A relative of the NLS equation revisited

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based on arXiv:2202.04512, Journal of Physics A
(largely similar work for the NLS equation:
O. Chvartatskyi and F. M.-H., J. Phys. A 50 (2017) 155204)

December 14, 2022
Workshop on the occasion of Maxim Pavlov's 60th birthday

Introduction

The subject of this talk is the third-order nonlinear **PDE**

$$\left(\frac{f_{xt}}{f}\right)_t + 2\,(f^* f)_x = 0 \tag{1}$$

It is **completely integrable** (in the sense of possessing a Lax pair) **if f is real**. This is most likely **no longer true if f is complex**, without restriction. But in between, there is an integrable reduction of the latter case. Writing it as the **system**

$$a_t = (f^* f)_x \qquad f_{xt} + 2a\,f = 0 \tag{2}$$

the integrable reduction is obtained by restricting the function a to be **real** (which in fact still allows complex f).

A.L. Sakovich (arXiv:2205.09538): the reduction of the above system, where $a = a_1 + \mathrm{i}\, a_2$, with real functions a_j, $j = 1, 2$, $a_2 \neq 0$, does **not** pass the **Painlevé test** of integrability.

A peculiar property of the PDE

A generalization of PDE (1) to higher dimensions is the system

$$\frac{\partial}{\partial t}\Big(f^{-1}\,\frac{\partial}{\partial t}\frac{\partial}{\partial x^\mu}f\Big)+2\,\frac{\partial}{\partial x^\mu}(f^*f)=0 \qquad \mu=1,\ldots,m$$

The left hand side behaves as the components of a covector (tensor of type (0,1)) under general coordinate transformations in m dimensions, if f is a scalar, also depending on a parameter t. This system thus defines dynamics of a scalar field on an m-dimensional differentiable manifold.

PDE (1) (and system (2)) is thus **invariant under general coordinate transformations** $x \mapsto x'$ in one dimension.

Kamchatnov and Pavlov, Phys. Lett. A 301 (2002) 269

Lax pair for a generalization of the SIT equations:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_\tau = \frac{1}{\lambda - \zeta} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Expansion in powers of ζ^{-1},

$$\frac{\partial}{\partial \tau} = \sum_{n \geq 0} \zeta^{-n} \frac{\partial}{\partial t_n} \,, \quad a = \sum_n \zeta^{-n} a_n \,, \quad b = \sum_n \zeta^{-n} b_n \,, \quad c = \sum_n \zeta^{-n} c_n$$

leads to recursion relations for the AKNS hierarchy equations.

Using instead

$$\frac{\partial}{\partial \tau} = \sum_{n \geq 1} \zeta^{n} \frac{\partial}{\partial t_{-n}} \,, \quad a = \sum_n \zeta^{n} a_{-n} \,, \quad b = \sum_n \zeta^{n} b_{-n} \quad c = \sum_n \zeta^{n} c_{-n} \,,$$

expansion in positive powers of ζ leads to recursion relations for the "**negative**" (or "reciprocal") AKNS hierarchy equations.

First "negative" AKNS flow:

$$(q\,r)_{t_1} = 2a_{-1,x}\,, \qquad q_{t_{-1}x} = 4\,a_{-1}\,q\,, \qquad r_{t_{-1}x} = 4\,a_{-1}\,r$$

Via $r = q^*$ and renamings, this becomes our system (2),

$$\boxed{a_t = (f^* f)_x \qquad f_{xt} + 2a\,f = 0}$$

The first "negative" AKNS flow has been studied by various scientists, including
H. Aratyn, L.A. Ferreira, J.F. Gomes, A.H. Zimerman 2000
M. Chen, S.-Q. Liu, Y. Zhang 2006
Soliton solutions:
J. Ji, J.B. Zhang and D.-J. Zhang 2009 via Hirota bilinearization
A. Dimakis, F. M.-H. 2010, 2011, V.E. Vekslerchik 2012

A vectorial binary Darboux transformation

Theorem

Let a_0, f_0 be a solution of the system (2) with real a_0. Let n-comp. column vectors η_i, $i = 1, 2$, be solutions of the **linear system**

$$\Gamma\,\eta_{1x} = a_0\,\eta_1 + f_{0x}^*\,\eta_2 \qquad \Gamma\,\eta_{2x} = -a_0\,\eta_2 + f_{0x}\,\eta_1$$
$$\eta_{1t} = -\frac{1}{2}\Gamma\,\eta_1 + f_0^*\,\eta_2 \qquad \eta_{2t} = \frac{1}{2}\Gamma\,\eta_2 - f_0\,\eta_1$$

where Γ is an invertible constant $n \times n$ matrix. Requiring the spectrum condition $\mathrm{spec}(\Gamma) \cap \mathrm{spec}(-\Gamma^\dagger) = \emptyset$, let Ω be the unique solution of the rank 2 **Lyapunov equation**

$$\Gamma\,\Omega + \Omega\,\Gamma^\dagger = \eta_1\eta_1^\dagger + \eta_2\eta_2^\dagger$$

Where Ω^{-1} exists, $\quad a = a_0 - (\eta_1^\dagger\,\Omega^{-1}\eta_1)_x\,, \quad \boxed{f = f_0 - \eta_1^\dagger\,\Omega^{-1}\,\eta_2}$

is also a solution of the system (2). f then solves PDE (1). $\square$

Lax pair

Writing the linear system in the form

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}_x = \begin{pmatrix} a_0\,\Gamma^{-1} & f_{0x}^*\,\Gamma^{-1} \\ f_{0x}\,\Gamma^{-1} & -a_0\,\Gamma^{-1} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$
$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}\Gamma & f_0^*\,I_n \\ -f_0\,I_n & \frac{1}{2}\Gamma \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix, constitutes a **Lax pair** for the system (2) with **real** a, since its integrability condition is equivalent to a_0, f_0 satisfying the system (2) and $a_0^* = a_0$.

Note: the usual **spectral parameter** is promoted to a **matrix** Γ. This is typical for a **vectorial** generalization of a (binary) Darboux transformation. Basically, this can be traced back to Marchenko's work in the 1980s. (Also see, e.g., A.L. Sakhnovich 1994, F. Guil and M. Mañas 1996.)

Trivial seed solutions

If $f_0 = 0$, we choose $a_0 = -1/2$. Solution of the linear system:

$$\eta_1 = \exp\Big(-\frac{1}{2}(\Gamma^{-1}x + \Gamma\, t)\Big)\, v\,, \quad \eta_2 = \exp\Big(\frac{1}{2}(\Gamma^{-1}x + \Gamma\, t)\Big)\, w\,,$$

where v, w are constant n-component column vectors. Let Ω be the (unique) solution of the Lyapunov equation

$$\Gamma\,\Omega + \Omega\,\Gamma^\dagger = \eta_1\eta_1^\dagger + \eta_2\eta_2^\dagger\,.$$

Then

$$f = v^\dagger\, e^{-\frac{1}{2}(\Gamma^{\dagger -1}x + \Gamma^\dagger\, t)}\,\Omega^{-1}\, e^{\frac{1}{2}(\Gamma^{-1}x + \Gamma\, t)}\, w$$

solves PDE (1).

... with diagonal Γ

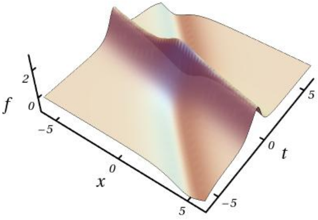
In this case we have

$$\Omega = (\Omega_{ij}) = \left(\frac{\eta_{1i}\,\eta_{1j}^* + \eta_{2i}\,\eta_{2j}^*}{\gamma_i + \gamma_j^*} \right) \quad \text{sum of 2 Cauchy-like matrices}$$

Example

Choosing $n = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $v_1 = v_2 = w_1 = w_2 = 1$, the above formula yields the special real 2-soliton solution

$$f = 6\,\frac{\cosh(2t + \frac{1}{2}x) - 2\cosh(t + x)}{\cosh(3t + \frac{3}{2}x) + 9\cosh(t - \frac{1}{2}x) - 8}$$



Trivial seed, non-diagonal Γ

Without restriction, Γ can be chosen in Jordan normal form. In contrast to "simple solitons", obtained with diagonal Γ, solutions determined by the above formula, with a non-diagonal Jordan matrix, depend also rationally on x and t.

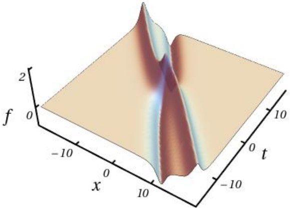
Example

$$\Gamma = \begin{pmatrix} \gamma & 0 \\ 1 & \gamma \end{pmatrix}, \qquad \kappa := 2\,\mathrm{Re}(\gamma)$$

$$\Longrightarrow \Omega = \frac{1}{\kappa} \sum_{i=1}^{2} \begin{pmatrix} |\eta_{i1}|^2 & \eta_{i1}(\eta_{i2} - \kappa^{-1}\eta_{i1})^* \\ \eta_{i1}^*(\eta_{i2} - \kappa^{-1}\eta_{i1}) & |\eta_{i2} - \kappa^{-1}\eta_{i1}|^2 + \kappa^{-2}|\eta_{i1}|^2 \end{pmatrix}$$

With $\gamma = v_1 = v_2 = w_1 = w_2 = 1$:

$$f = 4\,\frac{\cosh(x+t) + (x-t)\,\sinh(x+t)}{1 + 2(x-t)^2 + \cosh(2(x+t))}$$



Jordan matrix solutions of the Lyapunov equation

$k \times k$ lower triangular Jordan matrix

$$\Gamma_{(k)} = \begin{pmatrix} \gamma & 0 & \cdots & \cdots & 0 \\ 1 & \gamma & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \gamma \end{pmatrix}$$

For $k \leq n$, let $\Omega_{(k)}$ be the solution of the Lyapunov equation with $\Gamma_{(k)}$ and with η_i replaced by $\eta_{i(k)} := (\eta_{i1}, \eta_{i2}, \ldots, \eta_{ik})^T$, $i = 1, 2$.

$$\Omega_{(k+1)} = \begin{pmatrix} \Omega_{(k)} & B_{k+1} \\ B_{k+1}^{\dagger} & \omega_{k+1} \end{pmatrix}, \quad K_{(k)} := \Gamma_{(k)}\Big|_{\gamma \mapsto \kappa}, \ \kappa := 2\,\mathrm{Re}(\gamma)$$

$$B_{k+1} := K_{(k)}^{-1}\Big(\eta_{1(k)}\,\eta_{1,k+1}^{*} + \eta_{2(k)}\,\eta_{2,k+1}^{*} - \Omega_{(k)}(0,\ldots,0,1)^T\Big)$$

$$\omega_{k+1} := \frac{1}{\kappa}\Big(|\eta_{1,k+1}|^2 + |\eta_{2,k+1}|^2 - 2\,\mathrm{Re}[(0,\ldots,0,1)B_{k+1}]\Big)$$

... and their inverses

Solutions of the Lyapunov equation with Jordan matrices are thus **nested** and can be recursively computed.

For their inverses we have

$$\Omega_{(k+1)}^{-1} = \begin{pmatrix} \Omega_{(k)}^{-1} - S_{\Omega_{(k)}}^{-1} \Omega_{(k)}^{-1} B_{k+1} B_{k+1}^{\dagger} \Omega_{(k)}^{-1} & -S_{\Omega_{(k)}}^{-1} \Omega_{(k)}^{-1} B_{k+1} \\ -S_{\Omega_{(k)}}^{-1} B_{k+1}^{\dagger} \Omega_{(k)}^{-1} & S_{\Omega_{(k)}}^{-1} \end{pmatrix}$$

with the scalar Schur complement

$$S_{\Omega_{(k)}} = \omega_{k+1} - B_{n+1}^{\dagger} \Omega_{(k)}^{-1} B_{k+1}$$

(a special quasi-determinant). Hence also the inverses can be recursively computed.

Solutions with plane wave background

$$f_0 = C\, e^{\mathrm{i}\,(\alpha\, x - \beta\, t)} \qquad \alpha, \beta \in \mathbb{R}$$

Then $a_0 = -\alpha\beta/2$. Writing $\eta_1 = e^{-\frac{1}{2}\mathrm{i}\,(\alpha\, x - \beta\, t)}\,\tilde{\eta}_1$, $\eta_2 = e^{\frac{1}{2}\mathrm{i}\,(\alpha\, x - \beta\, t)}\,\tilde{\eta}_2$, the linear system can be simplified to

$$\tilde{\eta}_{1tt} - (\frac{1}{4}\tilde{\Gamma}^2 - |C|^2)\,\tilde{\eta}_1 = 0\,, \qquad \mathrm{i}\,\alpha\,\tilde{\eta}_{1t} + \Gamma\,\tilde{\eta}_{1x} = 0$$

$$\tilde{\eta}_2 = \frac{1}{C^*}\Big(\tilde{\eta}_{1t} + \frac{1}{2}\tilde{\Gamma}\,\tilde{\eta}_1\Big) \qquad \tilde{\Gamma} := \Gamma + \mathrm{i}\beta\, I_n$$

If R is an invertible **matrix root** of

$$R^2 = \frac{1}{4}\tilde{\Gamma}^2 - |C|^2 I_n$$

this is solved by

$$\tilde{\eta}_1 = \cosh(\Theta)\, V\,, \qquad \tilde{\eta}_2 = \frac{1}{2C^*}\Big(\cosh(\Theta)\,\tilde{\Gamma} - 2R\,\sinh(\Theta)\Big)V$$

with a constant n-component column vector V and

$$\Theta := \mathrm{i}\,\alpha\, x\,(\tilde{\Gamma} - \mathrm{i}\beta\, I_n)^{-1} R - t\, R + K$$

with a constant $n \times n$ matrix K that commutes with Γ (and R).

Solution of the PDE according to Theorem:

$$f = f_0 \left(1 - \frac{1}{|C|^2} V^\dagger \cosh(\Theta^\dagger)\, \Omega^{-1} \left(\cosh(\Theta)\, \tilde{\Gamma} - 2R \sinh(\Theta)\right) V\right)$$

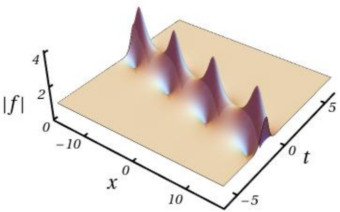
The further elaboration only requires the solution Ω of the Lyapunov equation for specified Γ.

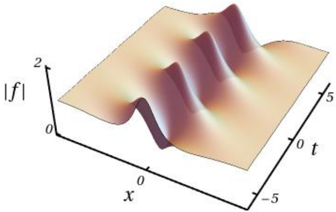
This leads to counterparts of **Akhmediev** and **Kuznetsov-Ma** breathers of the NLS equation.

For $n = 1$, we find

$$f = C\, e^{\mathrm{i}\,(\alpha\, x - \beta\, t)} \left(1 - \frac{\mathrm{Re}(\tilde{\gamma})\, \cosh(\Theta^*) \big(\tilde{\gamma} \cosh(\Theta) - 2\, r \sinh(\Theta)\big)}{|C|^2\, |\cosh(\Theta)|^2 + \frac{1}{4} |\tilde{\gamma}\, \cosh(\Theta) - 2\, r\, \sinh(\Theta)|^2} \right)$$

where now $\Theta = (\mathrm{i}\,\alpha\, x/(\tilde{\gamma} - \mathrm{i}\beta) - t) r + K$ with $r = \pm\sqrt{\frac{1}{4}\tilde{\gamma}^2 - |C|^2}$.
This is the counterpart of a single **Akhmediev breather** if $|\tilde{\gamma}| < 2|C|$ and a **Kuznetsov-Ma breather** if $|\tilde{\gamma}| > 2|C|$.





If Γ (and then also $\tilde{\Gamma}$) is an $n \times n$ Jordan matrix, a root of the quadratic equation is given by the Toeplitz matrix

$$R = \frac{1}{2}\sqrt{\tilde{\gamma}^2 - 4|C|^2} \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \tilde{\gamma}\,(\tilde{\gamma}^2 - 4|C|^2)^{-1} & 1 & \ddots & \ddots & \ddots & \vdots \\ -2\,(\tilde{\gamma}^2 - 4|C|^2)^{-2} & \ddots & \ddots & \ddots & \ddots & \vdots \\ 2\,\tilde{\gamma}\,(\tilde{\gamma}^2 - 4|C|^2)^{-3} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \end{pmatrix}$$

Regularity of higher Akhmediev and Kuznetsov-Ma breathers

Proposition. Let $\tilde{\Gamma}$ be an $n \times n$ (lower triangular) Jordan matrix with $\mathrm{Re}(\tilde{\gamma}) \neq 0$ and $\tilde{\gamma} \neq \pm 2|C|$. Then the solution

$$f = f_0 \left(1 - \frac{1}{|C|^2} V^\dagger \cosh(\Theta^\dagger)\, \Omega^{-1} \left(\cosh(\Theta)\, \tilde{\Gamma} - 2R \sinh(\Theta)\right) V\right)$$

of PDE (1) is **regular** if the first component of the vector V is different from zero.
Without restriction of generality one can set $V = (1, 0, \ldots, 0)^T$.

□

This mainly rests on the following (cf. Chvartatskyi and M.-H., J. Phys. A **50** (2017) 155204, Appendices A and B).

Proposition. Let Γ be a lower triangular $n \times n$ Jordan block with eigenvalue γ and $\mathrm{Re}(\gamma) \neq 0$. If the first component of one of the vectors η_i is different from zero, then the solution Ω of the rank k Lyapunov equation

$$\Gamma\,\Omega + \Omega\,\Gamma^\dagger = \sum_{i=1}^{k} \eta_i\,\eta_i^\dagger$$

is invertible.

Sketch of Proof: $\Omega = \sum_{i=1}^{k} \Omega_i$. If $\eta_{11} \neq 0$, then $\det(\Omega_1) \neq 0$. Furthermore, one can show that $\Omega_i = B_i^\dagger B_i$ with a matrix B_i.

$$\det(\Omega) = \underbrace{\det(B_1^\dagger)}_{\neq 0}\,\det(I_n + \underbrace{B_1^{-1\dagger}\,(\sum_{i=2}^{k} B_i^\dagger B_i)\,B_1^{-1}}_{\text{positive semi-definite}})\underbrace{\det(B_1)}_{\neq 0} \neq 0$$

□

A degenerate case of the linear system

Example

Let $n = 1$ and $\tilde{\gamma} = 2|C|$. Then we have

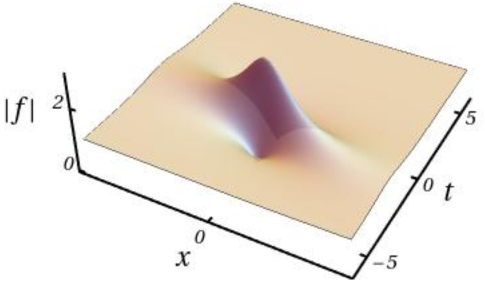
$$\tilde{\eta}_1 = c_0 + c_1 \left(\frac{\alpha\, x}{2\mathrm{i}\,|C| + \beta} + t\right), \qquad \tilde{\eta}_2 = \frac{C}{|C|}\,\tilde{\eta}_1 + \frac{c_1}{C^*}$$

$$\Omega = \frac{1}{4|C|}\Big(2\left|\tilde{\eta}_1 + \frac{c_1}{2|C|}\right|^2 + \frac{|c_1|^2}{2|C|^2}\Big)$$

$$f = C\, e^{\mathrm{i}\,(\alpha\, x - \beta\, t)}\Big[1 - \frac{1}{|C|\,\Omega}\Big(\Big|c_0 + c_1\,\big(\frac{\alpha\, x}{2\mathrm{i}\,|C| + \beta} + t\big)\Big|^2 + \frac{c_1}{|C|}\big[c_0^* + c_1^*\,\big(\frac{\alpha\, x}{-2\mathrm{i}\,|C| + \beta} + t\big)\big]\Big)\Big]$$

This quasi-rational solution is the counterpart of the **Peregrine breather** solution of the focusing NLS equation, which models a **rogue wave**.

Counterpart of Peregrine breather



Counterparts of **higher order Peregrine breathers** are obtained if $\tilde{\Gamma}$ is a Jordan matrix with eigenvalue $\tilde{\gamma} = 2|C|$.
(For NLS: exhaustively elaborated by V.B. Matveev.)

Proposition. Let

$$\mathcal{N} := \frac{1}{4}\tilde{\Gamma}^2 - |C|^2 I_n$$

be nilpotent of degree $N > 0$. The linear system is then solved by

$$\tilde{\eta}_1 = \big(R_1(\mathcal{N},t)\, R_1(-\alpha^2\Gamma^{-2}\mathcal{N},x) - \mathrm{i}\,\alpha\,\Gamma^{-1}\mathcal{N}\, R_2(\mathcal{N},t)\, R_2(-\alpha^2\Gamma^{-2}\mathcal{N},x)\big)\, v$$
$$+\big(R_1(\mathcal{N},t)\, R_2(-\alpha^2\Gamma^{-2}\mathcal{N},x) + \frac{\mathrm{i}}{\alpha}\Gamma\, R_2(\mathcal{N},t)\, R_1(-\alpha^2\Gamma^{-2}\mathcal{N},x)\big)\, w$$
$$\tilde{\eta}_2 = \frac{1}{C^*}\Big(\tilde{\eta}_{1t} + \frac{1}{2}\tilde{\Gamma}\,\tilde{\eta}_1\Big)$$

with constant n-component vectors v, w, and

$$R_1(\mathcal{N},t) := \sum_{k=0}^{N-1} \frac{t^{2k}}{(2k)!}\mathcal{N}^k\,, \qquad R_2(\mathcal{N},t) := \sum_{k=0}^{N-1} \frac{t^{2k+1}}{(2k+1)!}\mathcal{N}^k$$

□

Regularity of higher order Peregrine breathers

Proposition. Let $\tilde{\Gamma}$ be a (lower triangular) Jordan matrix with eigenvalue $\tilde{\gamma} = 2|C|$. Let η_1, η_2 be the above solution of the linear system. The solution of PDE (1), obtained via the vectorial binary Darboux transformation, is then regular if the first component of v is different from zero. Without restriction of generality, we can set $v = (1, 0, \ldots, 0)^T$. □

More general solutions

... can be generated by solving the Lyapunov equation with

$$\Gamma = \begin{pmatrix} \Gamma_{k_1} & 0 & \cdots & \cdots & 0 \\ 0 & \Gamma_{k_2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \Gamma_{k_m} \end{pmatrix}$$

where Γ_{k_i} is the $k_i \times k_i$ Jordan block with eigenvalue γ_i.

Origin of the above Darboux transformation: bidifferential calculus

A **graded associative algebra** is an associative algebra

$$\boldsymbol{\Omega} = \bigoplus_{r \geq 0} \boldsymbol{\Omega}^r$$

over a field $\mathbb{K}$ of characteristic zero, where $\mathcal{A} := \boldsymbol{\Omega}^0$ is an associative algebra over $\mathbb{K}$ and $\boldsymbol{\Omega}^r$, $r \geq 1$, are $\mathcal{A}$-bimodules such that $\boldsymbol{\Omega}^r \, \boldsymbol{\Omega}^s \subseteq \boldsymbol{\Omega}^{r+s}$. Elements of $\boldsymbol{\Omega}^r$ are called r-**forms**.

A **bidifferential calculus** is a unital graded associative algebra $\boldsymbol{\Omega}$, supplied with two $\mathbb{K}$-linear **graded derivations**

$$\mathrm{d}, \bar{\mathrm{d}} : \boldsymbol{\Omega}^r \to \boldsymbol{\Omega}^{r+1}$$

and such that

$$\mathrm{d}^2 = \bar{\mathrm{d}}^2 = \mathrm{d}\bar{\mathrm{d}} + \bar{\mathrm{d}}\mathrm{d} = 0$$

bDT in bidifferential calculus

Theorem. Given a bidifferential calculus, let 0-forms Δ, Γ and 1-forms κ, λ satisfy

$$\bar{\mathrm{d}}\Delta + [\lambda, \Delta] = (\mathrm{d}\Delta)\,\Delta\,, \qquad \bar{\mathrm{d}}\lambda + \lambda^2 = (\mathrm{d}\lambda)\,\Delta\,,$$
$$\bar{\mathrm{d}}\Gamma - [\kappa, \Gamma] = \Gamma\,\mathrm{d}\Gamma\,, \qquad \bar{\mathrm{d}}\kappa - \kappa^2 = \Gamma\,\mathrm{d}\kappa$$

Let 0-forms θ and η be solutions of the linear equations

$$\bar{\mathrm{d}}\theta = (\mathrm{d}\phi)\,\theta + (\mathrm{d}\theta)\,\Delta + \theta\,\lambda\,, \qquad \bar{\mathrm{d}}\eta = -\eta\,(\mathrm{d}\phi) + \Gamma\,\mathrm{d}\eta + \kappa\,\eta$$

where the 0-form ϕ satisfies

$$\boxed{\bar{\mathrm{d}}\mathrm{d}\phi = \mathrm{d}\phi\,\mathrm{d}\phi}$$

Furthermore, let Ω be an invertible solution of the linear system

$$\boxed{\Gamma\,\Omega - \Omega\,\Delta = \eta\,\theta} \quad \text{(generalization of Sylvester equation)}$$
$$\bar{\mathrm{d}}\Omega = (\mathrm{d}\Omega)\,\Delta - (\mathrm{d}\Gamma)\,\Omega + \kappa\,\Omega + \Omega\,\lambda + (\mathrm{d}\eta)\,\theta$$

Then

$$\boxed{\phi' = \phi - \theta\,\Omega^{-1}\eta + K}$$

with $\mathrm{d}K = 0$ solves the same equation. □

Application to the first negative AKNS flow

Let $\mathcal{A}$ be the commutative algebra of smooth functions on $\mathbb{R}^2$ and $\mathrm{Mat}(\mathcal{A})$ the algebra of all matrices over $\mathcal{A}$, where the product of two matrices is defined to be zero whenever their dimensions do not fit. We choose

$$\mathbf{\Omega} = \mathrm{Mat}(\mathcal{A}) \otimes \bigwedge \mathbb{C}^2 \,,$$

where $\bigwedge \mathbb{C}^2$ is the exterior algebra of the vector space $\mathbb{C}^2$. Let ξ_1, ξ_2 be a basis of $\bigwedge^1 \mathbb{C}^2$. For an $m \times n$ matrix F over $\mathcal{A}$, let

$$\mathrm{d}F = F_x \, \xi_1 + \frac{1}{2}(J_m F - F J_n) \, \xi_2 \,, \quad \bar{\mathrm{d}}F = \frac{1}{2}(J_m F - F J_n) \, \xi_1 + F_t \, \xi_2$$

where $J_2 = \mathrm{diag}(1, -1)$ etc. (A. Dimakis, F. M.-H. 2010).

Then $\bar{\mathrm{d}}\mathrm{d}\phi = \mathrm{d}\phi\,\mathrm{d}\phi$ with

$$\phi = \begin{pmatrix} p & f \\ q & -p \end{pmatrix} \qquad a := p_x - \frac{1}{2}$$

becomes the **first negative flow of the AKNS hierarchy**

$$a_t = (fq)_x\,, \qquad f_{xt} = -2a\,f\,, \qquad q_{xt} = -2a\,q\,,$$

and the last theorem can be worked out to obtain a vectorial binary Darboux transformation for it.
Finally, we impose the **reduction**

$$q = f^*$$

to obtain our vectorial Darboux transformation for system (2).

Remarks

- The PDE admits generalizations of the 1-soliton solution to Jacobi elliptic functions, which could also serve as seed for the bDT. Can this be worked out in some detail ?
- There are d and $\bar{\mathrm{d}}$ such that $\bar{\mathrm{d}}\mathrm{d}\phi = \mathrm{d}\phi\,\mathrm{d}\phi$ becomes the whole "negative" (or "reciprocal") AKNS hierarchy, see Dimakis and M.-H., SIGMA **6** (2010) 055.
- Rusuo Ye (2022): a Riccati-type Miura transformation from **Fokas-Lenells system**

$$q + q_{xt} - 2i\, q\, r\, q_x = 0 \qquad r + r_{xt} + 2i\, r\, q\, r_x = 0$$

  to the first negative AKNS system.
  Still open: a bidifferential calculus for the FL system.
- What more can be concluded about the **non**-integrable reduction of PDE (1) ?

Thanks for your attention !

Many thanks to Maxim for valuable help over the years and very pleasant times together !

All the best for Maxim for the next decades !

And thanks to Joseph for organizing this meeting !