Tropical limit of solitons,

Yang-Baxter maps and beyond

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Prelude: Geometry and Integrability

This section explains in particular how we construct exact solutions which are then explored in a "tropical limit".
It also bridges to a main topic of this meeting.

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- Integrability:

  $$\text{Equation (e.g., PDDE)} \Longleftrightarrow \text{zero curvature}$$

  Here the connection depends on constituents of the equation (dependent and possibly independent variables of a PDDE).

Zero curvature condition on associative algebras

$\mathcal{A}$ unital associative algebra, over a field $\mathbb{K}$
$(\boldsymbol{\Omega}, \delta)$ a differential calculus over $\mathcal{A}$

Here $\boldsymbol{\Omega} = \bigoplus_{k \geq 0} \boldsymbol{\Omega}^k$ is a graded algebra with $\boldsymbol{\Omega}^0 = \mathcal{A}$ and $\mathcal{A}$-bimodules $\boldsymbol{\Omega}^k$, and δ is a derivation of degree one.

A **connection** on a left $\mathcal{A}$-module $\boldsymbol{\Gamma}$ is a linear map $\nabla : \boldsymbol{\Gamma} \to \boldsymbol{\Omega}^1 \otimes_{\mathcal{A}} \boldsymbol{\Gamma}$, such that $\nabla(f\gamma) = \delta f \otimes_{\mathcal{A}} \gamma + f \nabla \gamma$, for all $f \in \mathcal{A}$ and $\gamma \in \boldsymbol{\Gamma}$. It extends to $\boldsymbol{\Omega} \otimes_{\mathcal{A}} \boldsymbol{\Gamma}$, hence we can define the **curvature** as ∇^2. The essence of integrability of some equation may then be expressed as

$$\nabla^2 = 0 \quad \Longleftrightarrow \quad \text{equation}$$

Simplifying assumption: $\boldsymbol{\Gamma}$ has a basis b^μ, $\mu = 1, \ldots, m$.
Using the summation convention, we have $\gamma = \gamma_\mu \, b^\mu$ and

$$\nabla\gamma = (\delta\gamma_\mu - \gamma_\nu \, \boldsymbol{A}^\nu{}_\mu) \otimes_{\mathcal{A}} b^\nu \qquad \nabla^2\gamma = \gamma_\mu \, F_\delta[\boldsymbol{A}]^\mu{}_\nu \otimes_{\mathcal{A}} b^\nu$$
$$F_\delta[\boldsymbol{A}]^\mu{}_\nu = \delta\boldsymbol{A}^\mu{}_\nu - \boldsymbol{A}^\mu{}_\kappa \, \boldsymbol{A}^\kappa{}_\nu$$

where $\boldsymbol{A}$ is the $m \times m$ matrix of elements of $\boldsymbol{\Omega}^1$ defined by $\nabla b^\mu = \boldsymbol{A}^\mu{}_\nu \otimes_{\mathcal{A}} b^\nu$.

However, in most cases the above characterization of integrability is not strong enough. Rather, one needs a connection that depends on a ("spectral") parameter and the stronger condition that the curvature vanishes for all of its values.

Bidifferential calculus (Dimakis & M-H 2000) is the special case where ∇ is linear in such a parameter. In particular, one meets this situation in case of the (anti-) self-dual Yang-Mills equation.

Bidifferential calculus

$$\delta = \bar{\mathrm{d}} + \nu \, \mathrm{d} \, , \qquad \boldsymbol{A} = A + \nu \, B \, .$$

The zero curvature condition $(\forall \nu)$ is then equivalent to

$$F_{\bar{\mathrm{d}}}[A] = 0 = F_{\mathrm{d}}[B] \qquad \mathrm{d}A + \bar{\mathrm{d}}B + A\,B + B\,A = 0$$

Setting $B = 0$ (typically one of the two "gauge potentials" A and B can be transformed to zero by a gauge transformation), this is

$$\mathrm{d}A = 0 \qquad F_{\bar{\mathrm{d}}}[A] = 0$$

The first equation can be solved by setting $A = \mathrm{d}\phi$, with $\phi \in \mathrm{Mat}(m, m, \mathcal{A})$, the algebra of $m \times m$ matrices over $\mathcal{A}$. Then

$$\boxed{\mathrm{d}\,\bar{\mathrm{d}}\,\phi + \mathrm{d}\phi\;\mathrm{d}\phi = 0}$$

Alternatively, we can solve $F_{\bar{\mathrm{d}}}[A] = 0$ by setting $A = (\bar{\mathrm{d}}g)\,g^{-1}$ with an invertible $g \in \mathrm{Mat}(m, m, \mathcal{A})$. Then $\mathrm{d}A = 0$ results in

$$\boxed{\mathrm{d}\,[(\bar{\mathrm{d}}g)\,g^{-1}] = 0}$$

Many integrable equations are **realizations** of one of these eqs.

Remark: Frölicher-Nijenhuis theory

Let $\mathbf{\Omega}$ be the algebra of differential forms on a manifold $\mathcal{M}$ and d the exterior derivative. A **Nijenhuis tensor** is a tensor field N of type (1,1) (or a fiber preserving endomorphism of TM) on $\mathcal{M}$, with **vanishing Nijenhuis torsion**

$$T(N)(X, Y) := [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]),$$

where X, Y are any vector fields on $\mathcal{M}$. Then

$$\bar{\mathrm{d}} = i_N \, \mathrm{d}$$

extends the above to a bidifferential calculus. Here we look at N as a vector-valued 1-form, and i_N acts via contraction of a vector field and a 1-form. According to Frölicher-Nijenhuis theory, any derivation $\bar{\mathrm{d}}$ of degree one, which anti-commutes with d and satisfies $\bar{\mathrm{d}}^2 = 0$, is of the form d_N.
There are applications to integrable systems (bi-Hamiltonian structures). **Magri**: construction of "Lenard chain" of conserved quantities. Works more generally in bidifferential calculus.

Integrability

Let $\Delta \in \mathrm{Mat}(n, n, \mathcal{A})$ and $\alpha \in \mathrm{Mat}(n, n, \mathbf{\Omega}^1)$ satisfy

$$\bar{\mathrm{d}}\Delta + [\alpha, \Delta] = (\mathrm{d}\Delta)\,\Delta \qquad \bar{\mathrm{d}}\alpha + \alpha^2 = (\mathrm{d}\alpha)\,\Delta$$

The linear system

$$\bar{\mathrm{d}}\theta = A\,\theta + (\mathrm{d}\theta)\,\Delta + \theta\,\alpha$$

for $\theta \in \mathrm{Mat}(m, n, \mathcal{A})$, then has the integrability condition

$$F_{\bar{\mathrm{d}}}[A]\,\theta - (\mathrm{d}A)\,\theta\,\Delta = 0$$

If Δ is such that this implies separate vanishing of both summands, $\mathrm{d}A = 0 = F_{\bar{\mathrm{d}}}[A]$ is **integrable** in the sense that it arises as the integrability condition of a linear system. Also:

$$\bar{\mathrm{d}}\eta = -\eta\,A + \Gamma\,\mathrm{d}\eta + \beta\,\eta \quad \Longrightarrow \quad \mathrm{d}A = 0 = F_{\bar{\mathrm{d}}}[A]$$

for $\eta \in \mathrm{Mat}(n, m, \mathcal{A})$, if $\Gamma \in \mathrm{Mat}(n, n, \mathcal{A})$ and $\beta \in \mathrm{Mat}(n, n, \mathbf{\Omega}^1)$ satisfy

$$\bar{\mathrm{d}}\Gamma - [\beta, \Gamma] = \Gamma\,\mathrm{d}\Gamma \qquad \bar{\mathrm{d}}\beta - \beta^2 = \Gamma\,\mathrm{d}\beta$$

A binary Darboux transformation

Let $\Delta, \lambda, \Gamma, \kappa$ satisfy the respective equations and ϕ_0 be a solution of

$$\mathrm{d}\,\bar{\mathrm{d}}\,\phi + \mathrm{d}\phi\,\mathrm{d}\phi = 0$$

Furthermore, let θ, η be solutions of the linear systems

$$\bar{\mathrm{d}}\theta = (\mathrm{d}\phi)\,\theta + (\mathrm{d}\theta)\,\Delta + \theta\,\alpha \qquad \bar{\mathrm{d}}\eta = -\eta\,\mathrm{d}\phi + \Gamma\,\mathrm{d}\eta + \beta\,\eta$$

Let Ω be a solution of the (compatible) linear equations

$$\Gamma\,\Omega - \Omega\,\Delta = \eta\,\theta$$
$$\bar{\mathrm{d}}\Omega = (\mathrm{d}\Omega)\,\Delta - (\mathrm{d}\Gamma)\,\Omega + \beta\,\Omega + \Omega\,\alpha + (\mathrm{d}\eta)\,\theta$$

Then

$$\phi = \phi_0 - \theta\,\Omega^{-1}\,\eta$$

is a new solution of $\mathrm{d}\,\bar{\mathrm{d}}\,\phi + \mathrm{d}\phi\,\mathrm{d}\phi = 0$.

There is an analogous result for $\mathrm{d}\,[(\bar{\mathrm{d}}g)\,g^{-1}] = 0$.

Generating solutions for matrix KdV, KP, ...

Let $u = 2\,\phi_x$ ($m \times n$) be the dependent variable.

$$
\begin{array}{ccc}
 & \text{seed solution } \phi_0 & \\
\swarrow & & \searrow \\
\begin{array}{c}\text{linear system}\\ \text{for } \theta\ (m \times N)\end{array} & & \begin{array}{c}\text{adjoint linear system}\\ \text{for } \chi\ (N \times n)\end{array} \\
\searrow & & \swarrow \\
 & \begin{array}{c}\text{compatible linear system}\\ \text{for ``Darboux potential'' } \Omega\end{array} & \\
 & \downarrow & \\
 & \begin{array}{c}\text{new solution}\\ \phi = \phi_0 - \theta\,\Omega^{-1}\,\chi\end{array} &
\end{array}
$$

For *soliton solutions* of matrix KdV: $\phi_0 = 0$.

$\curvearrowright$ $\theta = \theta_0\, e^{\vartheta(P)}$, $\chi = e^{\vartheta(P)}\,\chi_0$, with $\vartheta(P) = x\,P + t\,P^3$.

P is a constant $N \times N$ matrix.

Similar, but more involved, for Boussinesq, KP, etc.

Tropical limit of solitons and "particle" interactions

Main thoughts

- "Tropical limit" of a soliton solution in two space-time dimensions: wave crest limit.
- It associates with the wave solution a **classical point particle picture**.
- In case of matrix KdV, e.g., matrix data are related by a **Yang-Baxter map** along the tropical limit graph.
- In 2d integrable QFT models, **YB** equation expresses factorization of the multi-particle scattering matrix: the latter decomposes into 2-particle interactions. Similarly, we think of a multi-soliton solution also as being composed of 2-soliton interactions. However, because of the wave nature of solitons, there are no definite events at which the interaction takes place. But the tropical limit takes waves to "point particles" and then indeed determines events at which an interaction occurs. Does this mean that **YB** holds?

"Tropical limit" of KP soliton solutions

Writing

$$u = 2\,(\log\tau)_{xx}\,,$$

the function τ of a soliton solution of the **scalar** KP hierarchy, or some of its reductions, has the form

$$\tau = \sum_I e^{\Theta_I} \quad \text{with} \quad \Theta_I \quad \textbf{linear} \text{ in} \quad x, y, t$$

Then (Maslov dequantization)

$$\lim_{\epsilon\to 0}\ \epsilon \log \sum_I e^{\Theta_I/\epsilon} = \max\{\Theta_I\}$$

defines the **tropical limit** of $\log\tau$.
In the region, where $\Theta_I > \Theta_J$ for all $J \neq I$, we have $u = 0$ (since Θ_I is linear in x). Thus, the tropical limit of solitons has support on the **boundaries of dominating phase regions**.

Remark

"Tropicalization":

$$\begin{aligned} + \quad &\mapsto \quad \oplus := \max \\ \cdot \quad &\mapsto \quad \odot := + \end{aligned}$$

Characteristic of tropical geometry: *piecewise linear*.

KdV: 2-soliton solution

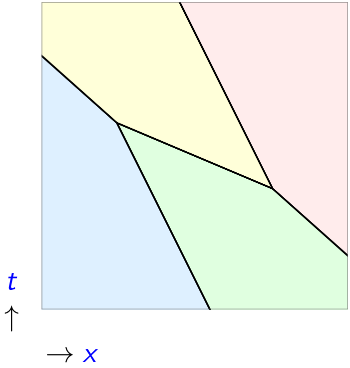
Korteweg-deVries (**KdV**) equation:

$$4\, u_t - u_{xxx} - 3\,(u^2)_x = 0$$

The black line segments in the figure constitute the

tropical limit graph

in the xt-plane, of a 2-soliton solution. It is the support of the variable u in the tropical limit.



This looks like a space-time scattering diagram for two point particles (exchanging a "virtual" particle).
The tropical limit gives a precise meaning to what contour plots have shown us long ago ("resonance").

Matrix KdV

$$4\,u_t - u_{xxx} - 3\,(u\,K\,u)_x = 0$$

where K is a constant $n \times m$ matrix and u an $m \times n$ matrix.

The **tropical limit** of a soliton solution, at fixed time, now consists of a **graph with matrix data attached to its edges**.

What are the rules according to which such data are distributed over the graph ?

Tropical limit of matrix KdV solitons

We have

$$\phi = \frac{F}{\tau} \quad \text{where} \quad \tau = \det \Omega$$

and F is a matrix. Now we **define the tropical limit graph using the function** τ, which is **not** in general a "tau function" of the scalar KdV equation.

Observation: support of matrix solution generically coincides with support of τ.

Yang-Baxter maps

Let X be a set. A map

$$R : X \times X \longrightarrow X \times X$$

is called a **Yang-Baxter map** if it satisfies the (quantum) **Yang-Baxter equation** (Yang 1967, Baxter 1972)

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$$

Both sides act on $X \times X \times X$. The indices of R_{ij} specify on which factors the map R acts. In general, R will depend on parameters that change with these indices.

Solutions of the Yang-Baxter equation play a crucial role in 2-dimensional quantum integrable models and exactly solvable models of statistical mechanics. Here mostly:
$R : V \otimes V \longrightarrow V \otimes V$, or $R : V \oplus V \longrightarrow V \oplus V$, V vector space.

A simple example: particles in one dimension

Classical Mechanics. Scattering of two particles on the real line, with masses m_1, m_2, incoming velocities v_1, v_2, outgoing velocities v_1', v_2'. They are subject to

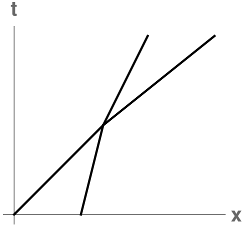
- momentum conservation: $m_1\, v_1 + m_2\, v_2 = m_1\, v_1' + m_2\, v_2'$
- energy conservation: $\frac{1}{2} m_1\, v_1^2 + \frac{1}{2} m_2\, v_2^2 = \frac{1}{2} m_1\, {v_1'}^2 + \frac{1}{2} m_2\, {v_2'}^2$

Solving for the outgoing velocities:

$$(v_1', v_2') = (v_1, v_2)\, R(m_1, m_2)$$

with

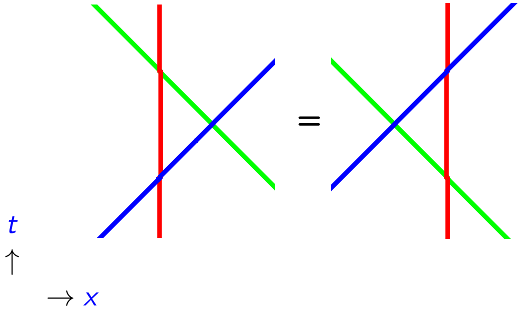
$$R(m_1, m_2) = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2\, m_1}{m_1 + m_2} \\ \frac{2\, m_2}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$



(see, e.g., *T.E. Kouloukas* 2017)

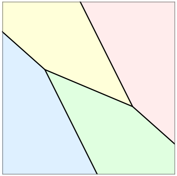
This matrix is an R-matrix, corresponding to a linear map solution of the Yang-Baxter equation. *We'll meet it again later on.*

Yang-Baxter relation in 3-particle scattering, *schematically*:



Matrix KdV Yang-Baxter map

Normalized ($\mathrm{tr}(K\hat{u})=1$) values of u along boundary segments: $u_{1,\mathrm{in}}$, $u_{2,\mathrm{in}}$ and $u_{1,\mathrm{out}}$, $u_{2,\mathrm{out}}$ for $N=2$. Writing



$$u_{j,\mathrm{in}} = \frac{\xi_{j,\mathrm{in}} \otimes \eta_{j,\mathrm{in}}}{\eta_{j,\mathrm{in}} K \xi_{j,\mathrm{in}}} \, , \qquad u_{j,\mathrm{out}} = \frac{\xi_{j,\mathrm{out}} \otimes \eta_{j,\mathrm{out}}}{\eta_{j,\mathrm{out}} K \xi_{j,\mathrm{out}}}$$

determines $\xi_{1,\mathrm{in/out}}$ and $\eta_{1,\mathrm{in/out}}$ up to scalings.

$$\xi_{1,\mathrm{out}} = \alpha_{\mathrm{in}}^{-1/2} \Big(1_m - \frac{2p_2}{p_2-p_1} \, \frac{\xi_{2,\mathrm{in}} \otimes \eta_{2,\mathrm{in}}}{\eta_{2,\mathrm{in}} K \xi_{2,\mathrm{in}}} K \Big) \xi_{1,\mathrm{in}}$$

$$\xi_{2,\mathrm{out}} = \alpha_{\mathrm{in}}^{-1/2} \Big(1_m - \frac{2p_1}{p_1-p_2} \, \frac{\xi_{1,\mathrm{in}} \otimes \eta_{1,\mathrm{in}}}{\eta_{1,\mathrm{in}} K \xi_{1,\mathrm{in}}} K \Big) \xi_{2,\mathrm{in}}$$

$$\eta_{1,\mathrm{out}} = \alpha_{\mathrm{in}}^{-1/2} \eta_{1,\mathrm{in}} \Big(1_n - \frac{2p_2}{p_2-p_1} \, K \, \frac{\xi_{2,\mathrm{in}} \otimes \eta_{2,\mathrm{in}}}{\eta_{2,\mathrm{in}} K \xi_{2,\mathrm{in}}} \Big)$$

$$\eta_{2,\mathrm{out}} = \alpha_{\mathrm{in}}^{-1/2} \eta_{2,\mathrm{in}} \Big(1_n - \frac{2p_1}{p_1-p_2} \, K \, \frac{\xi_{1,\mathrm{in}} \otimes \eta_{1,\mathrm{in}}}{\eta_{1,\mathrm{in}} K \xi_{1,\mathrm{in}}} \Big)$$

with $\alpha_{\mathrm{in}} = 1 + \frac{4p_1p_2}{(p_1-p_2)^2} \, \frac{\eta_{1,\mathrm{in}} K \xi_{2,\mathrm{in}} \, \eta_{2,\mathrm{in}} K \xi_{1,\mathrm{in}}}{\eta_{1,\mathrm{in}} K \xi_{1,\mathrm{in}} \, \eta_{2,\mathrm{in}} K \xi_{2,\mathrm{in}}}$.

If $K=I$: *A. Veselov* and *V.M. Goncharenko* (2003, 2004).

The vector KdV R-matrix

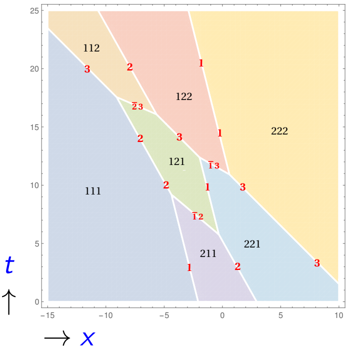
In this case ($n = 1$) we have a **linear** map

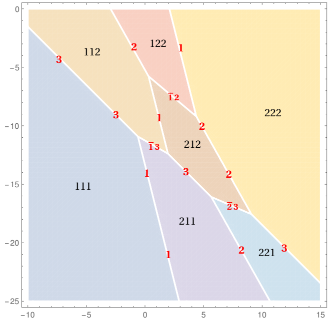
$$(u_{1,\mathrm{out}}, u_{2,\mathrm{out}}) = (u_{1,\mathrm{in}}, u_{2,\mathrm{in}})\, R(p_1, p_2)$$

$$R(p_i, p_j) = \begin{pmatrix} \frac{p_i - p_j}{p_i + p_j} & \frac{2p_i}{p_i + p_j} \\ \frac{2p_j}{p_i + p_j} & \frac{p_j - p_i}{p_i + p_j} \end{pmatrix}$$

Setting $p_i = m_i$ (masses), this is exactly the R-matrix governing the scattering of classical point particles in one dimension !
But the physical picture is different.

Why YB holds





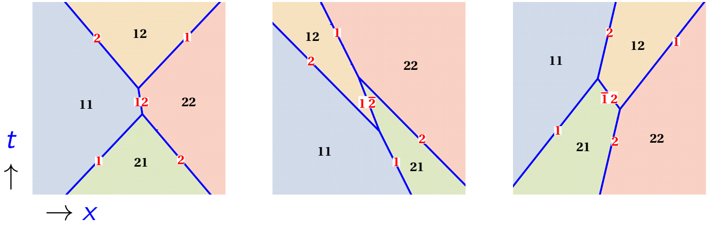
Tropical limit graphs of a **3-soliton solution** at a negative, respectively positive value of the next KdV hierarchy variable.
The polarizations do not depend on the variables!
Hence in both cases initial and final polarizations are the same.
Therefore the Yang-Baxter equation holds.

Matrix Boussinesq equation

Potential version ($u = 2\phi_x$):

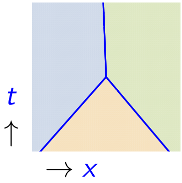
$$\phi_{tt} - 4\beta\phi_{xx} + \frac{1}{3}\phi_{xxxx} + 2\left(\phi_x K \phi_x\right)_x - 2\left(\phi_x K \phi_t - \phi_t K \phi_x\right) = 0$$

This is also a reduction of potential KP, but after a transformation that introduces the β term. We assume $\beta > 0$.



These are tropical limit graphs of 2-soliton solutions. Again, there is a Yang-Baxter map at work. But it is known that there is more.

Inelastic collision

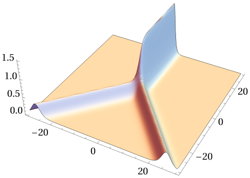


This is the most elementary binary tree.

Analog in case of KP: *Miles resonance*.

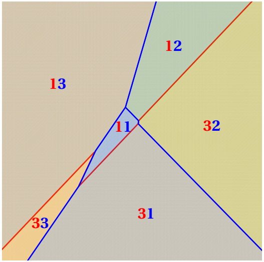
There is also the reverse process.

Not Yang-Baxter cases …



Scalar and vector Boussinesq do **not** admit regular solutions with more complicated trees.

Only "degenerate" cases, like the one shown on the right.



Solitons via binary Darboux ... and a trick

In the Boussinesq case, the linear system possesses solutions of the form

$$\theta = \sum_a \theta_a \, e^{\vartheta(P_a)} , \qquad \vartheta(P) = P\,x + P^2\,t ,$$

where

$$\boxed{P_a^3 = 3\beta\, P_a + C}$$

We only consider the case where C and P_a are diagonal. Parametrizing the entries of C as

$$c_i = 2\beta^{3/2} \frac{1 - 45\lambda_i^2 + 135\lambda_i^4 - 27\lambda_i^6}{(1+3\lambda_i^2)^3}$$

the roots can nicely be expressed as

$$p_{i,1} = -\frac{\sqrt{\beta}\,(1+6\lambda_i - 3\lambda_i^2)}{1+3\lambda_i^2} , \quad p_{i,2} = -\frac{\sqrt{\beta}\,(1-6\lambda_i - 3\lambda_i^2)}{1+3\lambda_i^2} ,$$

$$p_{i,3} = \frac{2\sqrt{\beta}\,(1-3\lambda_i^2)}{1+3\lambda_i^2} .$$

Correspondingly for adjoint linear system.

"Pure" soliton solutions

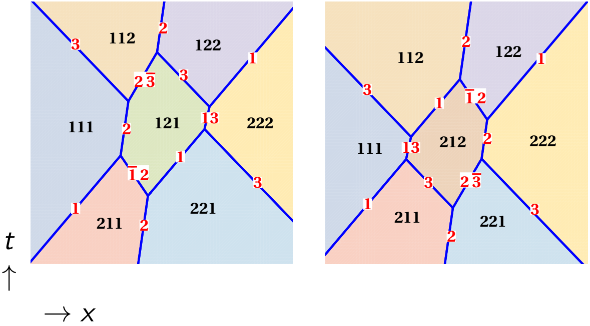
... essentially exclude inelastic collisions and splittings. Then Yang-Baxter holds along a tropical limit graph.

In the vector case, the corresponding R matrix reads

$$R(\lambda_i,\lambda_j) = \begin{pmatrix} \frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\frac{1-\lambda_i-\lambda_j-3\lambda_i\lambda_j}{1-\lambda_i+\lambda_j+3\lambda_i\lambda_j} & \frac{2\lambda_i}{\lambda_i+\lambda_j}\frac{1+3\lambda_j^2}{1-\lambda_i+\lambda_j+3\lambda_i\lambda_j} \\ \frac{2\lambda_j}{\lambda_i+\lambda_j}\frac{1+3\lambda_i^2}{1-\lambda_i+\lambda_j+3\lambda_i\lambda_j} & \frac{\lambda_j-\lambda_i}{\lambda_i+\lambda_j}\frac{1+\lambda_i+\lambda_j-3\lambda_i\lambda_j}{1-\lambda_i+\lambda_j+3\lambda_i\lambda_j} \end{pmatrix}$$

The KdV and the Boussinesq YB maps are reductions of a nonlinear YB map obtained for KP.

Why YB holds (for “pure” solitons)



Tropical limit graphs of 3-soliton solutions of a matrix Boussinesq equation. The graphs correspond to large negative, respectively positive values of the next hierarchy variable
($\vartheta(P) = P\,x + P^2\,t + P^4\,s$).

Matrix potential KP equation

Let $u = 2\phi_x$. We consider the $m \times n$ **matrix** potential KP equation

$$\boxed{4\phi_{xt} - \phi_{xxxx} - 3\phi_{yy} - 6(\phi_x K \phi_x)_x + 6(\phi_x K \phi_y - \phi_y K \phi_x) = 0}$$

Soliton solutions (via bDT with $\phi_0 = 0$)

$$\theta = \sum_{a=1}^{A} \theta_a \, e^{\vartheta(P_a)} \qquad \chi = \sum_{j=1}^{M} e^{-\vartheta(Q_j)} \, \chi_j$$

where P_a and Q_j are constant $N \times N$ matrices, θ_a, χ_j are constant $m \times N$, respectively $N \times n$ matrices, and

$$\vartheta(P) = x\,P + y\,P^2 + t\,P^3 \quad + s\,P^4 + \cdots$$

If, for all a, j, the matrices P_a and Q_j have no common eigenvalue, there are unique solutions W_{ja} of the Sylvester equations

$$Q_j W_{ja} - W_{ja} P_a = \chi_j K \theta_a \qquad a = 1, \ldots, A, \quad j = 1, \ldots, M\,.$$

Then the equations for Ω are solved by

$$\Omega = \Omega_0 + \sum_{a=1}^{A} \sum_{j=1}^{M} e^{-\vartheta(Q_j)} \, W_{ja} \, e^{\vartheta(P_a)}$$

and $\phi = -\theta\,\Omega^{-1}\chi$ yields a soliton solution if Ω^{-1} exists.

Pure soliton solutions

This is the subclass of soliton solutions with $A = M = 1$ and $P_1 = \mathrm{diag}(p_{1,1}, \ldots, p_{N,1}) = \mathrm{diag}(p_1, \ldots, p_N)$ and $Q_1 = \mathrm{diag}(p_{1,2}, \ldots, p_{N,2}) = \mathrm{diag}(q_1, \ldots, q_N)$. Then

$$\phi = \frac{F}{\tau} \qquad \tau = \sum_{I \in \{1,2\}^N} \mu_I \, e^{\vartheta_I} \,, \quad F = \sum_{I \in \{1,2\}^N} M_I \, e^{\vartheta_I}$$

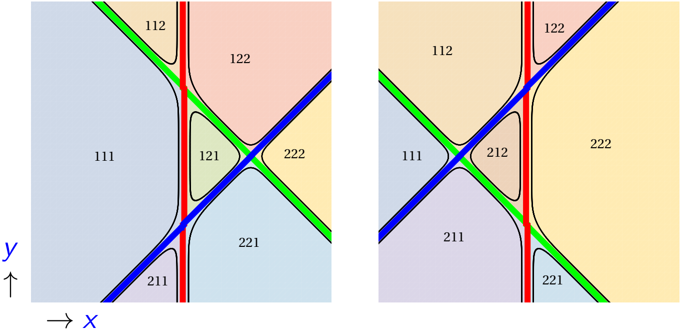
with constants μ_I (assumed > 0), constant $m \times n$ matrices M_I, and

$$\vartheta_I = \sum_{i=1}^{N} \vartheta(p_{i,a_i}) \qquad \text{if} \quad I = (a_1, \ldots, a_N) \in \{1,2\}^N$$

The tropical limit graph of a solution is defined as the tropical limit of τ.

- The matrix KdV solitons correspond to solutions from this class (via $q_i = -p_i$).
- The YB map generalizes that of KdV (and Boussinesq).

Why YB holds



These are contour plots in the *xy*-plane of a **3-soliton solution** at negative, respectively positive time t.
The polarizations do not depend on the independent variables.
Hence in both cases initial and finial polarizations are the same.
Therefore the Yang-Baxter equation holds.

Another subclass of KP solitons and the pentagon equation

Now we set $N = 1$, so that $P_a = p_a$ and $Q_j = q_j$. The bDT, with vanishing seed, then yields

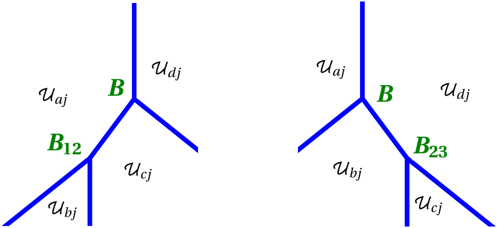
$$\phi = \frac{1}{\tau} \sum_{a=1}^{A} \sum_{i=1}^{M} \phi_{ai}\, \tau_{ai}$$

$$\tau = \sum_{a=1}^{A} \sum_{i=1}^{M} \tau_{ai}\,, \qquad \tau_{ai} = \mu_{ai}\, e^{\vartheta_{ai}}$$

$$\mu_{ai} = \frac{\chi_i K \theta_a}{p_a - q_i}\,, \qquad \phi_{ai} = \frac{\theta_a\, \chi_i}{\mu_{ai}}\,, \qquad \vartheta_{ai} = \vartheta(p_a) - \vartheta(q_i)$$

If $A = 1$ or $M = 1$, the tropical limit graph is a **rooted (generically) binary tree**.

Binary trees



The figures show tropical limit graphs, in the xy-plane, of a solution with $A = 4$ and $M = 1$ at $t \ll 0$ and $t \gg 0$, respectively.

A binary operation $\boldsymbol{B}(\lambda): \; V \times V \longrightarrow V$, $(\xi, \eta) \mapsto \lambda\, \xi + (1-\lambda)\, \eta$, enters the stage, satisfying the **local tetragon equation**

$$\boldsymbol{B}_{12}\Big(\frac{p_a - p_b}{p_a - p_c}\Big) \circ \boldsymbol{B}\Big(\frac{p_a - p_c}{p_a - p_d}\Big) = \boldsymbol{B}_{23}\Big(\frac{p_b - p_c}{p_b - p_d}\Big) \circ \boldsymbol{B}\Big(\frac{p_a - p_b}{p_a - p_d}\Big)$$

acting to the left.

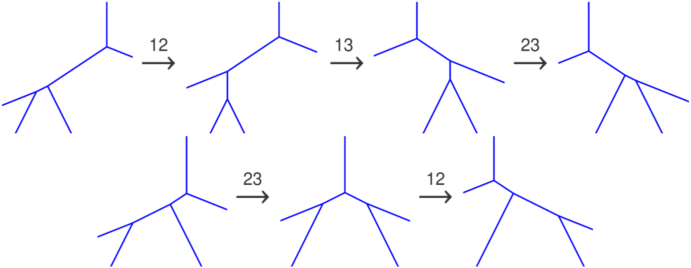
A pentagon relation

As a consequence,

$$\mathcal{T}\left(\frac{p_a - p_b}{p_a - p_c}, \frac{p_a - p_c}{p_a - p_d}\right) = \left(\frac{p_a - p_b}{p_a - p_d}, \frac{p_b - p_c}{p_b - p_d}\right)$$

then satisfies the **pentagon equation**

$$\mathcal{T}_{\mathbf{23}} \circ \mathcal{T}_{\mathbf{13}} \circ \mathcal{T}_{\mathbf{12}} = \mathcal{T}_{\mathbf{12}} \circ \mathcal{T}_{\mathbf{23}}$$



This is the pentagon Tamari lattice. The two chains correspond to a negative, resp. positive value of the next KP hierarchy variable.

Vector KP, and beyond trees

The **Yang-Baxter *R*-matrix and the binary map** jointly rule the distribution of polarizations over the limit graph. They satisfy a consistency condition,

$$
\begin{aligned}
& \boldsymbol{B}_{12}\Big(\frac{p_a-p_b}{p_a-p_c}\Big)\,\boldsymbol{R}\Big(\frac{p_a-p_c}{p_a-q_k},\frac{q_i-q_j}{p_a-q_k}\Big) \\
= \; & \boldsymbol{R}_{23}\Big(\frac{p_b-p_c}{p_b-q_j},\frac{q_i-q_j}{p_b-q_j}\Big)\,\boldsymbol{R}_{12}\Big(\frac{p_a-p_b}{p_a-q_j},\frac{q_i-q_j}{p_a-q_j}\Big)\,\boldsymbol{B}_{23}\Big(\frac{p_a-p_b}{p_a-p_c}\Big)
\end{aligned}
$$

Also this only applies to a subset of vector KP solutions!

To understand the **full** set of vector (and moreover matrix) KP solutions in such a way is still an open problem.

Conclusions

- For matrix **KdV**, in a space-time picture there are incoming ($t \to -\infty$) and outgoing ($t \to +\infty$) solitons, with attached polarizations. The corresponding map was found to satisfy the 2-simplex, i.e. Yang-Baxter equation (*Veselov*, *Goncharenko*). *The tropical limit yields a deeper understanding !*
- We addressed, more generally, the **matrix KP** equation (Dimakis, M-H: LMP (2018)), and explored its Boussinesq reduction (Dimakis, M-H, Chen: arXiv:1805.09711). Only a subclass of soliton solutions is ruled by YB !
- Beyond "pure" Boussinesq (and KP) solitons: more structure. For binary-tree-shaped solutions, a binary map solving a local **tetragon equation**, implying a **pentagon equation**, enters the stage (Dimakis, M-H: TMP (2018)). These equations belong to the family of **polygon equations**.
- Vector KP: parameter-dependent R-matrix $\curvearrowright$ new solutions of the 3-simplex (tetrahedron, Zamolodchikov) equation.

References

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Thanks for your attention !

and all the best to **Joseph**
for his post 70 period !