Xi_3 \end{array}$$

An optimal system of 2D Lie subalgebras is easily seen from $\mathcal{G}(\mathcal{L}_3)$:

$$\Theta_2 \equiv \{\{\Xi_1, \Xi_2\}, \ \{\Xi_1, \Xi_3\}, \ \{\Xi_2, \Xi_3\}, \ \{\Xi_1 + a_1\Xi_2, \Xi_3\}\}.$$

Lie algebra of KdV equation

Let $\mathcal{L}_4$ be the 4D Lie algebra of KdV equation spanned by

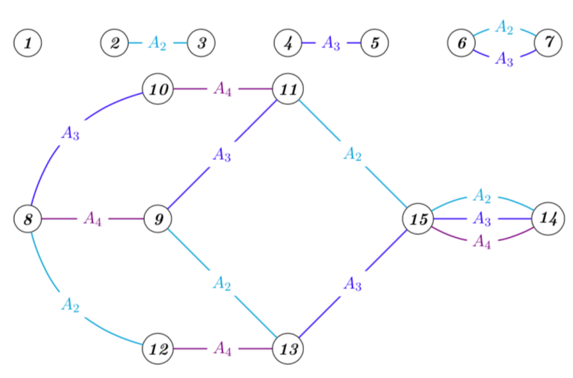
$$\{\Xi_1 = \partial_x, \quad \Xi_2 = \partial_t, \quad \Xi_3 = t\partial_x + \partial_u, \quad \Xi_4 = x\partial_x + 3t\partial_t - 2u\partial_u\}.$$

Olver (1986) shows the following optimal system of 1D subalgebras:

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 - \Xi_2\}, \{\Xi_3 + \Xi_2\}\}$$

Nevertheless, he adds a consideration: the list can be reduced slightly if we admit the discrete symmetry $(x, t, u) \mapsto (-x, -t, u)$, which maps $\Xi_3 - \Xi_2$ to $\Xi_3 + \Xi_2$, thereby reducing the number of inequivalent subalgebras to five.

Θ_1 for symmetries of KdV with SymboLie



There are 5 connected components giving an optimal system Θ_1 of $\mathcal{L}_4$:

$$\Theta_1 = \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_4\}\}.$$

Linear Heat Equation

The finite-dimensional Lie algebra of symmetries of $u_t - u_{xx} = 0$ is spanned by

$$\Xi_1 = \partial_x, \quad \Xi_2 = \partial_t, \quad \Xi_3 = u\partial_u, \quad \Xi_4 = x\partial_x + 2t\partial_t,$$
$$\Xi_5 = 2t\partial_x - xu\partial_u, \quad \Xi_6 = 4tx\partial_x, +4t^2\partial_t - (x^2 + 2t)u\partial_u,$$

with commutation table

$$\begin{bmatrix} 0 & 0 & 0 & \Xi_1 & -\Xi_3 & 2\Xi_5 \\ 0 & 0 & 0 & 2\Xi_2 & 2\Xi_1 & -2\Xi_3 + 4\Xi_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\Xi_1 & -2\Xi_2 & 0 & 0 & \Xi_5 & 2\Xi_6 \\ \Xi_3 & -2\Xi_1 & 0 & -\Xi_5 & 0 & 0 \\ -2\Xi_5 & 2\Xi_3 - 4\Xi_4 & 0 & -2\Xi_6 & 0 & 0 \end{bmatrix}$$

The centre is spanned by $\{\Xi_3\}$.

Linear Heat Equation

There is a system of 11 optimal 1D Lie subalgebras:

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_6\}, \{\Xi_1 + a_1\Xi_2\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_2 + a_1\Xi_5\}, \{\Xi_1 + a_1\Xi_6\}, \{\Xi_2 + a_1\Xi_6\}, \{\Xi_3 + a_1\Xi_6\}\},$$

a system of 20 optimal 2D Lie subalgebras:

$$\Theta_2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_3, \Xi_2\}, \{\Xi_2, \Xi_4\}, \{\Xi_2, \Xi_3 + a_1\Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_5, \Xi_3\}, \{\Xi_3, \Xi_6\}, \{\Xi_1 + a_1\Xi_6, \Xi_3\}, \{\Xi_4, \Xi_5\}, \{\Xi_4, \Xi_6\}, \{\Xi_3 + a_1\Xi_4, \Xi_5\}, \{\Xi_3 + a_1\Xi_4, \Xi_6\}, \{\Xi_5, \Xi_6\}, \{\Xi_3 + a_1\Xi_6, \Xi_5\}, \{\Xi_3 + a_1\Xi_5, \Xi_6\}\},$$

a system of 13 optimal 3D Lie subalgebras, ...

3D Lie Algebras [Patera & Winternitz, JMP, 1977]

Algebra	# 1D Opt. Subalg.	# 2D Opt. Subalg.
1	5	5
2	4	3
3	3	2
4	4	4
5	4	3
6	4	3
7	2	1
8	3 (P.-W.: 2)	1
9	3	2 (P.-W.: 1)
10	1	0

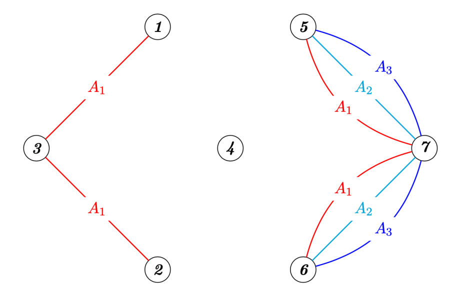
SymboLie is able to produce the optimal systems for all 10 3D Lie algebras in about 14 seconds (on a Notebook with I5 CPU).

Example (Patera & Winternitz – $\mathcal{L}_3$, #8)

Let $\mathcal{L}_3$ be the 3D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$ whose non-zero commutators are:

$$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2.$$

The multigraph $\mathcal{G}(\mathcal{L}_3)$ describing the equivalences between 1D subalgebras.



There are 3 connected components giving

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_1 + a_1\Xi_3\}\}.$$

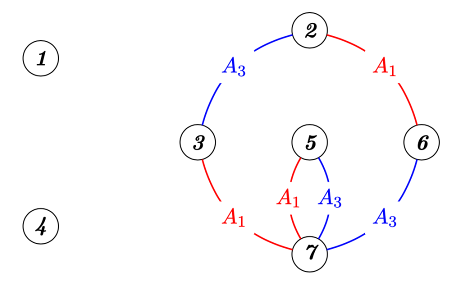
Patera & Winternitz list only two 1D subalgebras!

$$\Theta_2 = \{\{\Xi_1, \Xi_2\}\}.$$

Example (Patera-Winternitz – $\mathcal{L}_3$, # 9)

Let $\mathcal{L}_3$ be the 3D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$ with the non-zero commutators:

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3, \quad [\Xi_3, \Xi_1] = 2\Xi_2.$$



There are 3 connected components giving

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}\},$$
$$\Theta_2 = \{\{\Xi_1, \Xi_2\}, \{\Xi_2, \Xi_3\}\}.$$

Patera & Winternitz list only one 2D Lie subalgebra!

4D Lie Algebras [Patera & Winternitz, JMP,1977]

Algebra	# 1D Opt. Subalg.	# 2D Opt. Subalg.	# 3D Opt. Subalg.
1	15	35	15
2	11	17	8
3	8	10 (P.-W.: 11)	5
4	9	13	7
5	7	8	4
6	9	14	6
7	9	12	5
8	9	12	5
9	5	4	3
10	7 (P.-W.: 5)	5 (P.-W.: 4)	3
11	7	7 (P.-W.: 5)	3 (P.-W.: 2)
12	3	1	1
13	6	7 (P.-W.: 6)	3
14	6	7	3 (P.-W.: 4)
15	6	8	4

4D Lie Algebras [Patera & Winternitz, JMP,1977]

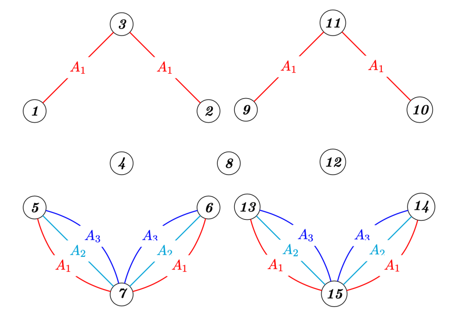
Algebra	# 1D Opt. Subalg.	# 2D Opt. Subalg.	# 3D Opt. Subalg.
16	7	9	4
17	5	6 (P.-W.: 5)	2
18	8 (P.-W.: 7)	10	4
19	8	11	5
20	8	11	5
21	8	13 (P.-W.: 14)	8
22	5 (P.-W.: 4)	5 (P.-W.: 4)	2
23	4	4	2
24	6	8	3
25	5	6	3 (P.-W.: 4)
26	5	7	4
27	6	7	4
28	4	2	1
29	4 (P.-W.: 3)	3 (P.-W.: 2)	1
30	5 (P.-W.: 4)	3	3

SYMBOLIE is able to produce the optimal systems for all 30 4D Lie algebras in about 210 seconds (on a Notebook with I5 CPU).

Example (Patera-Winternitz – $\mathcal{L}_4$, # 10)

Let $\mathcal{L}_4$ be the 4D Lie algebra apanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with non-zero commutators

$$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2.$$



There are 7 connected components whereupon

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_1 + a_1\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + a_1\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4\}\},$$

$$\Theta_2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, a_1\Xi_2 + a_2\Xi_4\}, \{\Xi_3, \Xi_4\}, \{a_1\Xi_1 + a_2\Xi_3, \Xi_4\}\}.$$

Patera & Winternitz report only five inequivalent 1D and four inequivalent 2D Lie subalgebras.

Patera & Winternitz – $\mathcal{L}_4$, #11

Let $\mathcal{L}_4$ be the Lie algebra spanned $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with the following non-zero commutators:

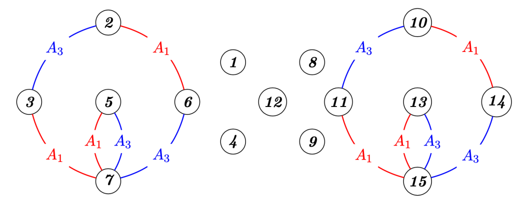
$$[\Xi_3, \Xi_1] = 2\Xi_2, \quad [\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3.$$

We note that $\mathcal{L}_4$ has a non-trivial centre. Indeed, $Z(\mathcal{L}_4) = \{\Xi_4\}$. In such a case, the inner automorphism related to the adjoint representation ad_{Ξ_4} is the identity matrix. Hence, we can limit to analyze the 3 other inner automorphisms of $\mathcal{L}_4$ labeled by A_1, A_2 and A_3.
Then we compute the adjacency matrix and construct the associated multigraph.

Example (Patera & Winternitz – $\mathcal{L}_4$, # 11)

Let $\mathcal{L}_4$ be the Lie algebra spanned $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ such that

$$[\Xi_3, \Xi_1] = 2\Xi_2, \quad [\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3.$$



There are 7 connected components, so that

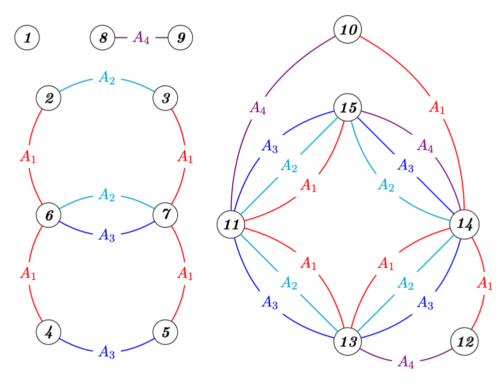
$$\begin{aligned}
\Theta_1 &\equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{a_1\Xi_1 + a_2\Xi_4\}, \{a_1\Xi_2 + a_2\Xi_4\}, \{a_1\Xi_3 + a_2\Xi_4\}, \\
\Theta_2 &\equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \\
&\quad \{\Xi_3, \Xi_4\}, \{\Xi_3, b_1\Xi_2 + b_2\Xi_4\}\}, \\
\Theta_3 &\equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}\}.
\end{aligned}$$

Patera & Winternitz listed five 2D and two 3D inequivalent Lie subalgebras!

Example (Patera-Winternitz – $\mathcal{L}_4$, # 29)

Let us consider the Lie algebra $\mathcal{L}_4$ spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ such that

$$[\Xi_1, \Xi_4] = 2a\Xi_1, \quad [\Xi_2, \Xi_4] = a\Xi_2 - \Xi_3, \quad [\Xi_3, \Xi_4] = \Xi_2 + a\Xi_3.$$



The multigraph has 4 connected components giving

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_4\}, \{a_1\Xi_2 + a_2\Xi_4\}\}.$$

For 2D subalgebras: 1 is an isolated vertex and so Ξ_1 must appear in all the representatives of Θ_2, whereupon we have

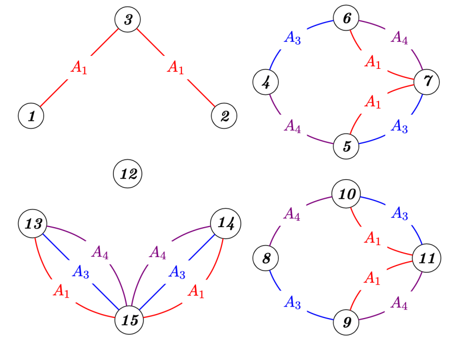
$$\Theta_2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, a_1\Xi_2 + a_2\Xi_4\}\}.$$

The subalgebra $\{\Xi_1, \alpha_1\Xi_2 + \alpha_2\Xi_4\}$ is not equivalent to the other two subalgebras, and so must be counted! (Patera & Winternitz list three 1D and two 2D inequivalent Lie subalgebras!)

Example (Patera-Winternitz – $\mathcal{L}_4$, # 30)

Let $\mathcal{L}_4$ be the 4D Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ with non-zero commutators:

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_2, \quad [\Xi_1, \Xi_4] = -\Xi_2, \quad [\Xi_2, \Xi_4] = \Xi_1.$$



There are 5 connected components, whereupon

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4\}\}.$$

Patera & Winternitz list only four 1D in-equivalent Lie subalgebras!

$$\Theta_2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_3, \Xi_4\}\},$$

$$\Theta_3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, a_1\Xi_3 + a_2\Xi_4\}\}.$$

Example ($\mathcal{L}_5$)

Let $\mathcal{L}_5$ be the 5D Lie algebra spanned by:

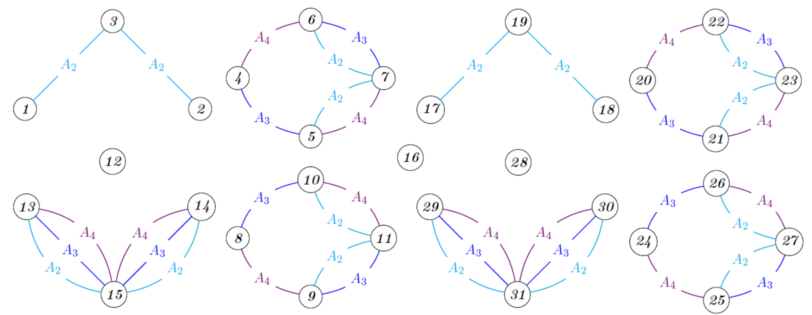
$$\{\Xi_1 = \partial_x,\ \Xi_2 = \partial_y,\ \Xi_3 = -y\partial_x + x\partial_y - v\partial_u + u\partial_v,\\ \Xi_4 = -x\partial_x - y\partial_y + \rho\partial_\rho + p\partial_p,\ \Xi_5 = u\partial_u + \partial_v + \rho\partial_\rho + p\partial_p\}.$$

These generators give the symmetries of the 2D steady ideal gas dynamics equations.

Remark

With SYMBOLIE *we obtain the complete system of optimal Lie subalgebras in about 467 seconds.*

Example ($\mathcal{L}_5$)



The multigraph has 11 connected components giving

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_3 + a_1\Xi_4\}, \{\Xi_1 + a_1\Xi_3 + a_2\Xi_4\}, \{\Xi_5\},$$
$$\{\Xi_1 + a_1\Xi_5\}, \{\Xi_3 + a_2\Xi_5\}, \{\Xi_4 + a_1\Xi_5\}, \{\Xi_3 + a_1\Xi_4 + a_2\Xi_5\},$$
$$\{\Xi_1 + a_1\Xi_3 + a_2\Xi_4 + a_3\Xi_5\}\}.$$

Moreover Θ_2 contains 13 inequivalent Lie subalgebras, Θ_3 9, and Θ_4 7.

$\mathcal{L}_6$: Ovsiannikov, 1993

Let $\mathcal{L}_6$ be a Lie algebra spanned by

$$\{\Xi_1 = \partial_x, \Xi_2 = \partial_y, \Xi_3 = t\partial_x, \Xi_4 = t\partial_y, \Xi_5 = y\partial_x - x\partial_y, \Xi_6 = \partial_t\}.$$

SymboLie finds the optimal system (0.85 sec., 6.05 sec., 120 sec., 2967 seconds.)

$$\Theta_1 \equiv \{\{\Xi_1\}, \{\Xi_3\}, \{a_1\Xi_2 + a_2\Xi_3\}, \{\Xi_5\}, \{\Xi_6\}, \{a_1\Xi_3 + a_2\Xi_6\}, \{a_1\Xi_5 + a_2\Xi_6\}\},$$

$$\Theta_2 \equiv \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, b_1\Xi_3 + b_2\Xi_4\}, \{\Xi_1, \Xi_6\}, \{\Xi_1, b_1\Xi_3 + b_2\Xi_6\}, \{\Xi_1, b_1\Xi_4 + b_2\Xi_6\}, \{\Xi_1, b_1\Xi_3 + b_2\Xi_4 + b_3\Xi_6\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, b_1\Xi_1 + b_2\Xi_4\}, \{\Xi_3, b_1\Xi_2 + b_2\Xi_4\}, \{\Xi_3, b_1\Xi_1 + b_2\Xi_2 + b_3\Xi_4\}, \{a_1\Xi_2 + a_2\Xi_3, b_1\Xi_1 + b_2\Xi_4\}, \{\Xi_5, \Xi_6\}\},$$

$$\Theta_3 \equiv \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, c_1\Xi_3 + c_2\Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_6\}, \{\Xi_1, \Xi_2, c_1\Xi_3 + c_2\Xi_6\}, \{\Xi_1, \Xi_2, c_1\Xi_3 + c_2\Xi_4 + c_3\Xi_6\}, \{\Xi_1, \Xi_2, c_1\Xi_5 + c_2\Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_6\}, \{\Xi_1, \Xi_3, c_1\Xi_4 + c_2\Xi_6\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_3, \Xi_4\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_3, \Xi_6\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_3, c_1\Xi_4 + c_2\Xi_6\}, \{a_1\Xi_1 + a_2\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_3, \Xi_4, \Xi_5\}\},$$

$$\Theta_4 \equiv \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_3, d_1\Xi_4 + d_2\Xi_6\}, \{\Xi_1, \Xi_2, c_1\Xi_3 + c_2\Xi_4, d_1\Xi_3 + d_2\Xi_6\}, \{\Xi_1, \Xi_2, \Xi_5, \Xi_6\}\}.$$

There are some discrepancies with the results by Ovsiannikov to be checked!

Projective algebra in $\mathbb{R}^2$

Let $\mathcal{L}_8$ the algebra $\mathcal{P}(\mathbb{R}^2)$ of projective transformations in $\mathbb{R}^2$:

$$\begin{aligned}\mathcal{L}_8 =\{&\Xi_1 = \partial_x, \quad \Xi_2 = \partial_y, \quad \Xi_3 = x\partial_x, \quad \Xi_4 = y\partial_x, \quad \Xi_5 = x\partial_y,\\ &\Xi_6 = y\partial_y, \quad \Xi_7 = x^2\partial_x + xy\partial_y, \quad \Xi_8 = xy\partial_x + y^2\partial_y\},\end{aligned}$$

with commutator table

$$\begin{bmatrix} 0 & 0 & \Xi_1 & 0 & \Xi_2 & 0 & 2\Xi_3+\Xi_6 & \Xi_4\\ 0 & 0 & 0 & \Xi_1 & 0 & \Xi_2 & \Xi_5 & \Xi_3+2\Xi_6\\ -\Xi_1 & 0 & 0 & -\Xi_4 & \Xi_5 & 0 & \Xi_7 & 0\\ 0 & -\Xi_1 & \Xi_4 & 0 & -\Xi_3+\Xi_6 & -\Xi_4 & \Xi_8 & 0\\ -\Xi_2 & 0 & -\Xi_5 & \Xi_3-\Xi_6 & 0 & \Xi_5 & 0 & \Xi_7\\ 0 & -\Xi_2 & 0 & \Xi_4 & -\Xi_5 & 0 & 0 & \Xi_8\\ -2\Xi_3-\Xi_6 & -\Xi_5 & -\Xi_7 & -\Xi_8 & 0 & 0 & 0 & 0\\ -\Xi_4 & -\Xi_3-2\Xi_6 & 0 & 0 & -\Xi_7 & -\Xi_8 & 0 & 0 \end{bmatrix}.$$

SymboLie finds Θ_1 (25 s.a) and Θ_2 (35 s.a.) of $\mathcal{P}(\mathbb{R}^2)$ in about 135 seconds.

$$
\begin{aligned}
\Theta_1 \equiv \{&\{\Xi_1\}, \{\Xi_3\}, \{a_1\Xi_2 + a_2\Xi_3\}, \{a_1\Xi_2 + a_2\Xi_4\}, \{a_1\Xi_1 + a_2\Xi_5\}, \{a_1\Xi_4 + a_2\Xi_5\},\\
&\{a_1\Xi_3 + a_2\Xi_4 + a_3\Xi_5\}, \{\Xi_6\}, \{a_1\Xi_1 + a_2\Xi_6\}, \{a_1\Xi_2 + a_2\Xi_7\}, \{a_1\Xi_4 + a_2\Xi_7\},\\
&\{a_1\Xi_2 + a_2\Xi_4 + a_3\Xi_7\}, \{a_1\Xi_1 + a_2\Xi_2 + a_3\Xi_4 + a_4\Xi_7\}, \{a_1\Xi_3 + a_2\Xi_4 + a_3\Xi_7\},\\
&\{a_1\Xi_2 + a_2\Xi_3 + a_3\Xi_4 + a_4\Xi_7\}, \{a_1\Xi_1 + a_2\Xi_2 + a_3\Xi_3 + a_4\Xi_4 + a_5\Xi_7\},\\
&\{a_1\Xi_6 + a_2\Xi_7\}, \{a_1\Xi_2 + a_2\Xi_4 + a_3\Xi_6 + a_4\Xi_7\}, \{a_1\Xi_1 + a_2\Xi_8\}, \{a_1\Xi_5 + a_2\Xi_8\},\\
&\{a_1\Xi_1 + a_2\Xi_5 + a_3\Xi_8\}, \{a_1\Xi_1 + a_2\Xi_2 + a_3\Xi_5 + a_4\Xi_8\},\\
&\{a_1\Xi_1 + a_2\Xi_3 + a_3\Xi_5 + a_4\Xi_8\}, \{a_1\Xi_1 + a_2\Xi_5 + a_3\Xi_6 + a_4\Xi_8\},\\
&\{a_1\Xi_1 + a_2\Xi_2 + a_3\Xi_4 + a_4\Xi_5 + a5\Xi_7 + a_6\Xi_8\}\},\\
\Theta_2 \equiv \{&\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, b_1\Xi_2 + b_2\Xi_4\}, \{\Xi_1, \Xi_6\},\\
&\{\Xi_1, b_1\Xi_3 + b_2\Xi_6\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_5\}, \{\Xi_2, b_1\Xi_1 + b_2\Xi_5\}, \{\Xi_2, \Xi_6\},\\
&\{\Xi_2, b_1\Xi_1 + b_2\Xi_6\}, \{\Xi_2, b_1\Xi_3 + b_2\Xi_6\}, \{\Xi_3, \Xi_5\}, \{\Xi_3, \Xi_6\}, \{\Xi_3, \Xi_8\},\\
&\{a_1\Xi_2 + a_2\Xi_3, \Xi_5\}, \{\Xi_4, \Xi_6\}, \{\Xi_4, b_1\Xi_1 + b_2\Xi_6\}, \{\Xi_4, b_1\Xi_3 + b_2\Xi_6\}, \{\Xi_4, \Xi_8\},\\
&\{\Xi_4, b_1\Xi_1 + b_2\Xi_8\}, \{\Xi_4, b_1\Xi_3 + b_2\Xi_8\}, \{\Xi_5, b_1\Xi_3 + b_2\Xi_6\}, \{\Xi_5, \Xi_7\},\\
&\{\Xi_5, b_1\Xi_2 + b_2\Xi_7\}, \{\Xi_5, b_1\Xi_6 + b_2\Xi_7\}, \{\Xi_6, \Xi_7\}, \{a_1\Xi_3 + a_2\Xi_6, \Xi_7\},\\
&\{a_1\Xi_3 + a_2\Xi_6, \Xi_8\}, \{\Xi_7, \Xi_8\}, \{\Xi_7, b_1\Xi_3 + b_2\Xi_8\}, \{\Xi_7, b_1\Xi_5 + b_2\Xi_8\},\\
&\{a_1\Xi_4 + a_2\Xi_7, \Xi_8\}, \{a_1\Xi_6 + a_2\Xi_7, \Xi_8\}\}.
\end{aligned}
$$

THANK YOU.