A non-trivial conservation law with a trivial characteristic

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Outline

- The presymplectic operator

$$\Delta = \overline{D}_x \tag{1}$$

of the potential mKdV (with t, x)

$$u_t = 4u_x^3 + u_{xxx} \tag{2}$$

doesn't originate from its cosymmetries. (Proof by infinite descent).

- The one-component conservation law

$$u_x^4 dt \wedge dx \tag{3}$$

of the overdetermined system (with t, x, y)

$$u_t - 4u_x^3 - u_{xxx} = 0\,, \qquad u_y = 0 \tag{4}$$

is non-trivial. One of its characteristics is $(u_{xy}, 0)$. This conservation law is an element of $\ker d_1^{0,n-1} = E_2^{0,n-1}$, $n = 3$.

The conservation of zero, speculations

- A variational interpretation of $E_2^{0,n-1}$: the homotopy formula.

Informally,

the space of boundary conditions is degenerate. Obstacles to compactly supported perturbations?

- The presymplectic structures $\overline{D}_x$ of the equations

$$u_t = \alpha u_x^{\alpha-1} + u_{xxx}\,, \qquad \alpha \neq 4 \tag{5}$$

do originate from their cosymmetries. The variational interpretation doesn't clarify the triviality of the respective conservation laws for

$$u_t = \alpha u_x^{\alpha-1} + u_{xxx}\,, \qquad u_y = 0\,. \tag{6}$$

Loosely speaking, if $E_2^{0,n-1} \neq 0$, then

the space of boundary conditions is degenerate in a special way.

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The potential mKdV

Let us denote by $\mathcal{E}$ the equation

$$u_t = 4u_x^3 + u_{xxx} \tag{7}$$

with all its differential consequences and adopt the following notation

$$u_0 = u\,, \qquad u_1 = u_x\,, \qquad u_2 = u_{xx}\,, \qquad \ldots \tag{8}$$

Then the variables t, x, u_0, u_1, u_2, ... can be taken as intrinsic coordinates on the system

$$\mathcal{E}: \quad u_t = 4u_1^3 + u_3, \;\; D_x(u_t - 4u_1^3 - u_3) = 0, \;\; D_t(u_t - 4u_1^3 - u_3) = 0, \;\; \ldots$$

The Cartan distribution of $\mathcal{E}$ is spanned by the restrictions of D_x and D_t

$$\overline{D}_x = \partial_x + u_{i+1}\partial_{u_i}\,, \qquad \overline{D}_t = \partial_t + \overline{D}_x^{\,i}(4u_1^3 + u_3)\partial_{u_i}\,. \tag{9}$$

Functions on $\mathcal{E}$

Elements of the algebra

$$\mathcal{F}(\mathcal{E}) \tag{10}$$

are smooth functions of a finite number of the variables t, x, u_0, u_1, u_2, ...

For $f \in \mathcal{F}(\mathcal{E})$, denote by $\operatorname{ord} f$ the highest order of the derivatives u_i among its arguments. For $g = g(t, x)$, we put $\operatorname{ord} g = -\infty$.

The total derivative $\bar{D}_x$ increases the order of any function by 1. Then

$$\bar{D}_x(f) = 0 \qquad \Leftrightarrow \qquad f = f(t)\,. \tag{11}$$

The presymplectic operator

The potential mKdV equation

$$u_t = 4u_x^3 + u_{xxx} \tag{12}$$

admits the Lagrangian

$$L = \lambda\, dt \wedge dx\,, \qquad \lambda = \frac{u_t u_x}{2} - u_x^4 + \frac{u_{xx}^2}{2}\,, \tag{13}$$

which gives rise to the presymplectic operator:

$$\frac{\delta\lambda}{\delta u} = -D_x(u_t - 4u_x^3 - u_{xxx}) \qquad \Rightarrow \qquad \Delta = (-D_x)^*|_{\mathcal{E}} = \bar{D}_x\,. \tag{14}$$

The relation with the mKdV:

$$v_t = 12v^2 v_x + v_{xxx}\,, \qquad v = u_x\,. \tag{15}$$

For the mKdV: Δ determines a non-local presymplectic operator $\Rightarrow$ a local Hamiltonian operator.

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The linearization operator reads

$$l_{\mathcal{E}} = \bar{D}_t - 12u_1^2\,\bar{D}_x - \bar{D}_x^{\,3}\,. \tag{16}$$

Its adjoint operator has the form

$$l_{\mathcal{E}}^{*} = -\bar{D}_t + 12u_1^2\bar{D}_x + 12\bar{D}_x(u_1^2) + \bar{D}_x^{\,3}\,. \tag{17}$$

It cannot increase the order of a function by 3 or more.

A function $\psi \in \mathcal{F}(\mathcal{E})$ is a cosymmetry of $\mathcal{E}$ if

$$l_{\mathcal{E}}^{*}(\psi) = 0\,. \tag{18}$$

Cosymmetries of $\mathcal{E}$ describe all its conservation laws and some of its presymplectic operators (= admissible variational principles),

$$\psi \mapsto l_{\psi} - l_{\psi}^{*}\,, \qquad l_{\psi} = \sum_i \partial_{u_i}\psi\,\bar{D}_x^{\,i} \tag{19}$$

Is $\Delta = \bar{D}_x$ produced by a cosymmetry?

Cartan (contact) 1-forms of the potential mKdV:

$$\mathcal{C}\Lambda^1(\mathcal{E}) \ni \omega^i \bar{\theta}_i\,, \qquad \bar{\theta}_i = du_i - u_{i+1}dx - \overline{D}_x^{\,i}(4u_1^3 + u_3)dt\,. \tag{20}$$

The ideal of Cartan forms:

$$\mathcal{C}\Lambda^*(\mathcal{E}) = \mathcal{C}\Lambda^1(\mathcal{E}) \wedge \Lambda^*(\mathcal{E})\,, \qquad d\mathcal{C}\Lambda^*(\mathcal{E}) \subset \mathcal{C}\Lambda^*(\mathcal{E}) \tag{21}$$

Its powers $\mathcal{C}^p\Lambda^*(\mathcal{E})$, $p = 1, 2, \dots$ give rise to the filtration

$$\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}^2\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}^3\Lambda^\bullet(\mathcal{E}) \supset \dots \tag{22}$$

The corresponding spectral sequence is the Vinogradov $\mathcal{C}$-spectral sequence

$$(E_r^{p,q}(\mathcal{E}), d_r^{p,q}) \tag{23}$$

- Cosymmetries of the potential mKdV are in one-to-one correspondence with its variational 1-forms $E_1^{1,1}(\mathcal{E})$

$$\psi \mapsto l_\psi - l_\psi^* \qquad \Leftrightarrow \qquad \omega_\psi \mapsto d_1\omega_\psi \tag{24}$$

- Presymplectic operators of the potential mKdV that don't involve $\bar{D}_t$ are in one-to-one correspondence with its presymplectic structures

$$\ker d_1^{2,1} \subset E_1^{2,1}(\mathcal{E}) \tag{25}$$

- The presymplectic structure Ω corresponding to Δ is not produced by a cosymmetry iff it represents a non-trivial element of

$$E_2^{2,1}(\mathcal{E}) \tag{26}$$

The Lagrangian

$$L = \lambda\, dt \wedge dx\,, \qquad \lambda = \frac{u_t u_x}{2} - u_x^4 + \frac{u_{xx}^2}{2}\,, \tag{27}$$

of the potential mKdV is scale-invariant,

$$(t, x, u, \dots) \mapsto (e^{3\epsilon} t, e^{\epsilon} x, u, \dots) \tag{28}$$

Then the equation $\mathcal{E}$ possesses the scaling symmetry

$$X = 3t\partial_t + x\partial_x - \sum_{j=0}^{+\infty} j u_j \partial_{u_j}\,. \tag{29}$$

Note that X cannot increase the order of a function. Its characteristic φ is

$$\varphi = -3t(4u_1^3 + u_3) - xu_1\,. \tag{30}$$

The presymplectic structure Ω given by $\Delta = \bar{D}_x$ is X-invariant,

$$l_{\Delta(\varphi)} - l^*_{\Delta(\varphi)} = 0\,. \tag{31}$$

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Useful formulas

It is convenient to use the commutator

$$[\partial_{u_j}, \bar{D}_t] = \partial_{u_j}\big(\bar{D}_x^{\,i}(4u_1^3) + u_{i+3}\big)\partial_{u_i} \tag{32}$$

and the simple combinatorial observation.

Proposition 1.

For any integers $i \geqslant 1$, $j \geqslant 0$,

$$\partial_{u_j}\bar{D}_x^{\,i} = \sum_{r=0}^{\min\{i,j\}} \binom{i}{r} \bar{D}_x^{\,i-r}\partial_{u_{j-r}}\,. \tag{33}$$

Another helpful fact is given by the following

Proposition 2.

If $\psi \in \mathcal{F}(\mathcal{E})$ and $s \geqslant 1$ is an integer such that $\operatorname{ord}\psi \leqslant s$, then

$$\partial_{u_{s+2}}\, l_{\mathcal{E}}^*(\psi) = 3\bar{D}_x(\partial_{u_s}\psi)\,. \tag{34}$$

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Theorem 1.

The operator $\Delta = \bar{D}_x$ is not produced by a cosymmetry.

- In order to prove this result by contradiction, we now assume that the presymplectic operator $\Delta = \bar{D}_x$ is produced by a cosymmetry.
- Let ψ_0 denote a cosymmetry such that $l_{\psi_0} - l^*_{\psi_0} = \Delta$, and let k denote its order, $\operatorname{ord}\psi_0 = k$.
- Direct computations of cosymmetries of orders $\leqslant 6$ show that

$$k \geqslant 7\,. \tag{35}$$

Lemma 1.

There exist functions $a = a(t)$ and $\psi_2 \in \mathcal{F}(\mathcal{E})$ such that $\psi_0 = au_k + \psi_2$, $\operatorname{ord}\psi_2 \leqslant k - 2$. Besides, k is even.

Proof. Proposition 2 + the condition $l_{\psi_0} - l^*_{\psi_0} = \Delta$

$$\partial_{u_{k+2}}\, l^*_{\mathcal{E}}(\psi_0) = 3\bar{D}_x(\partial_{u_k}\psi_0)\,. \tag{36}$$

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where $\operatorname{ord}\psi_3 \leqslant k-3$ and

$$B = \frac{\dot{a}}{3}x + 4(k-1)au_1^2 + b\,. \tag{38}$$

Lemma 3.

There exists a function $\psi_4 \in \mathcal{F}(\mathcal{E})$ such that

$$\psi_0 = au_k + Bu_{k-2} + Cu_{k-3} + \psi_4\,, \tag{39}$$

where $\operatorname{ord}\psi_4 \leqslant k-4$ and

$$C = \frac{k-2}{2}\bar{D}_x(B)\,. \tag{40}$$

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Finally, we obtain

Lemma 5.

There exist $\psi_1 \in \mathcal{F}(\mathcal{E})$ and $a_0 \in \mathbb{R}$ such that

$$\psi_0 = a_0 u_k + \psi_1\,, \tag{42}$$

$a_0 \neq 0$, and $\operatorname{ord}\psi_1 \leqslant k-1$.

Note that the scaling symmetry

$$X = 3t\partial_t + x\partial_x - u_1\partial_{u_1} - 2u_2\partial_{u_2} - \dots \tag{43}$$

acts on cosymmetries:

$$\omega_\psi \mapsto \mathcal{L}_X\omega_\psi \quad \Leftrightarrow \quad \psi \mapsto (X+1)\psi\,. \qquad \operatorname{ord}(X+1)\psi \leqslant \operatorname{ord}\psi\,. \tag{44}$$

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Proof by infinite descent

$$\psi_0 = a_0 u_k + \psi_1\,, \qquad (X+k)(a_0 u_k) = 0 \tag{45}$$

if ψ_0 produces Δ, then the cosymmetry

$$\frac{(X+k)\psi_0}{k-1} \qquad \Leftrightarrow \qquad \frac{\mathcal{L}_X \omega_{\psi_0}}{k-1} + \omega_{\psi_0} \tag{46}$$

having $\mathrm{ord} < k$ also produces Δ.

There must be a lowest order cosymmetry that produces Δ. This result contradicts to the computational observation

$$k \geqslant 7\,. \tag{47}$$

The main result for the potential mKdV $u_t = 4u_x^3 + u_{xxx}$

The presymplectic operator $\Delta = \bar{D}_x$ is not produced by a cosymmetry.

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The Lagrangian

$$L = \lambda\, dt \wedge dx\,, \qquad \lambda = \frac{u_t u_x}{2} - u_x^4 + \frac{u_{xx}^2}{2} \tag{48}$$

gives rise to a differential 2-form

$$\ell \in \Lambda^2(\mathcal{E}) \tag{49}$$

such that $\ell - L|_{\mathcal{E}} \in \mathcal{C}\Lambda^2(\mathcal{E})$ and the coset

$$d\ell + \mathcal{C}^3\Lambda^3(\mathcal{E}) \in E_0^{2,1}(\mathcal{E}) \tag{50}$$

represents the presymplectic structure Ω. Due to

$$d_2^{0,2}\colon E_2^{0,2}(\mathcal{E}) \to E_2^{2,1}(\mathcal{E})\,, \tag{51}$$

the element of $E_2^{0,2}(\mathcal{E})$ determined by ℓ is non-trivial. It coincides with the element of $E_2^{0,2}(\mathcal{E})$ determined by $L|_{\mathcal{E}}$.

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The trick

For the system $\mathcal{E}'_{\partial_y}$

$$u_t = 4u_x^3 + u_{xxx}\,, \qquad u_y = 0\,, \tag{52}$$

the 2-form $L|_{\mathcal{E}'_{\partial_y}}$

$$L = \lambda\, dt \wedge dx\,, \qquad \lambda = \frac{u_t u_x}{2} - u_x^4 + \frac{u_{xx}^2}{2} \tag{53}$$

determines the conservation law

$$L|_{\mathcal{E}'_{\partial_y}} + d_0\Big(\frac{u_x u_{xx}}{2} dt\Big) = u_x^4 dt \wedge dx \tag{54}$$

According to the decomposition

$$D_y(\lambda) = u_{xy}\,(u_t - 4u_x^3 - u_{xxx}) + 0 \cdot u_y + D_t\Big(\frac{u_x}{2}\,u_y\Big) + D_x\Big(u_{xx}u_{xy} - \frac{u_t}{2}\,u_y\Big),$$

the conservation law is associated, for example, with the characteristic Q

$$Q_1 = u_{xy}\,, \qquad Q_2 = 0\,.$$

The cosymmetry $Q|_{\mathcal{E}'_{\partial_y}}$ is zero.

The systems $\mathcal{E}$ and $\mathcal{E}'_{\partial_y}$ are related,

$$\mathcal{E}'_{\partial_y} = \mathcal{E} \times \mathbb{R}_{\partial_y} \,. \tag{55}$$

The homotopy equivalence:

$$\mathrm{pr}_{\mathcal{E}} \colon \mathcal{E} \times \mathbb{R}_{\partial_y} \to \mathcal{E} \,, \qquad \mathcal{E} \to \mathcal{E}'_{\partial_y} \,, \;\; \rho \mapsto (\rho, 0) \tag{56}$$

Then

$$E_2^{0,2}(\mathcal{E}) = E_2^{0,2}(\mathcal{E}'_{\partial_y}) \tag{57}$$

The conservation law

$$u_x^4 dt \wedge dx \tag{58}$$

of the system

$$u_t = 4u_x^3 + u_{xxx} \,, \qquad u_y = 0 \tag{59}$$

is non-trivial.

Variational interpretation

Let N be a compact, oriented smooth manifold, $\dim N = (n-1)$.

An embedding $\sigma\colon N \to \mathcal{S}$ is an almost boundary condition if it defines an integral manifold of the Cartan distribution $\mathcal{C}$.

Example

For the heat equation

$$u_t = u_{xx} \tag{60}$$

with intrinsic coordinates t, x, u, u_x, u_{xx}, ..., one can take

$$N\colon\ \ t \in [0;1],\ x = 0\,, \quad \sigma\colon\ \ u = c_1(t)\,,\ u_x = c_2(t)\,,\ u_{xx} = \dot{c}_1\,,\ \ldots \tag{61}$$

or

$$N\colon\ \ t = 0,\ x \in [0;1]\,, \quad \sigma\colon\ \ u = h(x)\,,\ u_x = h'\,,\ u_{xx} = h''\,,\ \ldots \tag{62}$$

or some condition on $N\colon\ \ t^2 + x^2 = 1$.

Variational interpretation

Let $\gamma\colon N \times [0;1] \to \mathcal{S}$ be a path in almost boundary conditions, i.e., each

$$\gamma_\tau\colon\ N \to \mathcal{S}\,, \quad p \mapsto \gamma(p,\tau) \tag{63}$$

is an almost boundary condition. Here $\gamma \circ s_\tau = \gamma_\tau$ for the section

$$s_\tau\colon\ N \to N \times [0;1]\,, \quad p \mapsto (p,\tau)\,. \tag{64}$$

The homotopy formula for $\omega \in \Lambda^{n-1}(\mathcal{S})$ reads

$$\gamma_1^*(\omega) - \gamma_0^*(\omega) = dK(\omega) + K(d\omega)\,, \quad K(\omega) = \int_0^1 s_\tau^*(\partial_\tau \lrcorner\, \gamma^*(\omega))d\tau \tag{65}$$

If ω determines a conservation law $\xi \in E_1^{0,n-1}(\mathcal{S})$, and ∂N is fixed, then

$$\int_N \gamma_1^*(\omega) - \int_N \gamma_0^*(\omega) = \int_N K(d\omega)\,, \tag{66}$$

and the perturbation is determined by $d_1\xi$, i.e., by a cosymmetry of ξ.

If the cosymmetry of ξ is trivial,

the integral functional

$$\sigma \mapsto \int_N \sigma^*(\omega) \qquad (67)$$

is indifferent to perturbations of an almost boundary condition $\sigma\colon N \to \mathcal{S}$.

For the system

$$u_t = 4u_x^3 + u_{xxx}\,, \qquad u_y = 0\,, \qquad (68)$$

we can take as $N \subset \mathbb{R}^3$ the disc

$$N\colon \quad t^2 + x^2 \leqslant 1\,, \qquad y = 0 \qquad (69)$$

Then σ is a solution to the potential mKdV $\Rightarrow$ problems with perturbations of σ such that ∂N is fixed.

If N is given by $t = 0$, $x^2 + y^2 \leqslant 1$, then ∂N is fixed iff N is fixed (due to the condition $u_y = 0$).

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The Lagrangian

$$\widetilde{L} = \tilde{\lambda}\, dt \wedge dx\,, \qquad \tilde{\lambda} = \frac{u_t\, u_x}{2} - u_x^{\alpha} + \frac{u_{xx}^2}{2}\,, \qquad 2 \neq \alpha \neq 4 \tag{70}$$

leads to the presymplectic operator $\overline{D}_x$ of the equation

$$u_t = \alpha u_x^{\alpha-1} + u_{xxx}\,. \tag{71}$$

The transformation

$$g^{\epsilon}\colon (t, x, u, \ldots) \mapsto (e^{3\epsilon} t, e^{\epsilon} x, e^{\beta\epsilon} u, \ldots)\,, \qquad \beta = \frac{\alpha - 4}{\alpha - 2} \tag{72}$$

scales $\widetilde{L}$:

$$(g^{\epsilon})^{*}(\widetilde{L}) = \exp(2\beta\epsilon)\, \widetilde{L}\,. \tag{73}$$

Then the corresponding presymplectic structure $\widetilde{\Omega}$ is produced by a variational 1-form,

$$\widetilde{\Omega} = d_1 \frac{\widetilde{X} \lrcorner \widetilde{\Omega}}{2\beta}\,, \qquad \widetilde{X} = 3t\partial_t + x\partial_x + \beta u \partial_u + \ldots \tag{74}$$

Possible Noether's theorem violation

The Noether correspondence for Lagrangian systems:

$$X \lrcorner \Omega = d_1 \xi \tag{75}$$

Some References

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Thank you!