Exact solutions and upscaling for 1D hyperbolic flows in micro heterogeneous media

$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s\right)}{\partial x}=0$$

$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s,x\right)}{\partial x}=0$$

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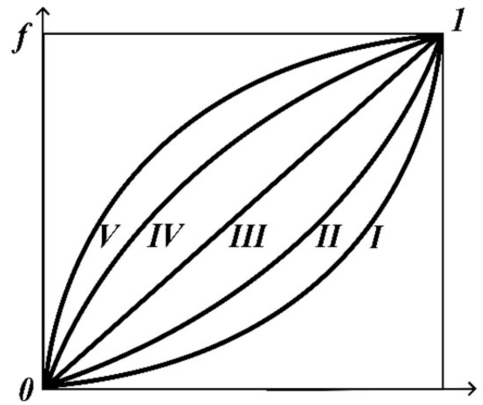


September 18, 2024

Mass conservation law with flux a function of density

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \qquad \rho v = f(\rho)$$

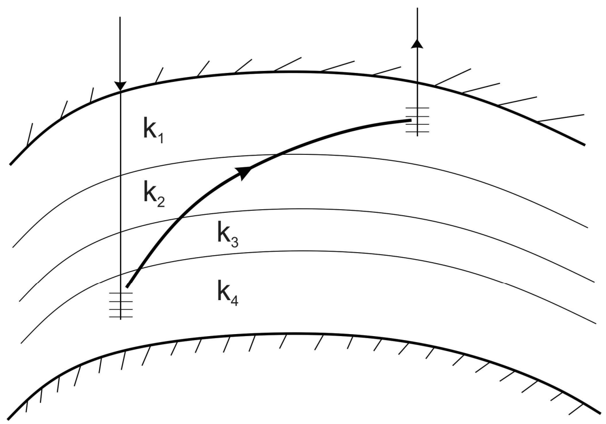
1D flow with density-dependent flux function

$$\rho v = f(\rho, x)$$

Scalar conservation law with density-dependent flux function

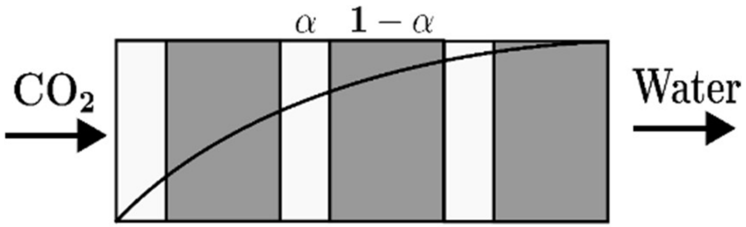
$$\frac{\partial s}{\partial t} + \frac{\partial f(s, x)}{\partial x} = 0$$

Upscaling in micro heterogeneous media



$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s,x\right)}{\partial x}=0$$

$$\frac{\partial S}{\partial t}+\frac{\partial F\left(S\right)}{\partial x}=0$$

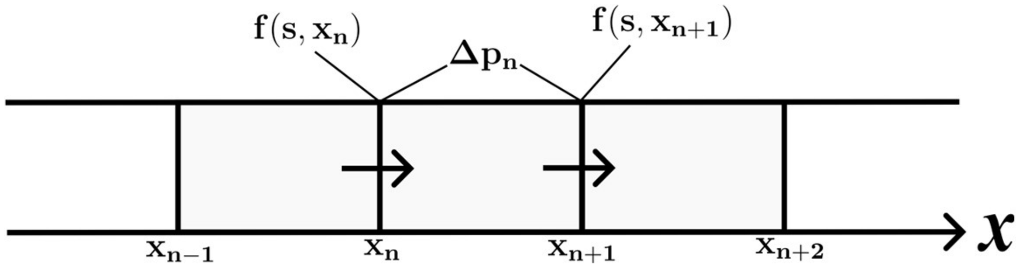


Composite core

Upscaling in numerical methods

$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s,x\right)}{\partial x}=0$$

$$\frac{\partial S}{\partial t}+\frac{\partial F\left(S\right)}{\partial x}=0$$

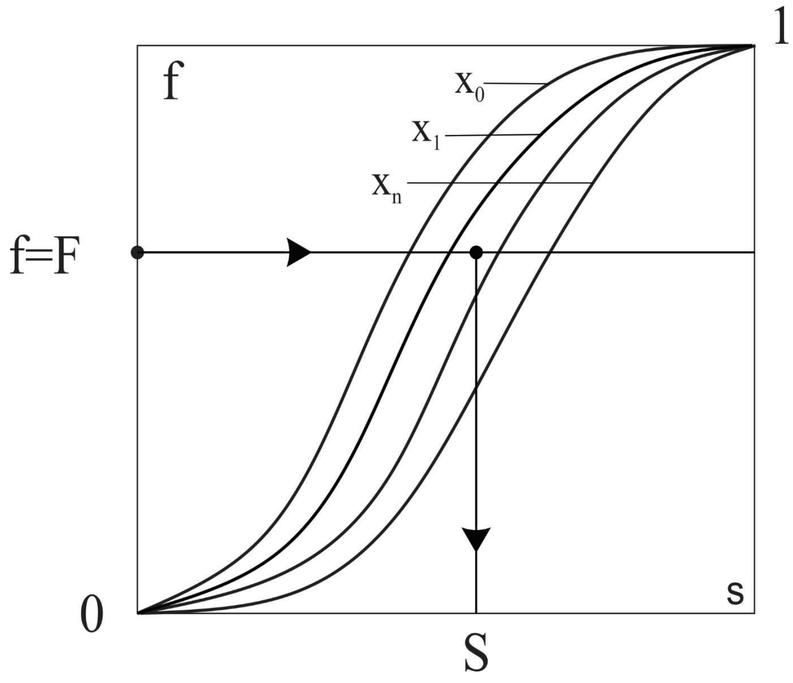


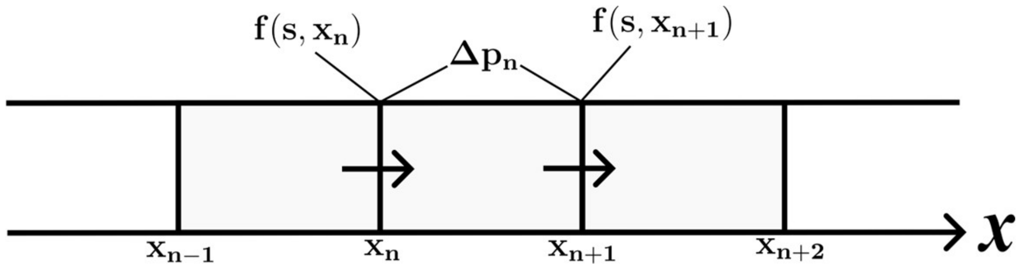
Numerical schema for characteristic finite-difference solution of 1D transport equation: how to transform from dense grid to coarse grid?

Schematic for upscaling

$$F = F(S), \; S = F^{-1}(f)$$

$$F^{-1}(f) = \int_0^{x_N} f^{-1}(f, y)\, dy$$





With application of the upscaling for each numerical cell $[x_{n}, x_{n+1}]$, the solution for microscale and upscaled systems, F(S) and f(s,x), coincide in all nodes x_n

Contents

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1. Reminder of Riemann solution for f=f(s)
2. For f=f(s,x), flux is a Riemann invariant
3. Exact solutions of Riemann problem: rarefaction, shock, transitional solutions
4. Exact solution for any problem with ICs and BCs
5. Upscaling

Extensions of the approach

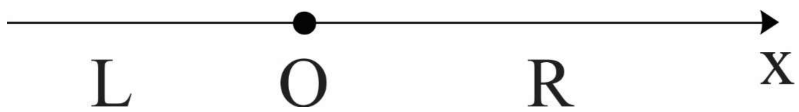
Conclusions

1. Riemann' problems for conservation law

$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} = 0$$

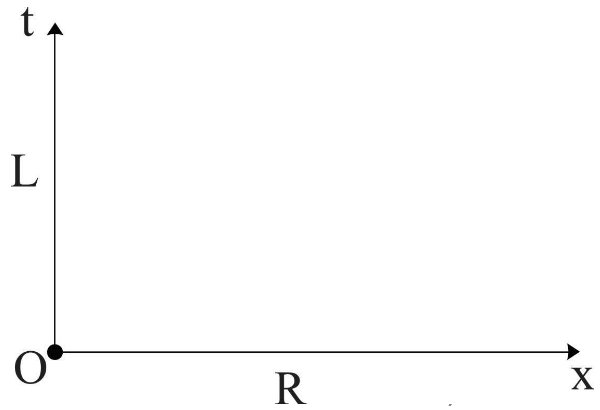
Cauchy' problem IC:

$$t = 0 \ : \ s(x,0) = \begin{cases} s_L, & x < 0 \\ s_R, & x > 0 \end{cases}$$



Initial-boundary value problem BC:

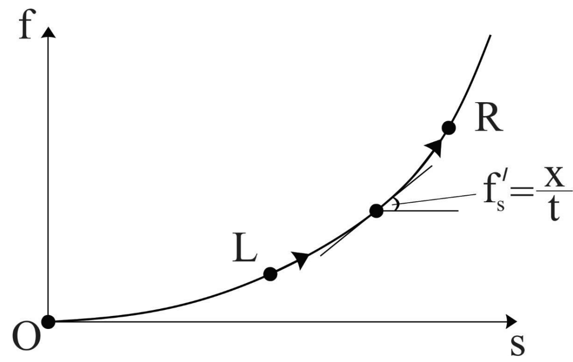
$$t = 0: s\left(x,0\right) = s_L, \quad x = 0, \ s\left(0,t\right) = s_R$$



$$\frac{\partial s}{\partial t}+\frac{\partial f(s)}{\partial x}=0$$

Self-similar solution:

$$x=0\ :\ s=s_L,\qquad t=0:\ s=s_R$$



$$s(x,t)=S(v),\ \ v=x/t,$$

$$(v=0):\ s=s_L,\qquad (v\to\infty):\ s=s_L$$

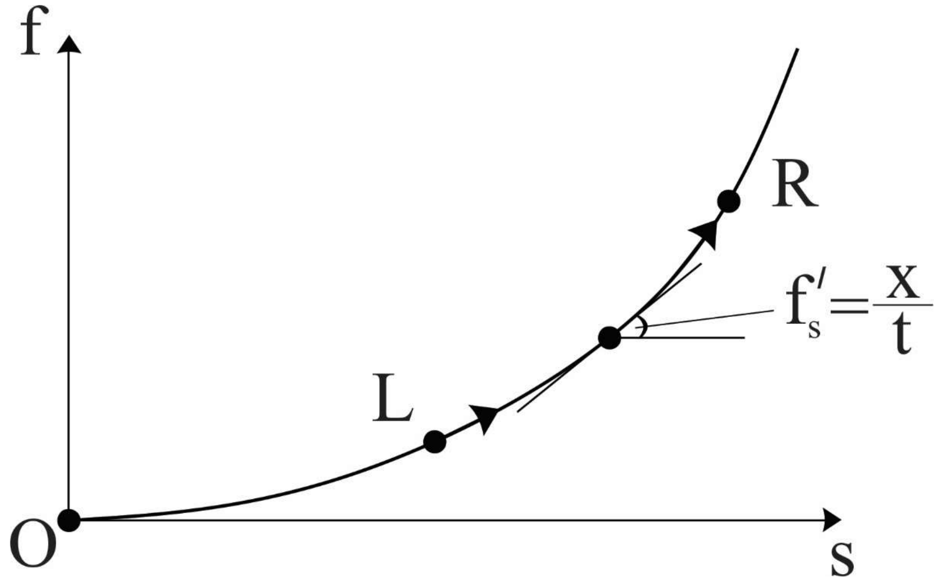
S(v)=const over x=vt, v is the velocity, so value S is transported with speed v

Two types of continuous solutions:

$$S(v) = const, \ \ v = f_s^{/}(s)$$

The Riemann solution consists of permanent state $s(v)=s_L$, rarefaction wave

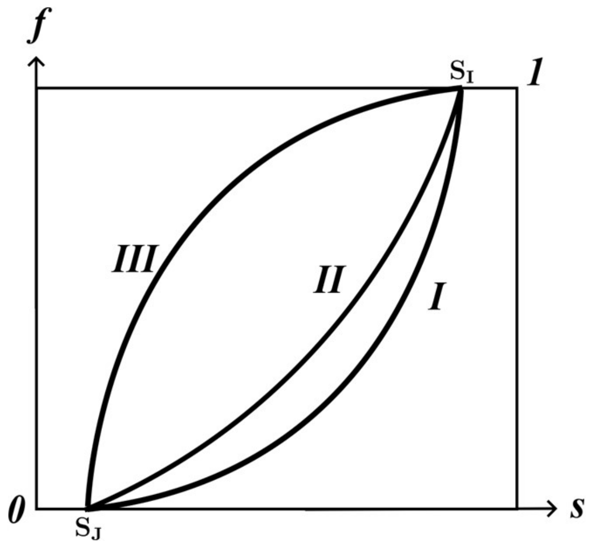
$$\frac{x}{t} = v = f_s^{/}(s)$$



and permanent state $s(v)=s_R$

This solution is continuous

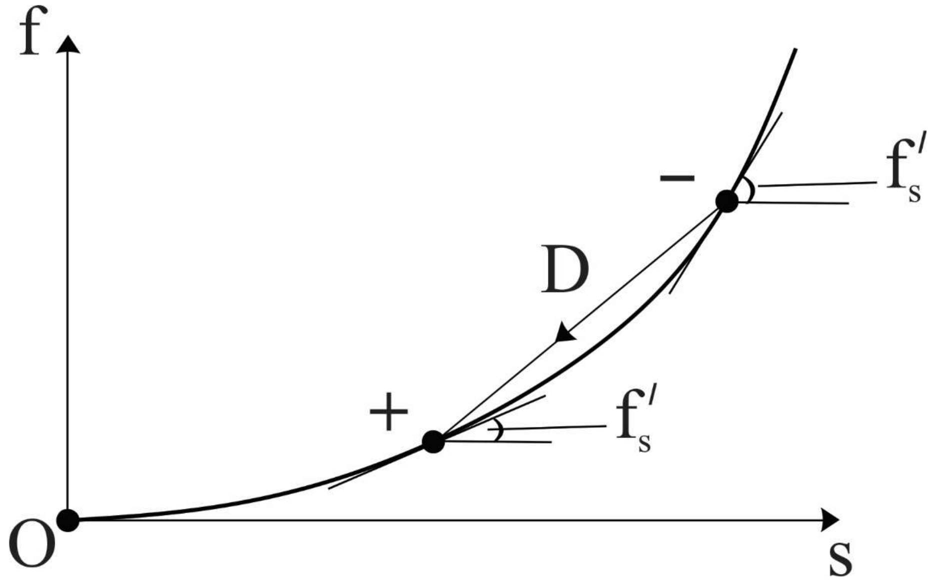
Exact solution for Riemann problem for convex f-f-function f=f(s) - I



$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} = 0$$

$$s(x,t) = \begin{cases} S_J, & 0 < \dfrac{x}{t} < f'(S_J) \\ \dfrac{x}{t} = \langle f \rangle'(s), & f'(S_J) < \dfrac{x}{t} < f'(S_I) \\ S_I, & f'(S_I) < \dfrac{x}{t} < \infty \end{cases}$$

In continuous solution s=s(v), speed v must increase from zero to infinity. If S_L (S^-) is less than S_R (S^+), v decreases, so there is no continuous solution.



We expand the space of admissible functions to discontinuous functions, which suffer shocks S^-->S^+ along fronts $x_f(t)$

A discontinuous solution of hyperbolic equation

$$\frac{\partial s}{\partial t}+\frac{\partial f(s)}{\partial x}=0$$

is admissible (stable)

if it is a limit of continuous solution of the equation with vanishing viscosity

$$\frac{\partial s}{\partial t}+\frac{\partial f(s)}{\partial x}=\varepsilon\frac{\partial^2 s}{\partial x^2}$$

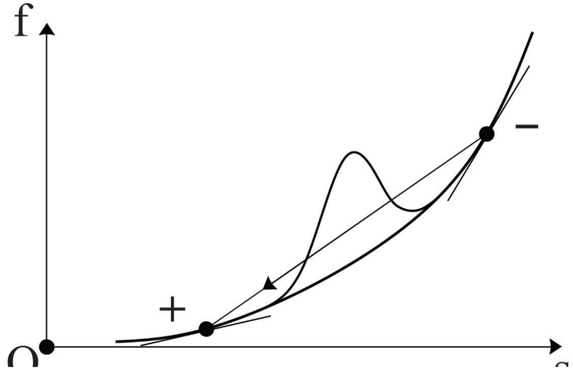
The admissibility conditions:
(i) Mass balance on the shock

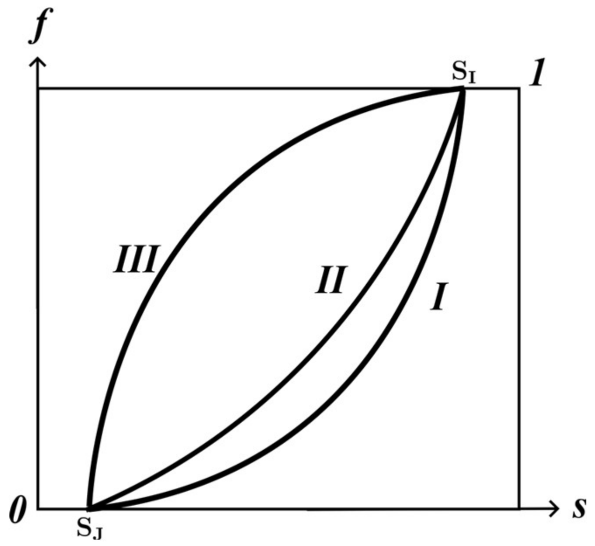
$$D=\frac{dx_f(t)}{dt},\quad f(s^-)-f(s^+)=D(s^- - s^+)$$

(ii) Shock stability with respect to linear perturbations (Lax)

$$f_s'(s^+)<D<f_s'(s^-)$$

Shock stability with respect to any perturbations (Oleinik)



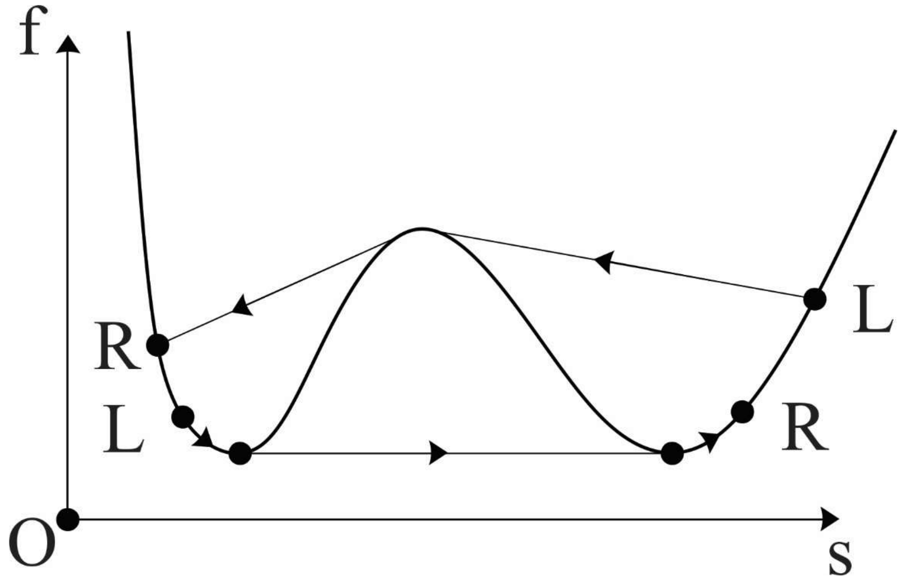


$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s\right)}{\partial x}=0$$

Exact solution for Riemann problem for concave f-f-function f=f(s) – III – is shock wave with volume balance of the front (Hugoniot condition)

$$s\left(x,t\right)=\begin{cases} s=S_J, & 0<\dfrac{x}{t}<D=\dfrac{1}{S_I-S_J} \\ s=S_I, & D<\dfrac{x}{t}<\infty \end{cases}$$

Riemann' self-similar solution L→R for hyperbolic equation



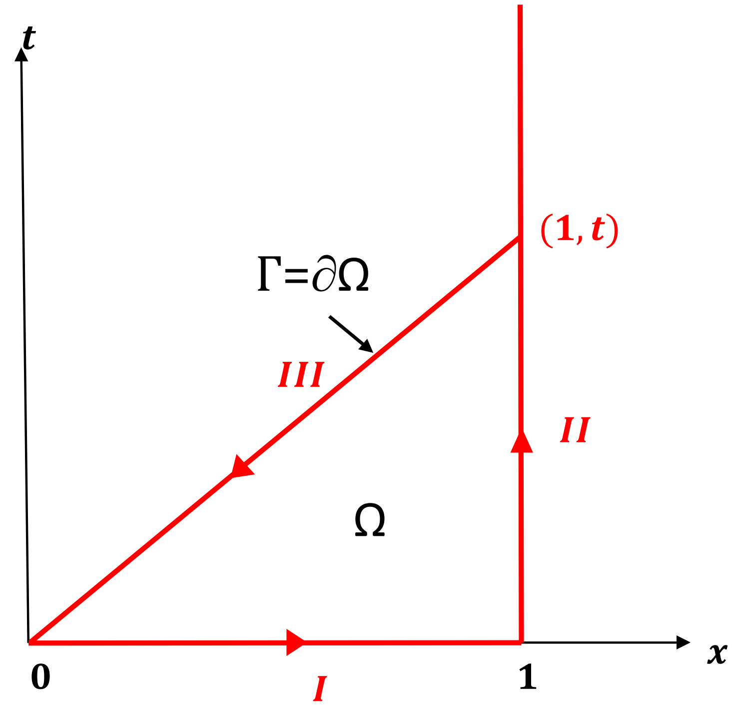
$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} = 0$$

Determining f(s) from lab f(s(1,t)), i.e. from f-data at x=1

$$\frac{\partial s}{\partial t}+\frac{\partial f(s)}{\partial x}=0$$

$$0=\iint_{\Omega}\left[\frac{\partial s}{\partial t}+\frac{\partial f(s)}{\partial x}\right]dxdt=\oint_{\Gamma} fdt-sdx$$

$$-S_I+\int_0^t f\left[s(1,y)\right]dy=f\left[s(1,t)\right]t-s(1,t)$$



Given f(s(1,t)), we calculate s(1,t) for all t>0

The inverse solution does not involve direct solution s(x,t) rather using its self-similarity alone

2. Analysis of microscale equation with f=f(s,x)

Multiplying by f'_s $\quad f'_s \frac{\partial s}{\partial t} + f'_s \frac{\partial f(s,x)}{\partial x} = 0 \qquad \frac{\partial f(s,x)}{\partial t} + f'_s \frac{\partial f(s,x)}{\partial x} = 0$

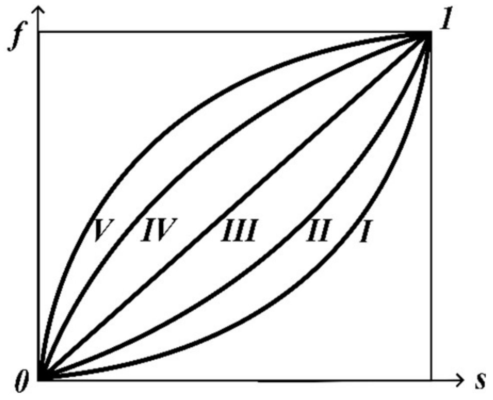
Characteristic's form:

$$\frac{dx}{dt} = f'_s, \ \frac{df}{dt} = 0, \qquad f = f(s,x), \ s = f^{-1}(f,x)$$

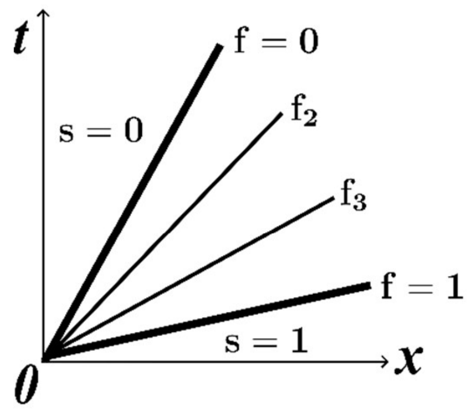
Implicit expression for f-characteristics $\quad t = \tau(x, f)$

$$\frac{dx}{dt} = f'_s\left(f^{-1}(f,x),x\right) \qquad t = \int_{x_0}^{x} \frac{dy}{f'_s\left(f^{-1}\left(f\left(s_0\left(x_0\right)\right),y\right),y\right)}$$

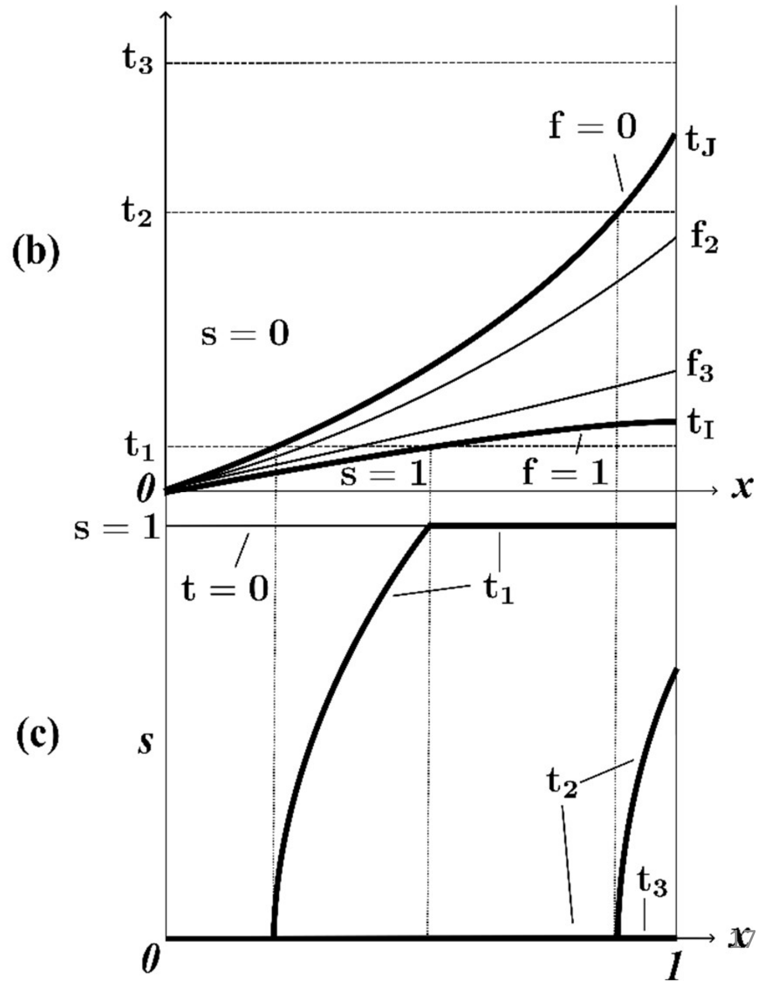
3. Continuous solution for convex FFF



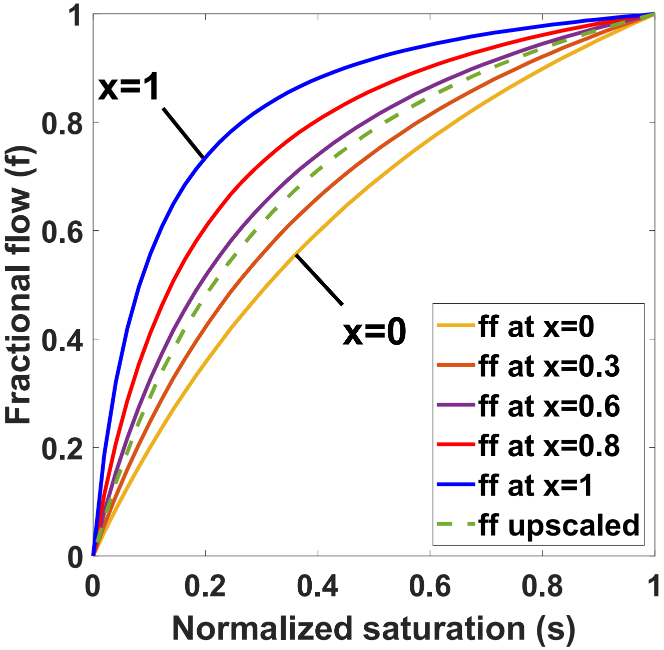
FFF curve II at *x=0*, and curve I – at *x=1*

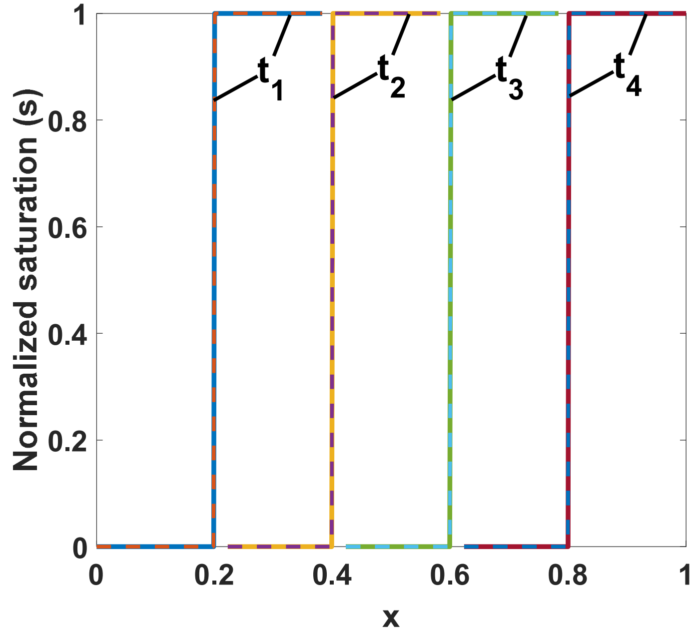


(a)

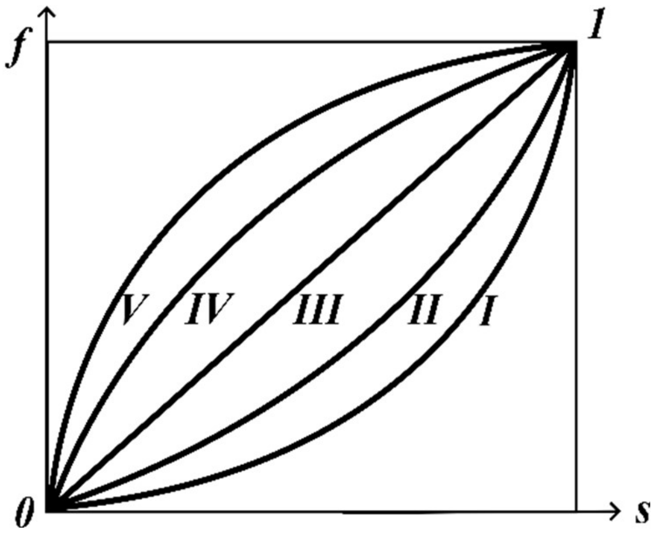


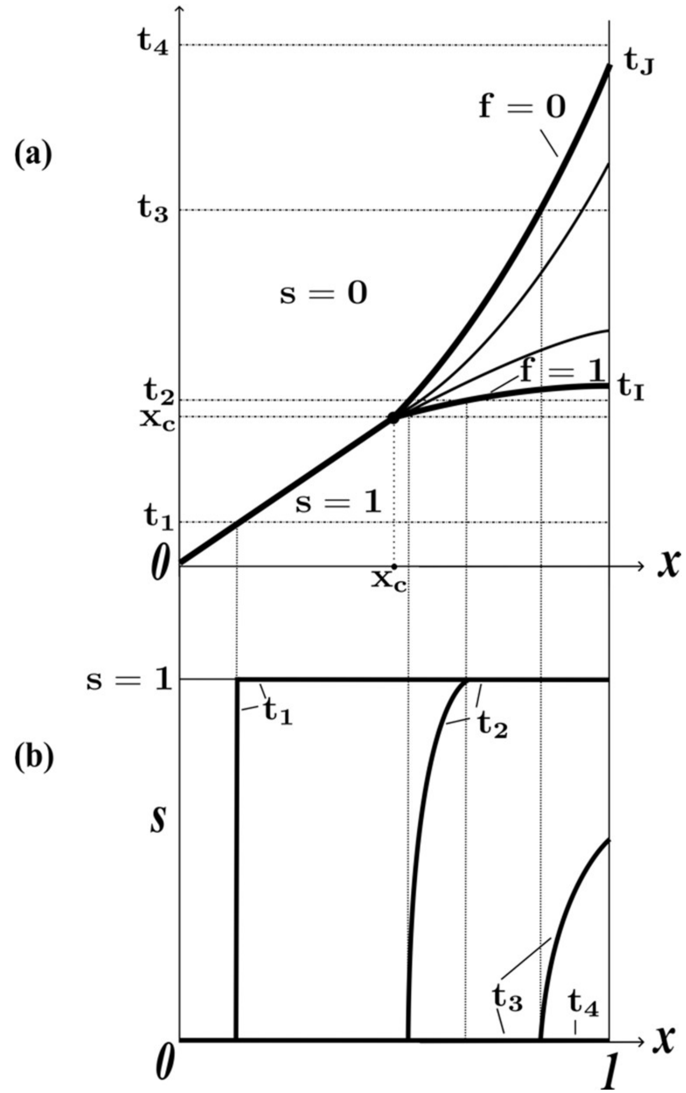
Shock wave solution for concave FFF





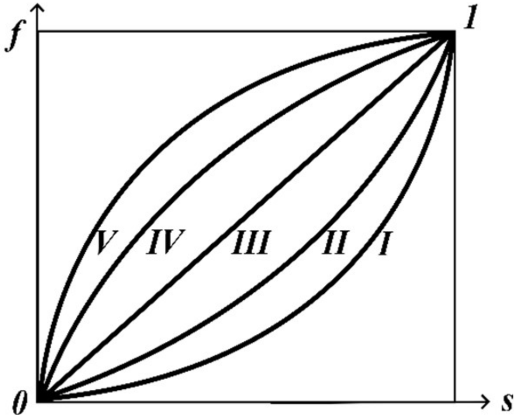
Transition from shock to continuous wave with FFF decreasing in x from concave to convex





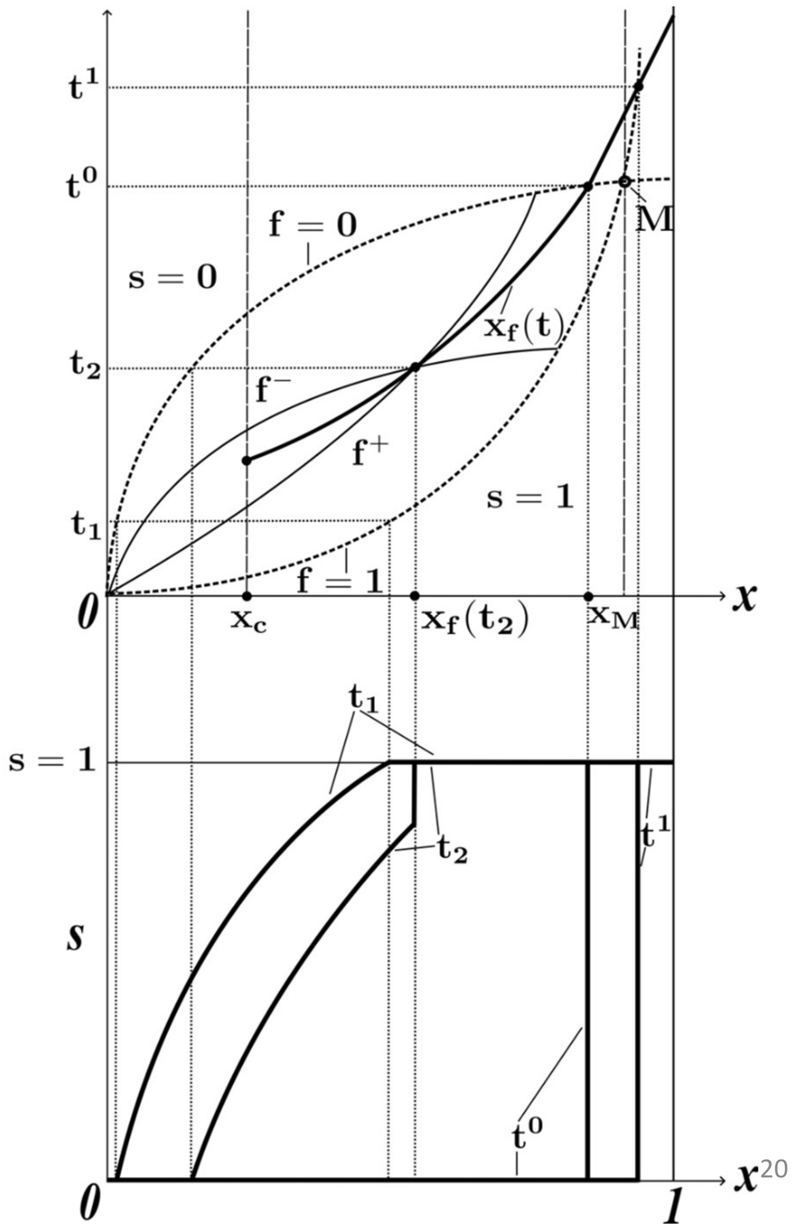
FFF V that is concave at $0<x<x_c$, straight line at $x=x_c$, and convex I at $x_c<x<1$

Transition from continuous wave to shock with FFF increasing in *x* from convex to concave



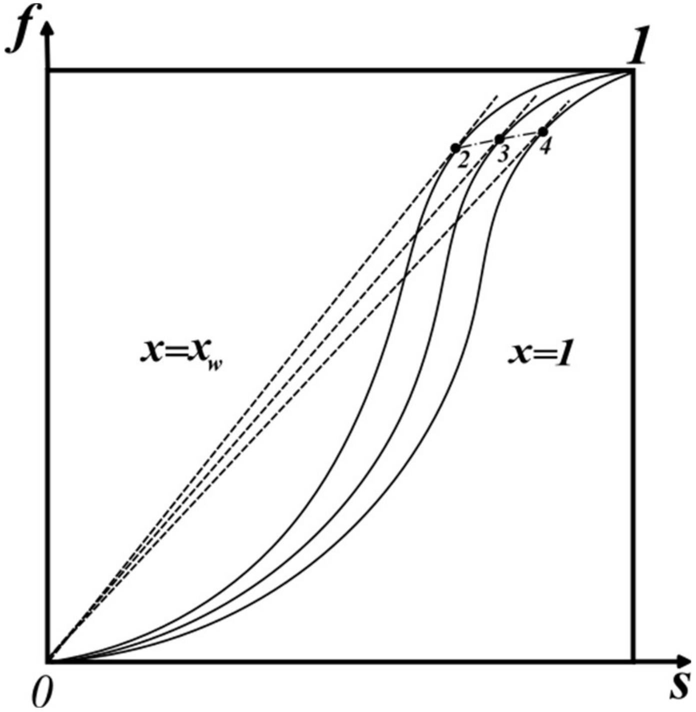
(a)

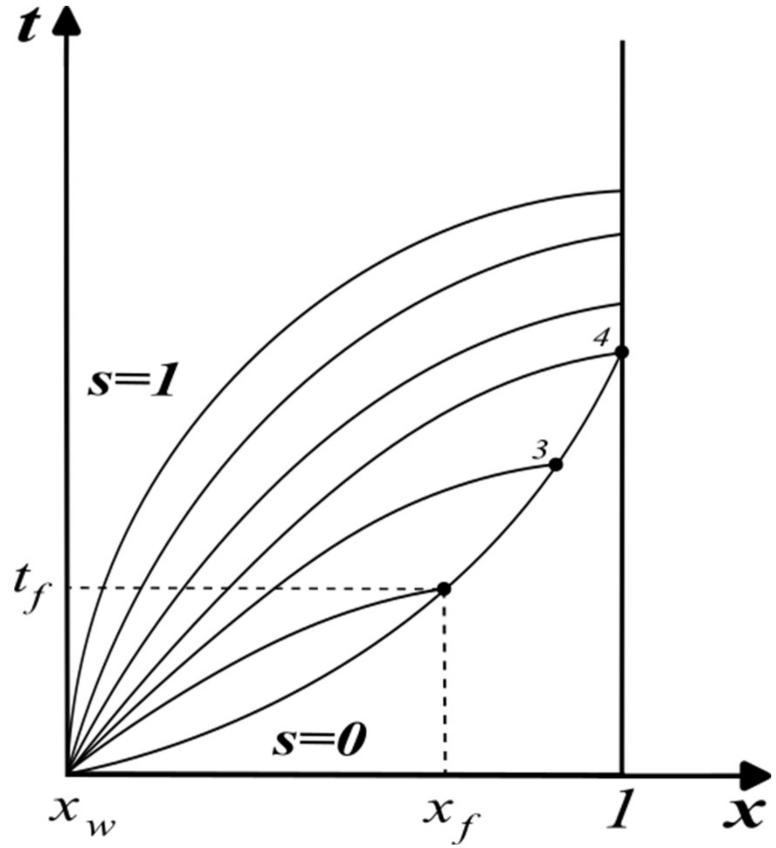
(b)



FFF I that is convex at $0<x<x_c$, straight line at $x=x_c$, and concave V at $x_c<x<1$

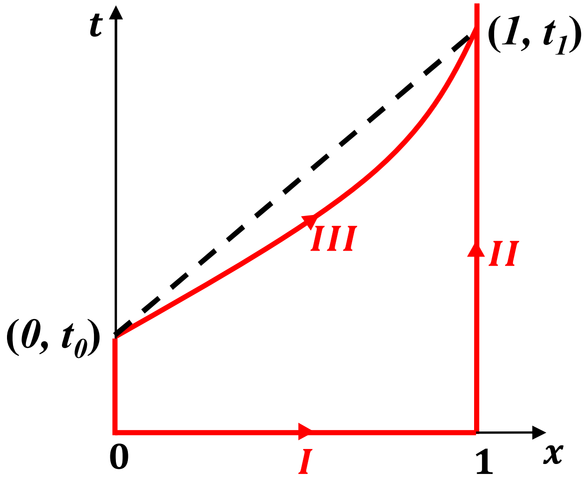
Riemann's solution for S-shaped FFF





4. Exact continuous solution for any initial-boundary value problem

$$t=0:\ s=s_0(x) \qquad\qquad x=0:\ f=f_0(t)$$



Trajectory of characteristic carrying flux f $\quad t=\tau(x,f)$

$$s(x,t)=\begin{cases} f^{-1}\left(f\left(s_0(x_0),x_0\right),x_0\right),\ t=\displaystyle\int_{x_0}^{x}\frac{dy}{f_s'\left(f^{-1}\left(f\left(s_0(x_0),x_0\right),y\right),y\right)}, & t<\tau\left(x,f_0(0)\right) \\ f^{-1}\left(f_0(t_0),x\right),\ t=t_0+\displaystyle\int_{0}^{x}\frac{dy}{f_s'\left(f^{-1}\left(f_0(t_0),y\right),y\right)}, & t>\tau\left(x,f_0(0)\right) \end{cases}$$

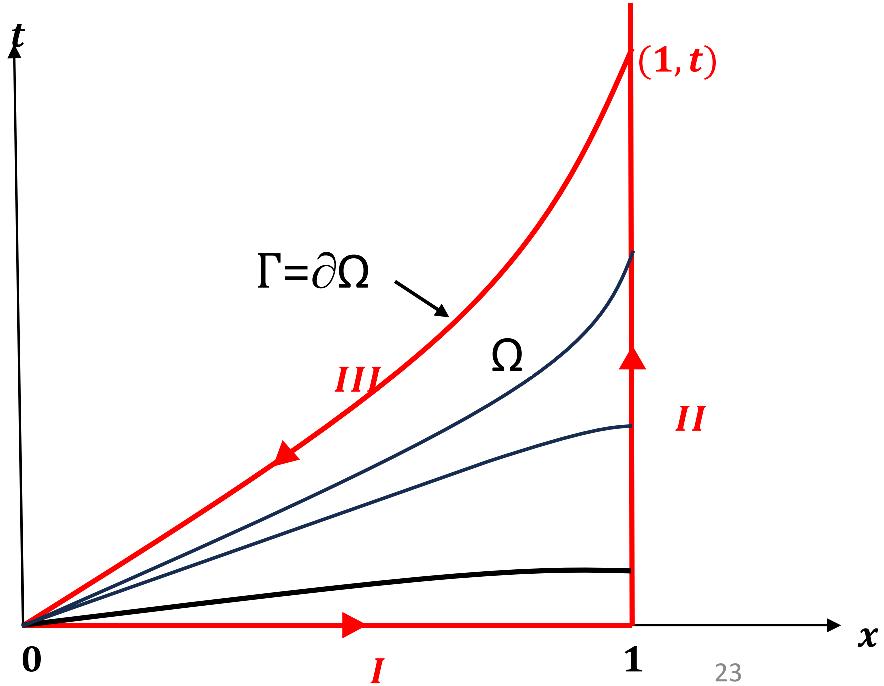
5. Upscaling of Riemann problem

Lab determining of fractional flow function f(s) from breakthrough water-cut history is a common method. It is valid for the case of micro-heterogeneous media f=f(s,x)?

In lab, upscaling must give the same values at the end of core x=1.

Applying Green's theorem:

$$\iint_{\Omega}\left[\frac{\partial s}{\partial t}+\frac{\partial f(s,x)}{\partial x}\right]dxdt=\oint_{\Gamma} fdt-sdx=0$$

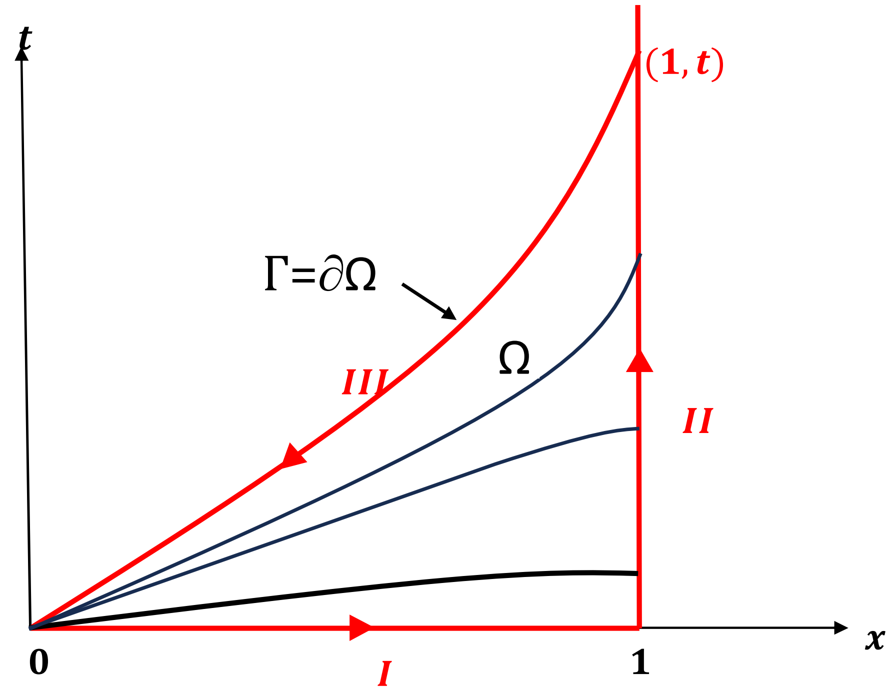


Integrals over the sides of curvilinear triangle:

$$I: \oint_{\Gamma} f dt - s dx = -S_I$$

$$II: \oint_{\Gamma} f dt - s dx = \int_0^t f\left[s(1,t),1\right] dt$$

$$III: \oint_{\Gamma} f dt - s dx = ft - \left\langle S(f,x)\right\rangle = ft - \left\langle f^{-1}(f,x)\right\rangle$$



Comparing with self-similar case

$$III: \oint_{\Gamma} F(S) dt - S dx = F\left(\frac{1}{t}\right) t - S\left(\frac{1}{t}\right)$$

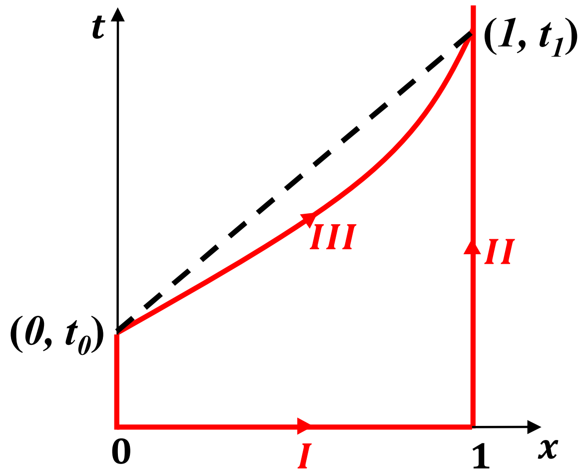
we obtain the upscaling formula

$$s(f) = \int_0^1 f^{-1}(f,x) dx$$

6. Upscaling of f(s,x) for any IC BC

Consider the case where $t > \tau(0, f_0(0))$ and domain Ω bounded by curvilinear rectangular $\Gamma = \partial\Omega$: $(0,0) \rightarrow (1,0) \rightarrow (1,t) \rightarrow (0,t_0) \rightarrow (0,0)$ where $f = const$ along the side $(0,t_0) \rightarrow (1,t)$

$$-\int_0^1 s_0(x,0)dx, \quad \int_0^t f_1(y)dy, \quad -f_1(t)t + \int_0^1 f^{-1}(f_1(t),x)dx, \; -\int_0^{t_0} f_0(y)dy$$



$$t = \tau(f_0(t_0), 1), \;\; f_1(t) = f_0(t_0)$$

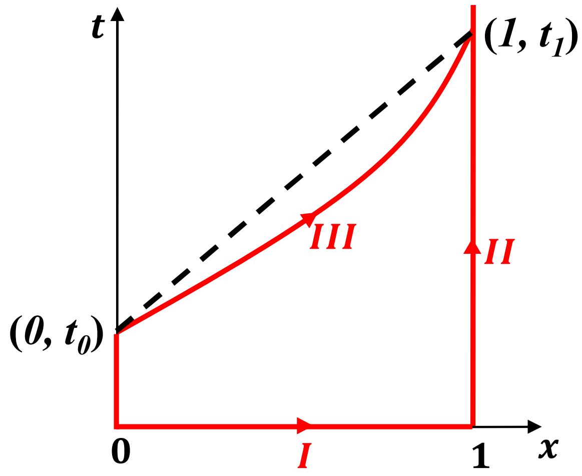
$$S(1,t) = F^{-1}(f_1(t)) = \int_0^1 f^{-1}(f_1(t), x)dx = \int_0^{t_0} f_0(y)dy - \int_0^t (f_1(t) - f_1(y))dy + \int_0^1 s_0(x,0)dx$$

Microscale

$$S(1,t) = F^{-1}(f_1(t)) = \int_0^1 f^{-1}(f_1(t),x)\,dx = \int_0^{t_0} f_0(y)\,dy - \int_0^t (f_1(t) - f_1(y))\,dy + \int_0^1 s_0(x,0)\,dx$$

Macroscale

$$f_0(t_0)t - \int_0^1 f^{-1}(f_0(t_0),y)\,dy = Ft - F^{-1}(F),\; F = f_0(t_0)$$

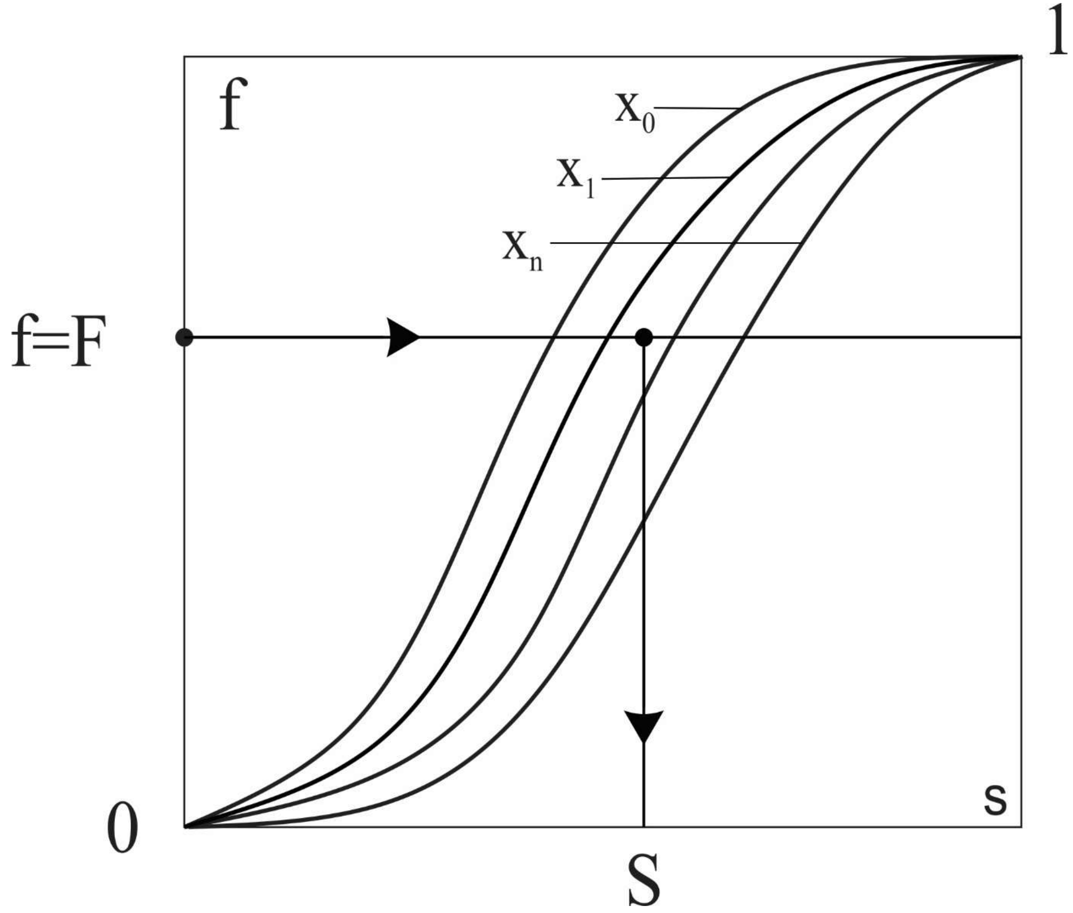


Upscaling formula:

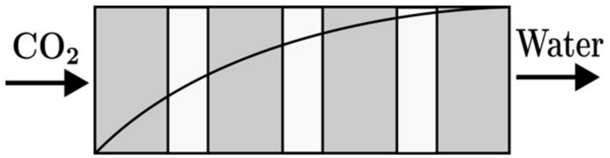
$$F^{-1}(f) = \int_0^1 f^{-1}(f,y)\,dy$$

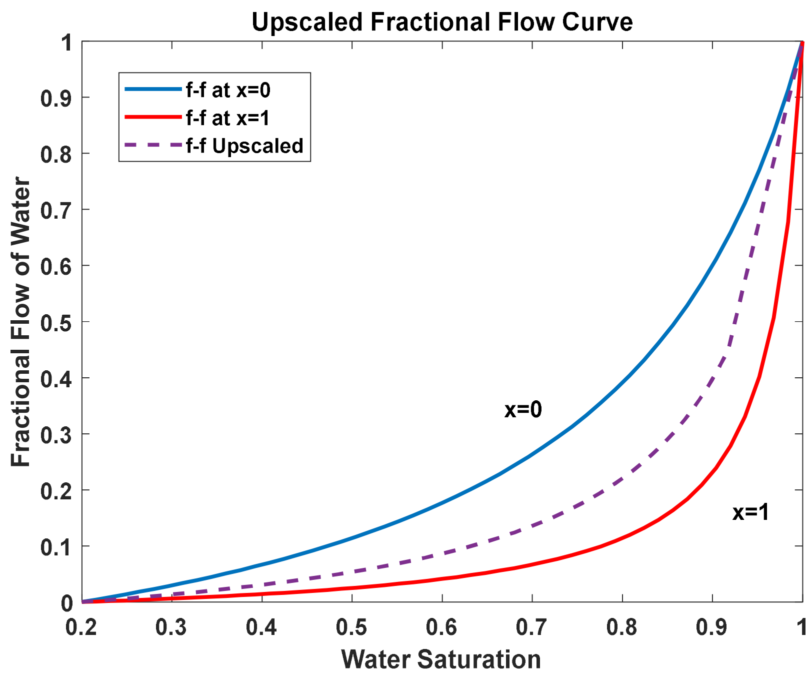
Schematic for upscaling

$$F^{-1}(f) = \int_{0}^{1} f^{-1}(f, y)\,dy$$



7. Upscaling of piecewise-constant periodical system (composite core) three periods





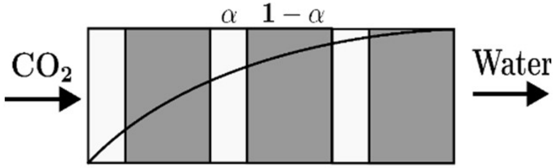
$$S(f) = \int_0^1 f^{-1}(f, x)\, dx$$

$$S(f) = \alpha s_1(f) + (1-\alpha) s_2(f)$$

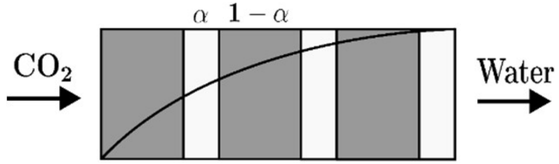
α – fraction of the rock with ff f_0 in the overall core
1-α – fraction of the rock with ff f_1 in the overall core
α=0.4

Flow in periodical two-piece (composite) porous media

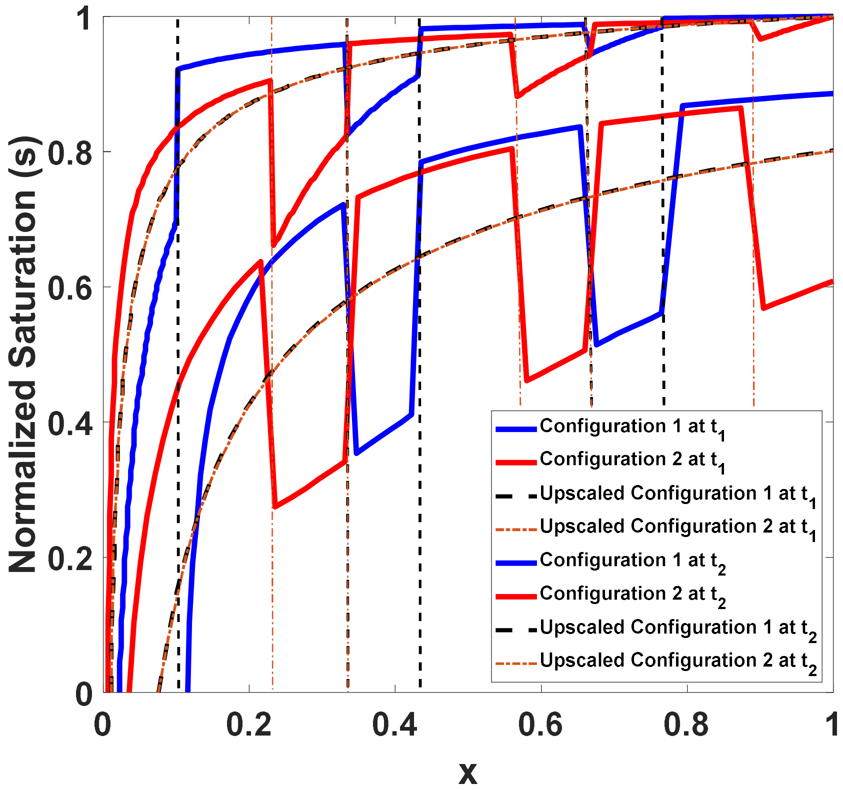
Blue core



Red core

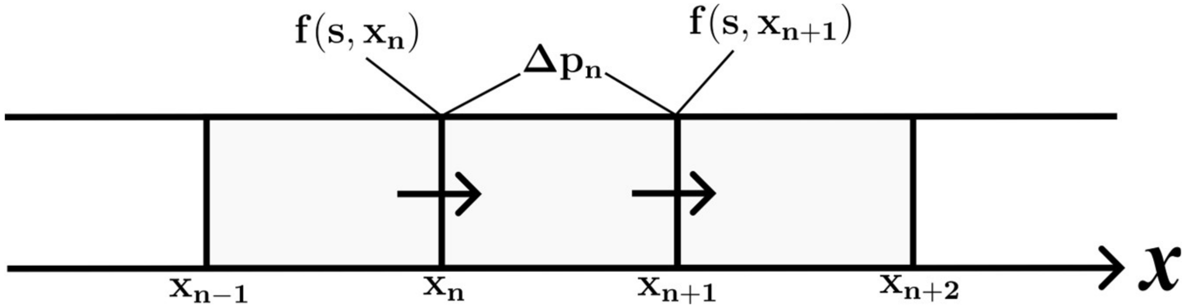


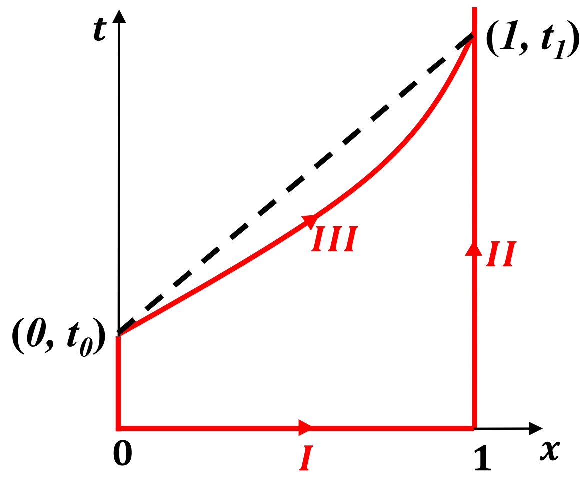
Microscale solutions are different for two cores.
They coincide at macro scale



8. Numerical

$$t=0:\ s=s_0(x)\quad x=0:\ f=f_0(t)$$





Numerical schema for characteristic finite-difference solution of 1D two-phase transport equation

$$f=F_n(s),\ s=f^{-1}{}_n(f),\ x\in[x_n,x_{n+1}],\ n=0,1\ldots N,\ x_0=0,\ x_N=1$$

$$f(s(x_n,t),x)=F_n(s(x_n,t))$$

Some extensions

Proposed Upscaling = exact solution at micro scale for f(s,x) + exact inverse solution at upper scale

Linear PDEs: exact solution by Green’s function and inverse problem for its integral equation

Scalar conservation laws

$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s,t\right)}{\partial x}=0$$

Time-dependent flux

$$\frac{\partial f\left(s,x\right)}{\partial t}+\frac{\partial s}{\partial x}=0$$

Space-dependent adsorption

$$\frac{\partial f\left(s,t\right)}{\partial t}+\frac{\partial s}{\partial x}=0$$

Time-dependent adsorption

“Multicomponent” flows

S is the density, f is the advective flux, c-concentration of an additive, cf is the advective flux of the additive, a – adsorption concentration

$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s,c\right)}{\partial x}=0, \qquad \frac{\partial\left(cs+a\left(c\right)\right)}{\partial t}+\frac{\partial\left(cf\right)}{\partial x}=0$$

$$d\varphi = fdt - sdf,\ \ \varphi=\int fdt - sdf,\ \ s\left(x,t\right)=S\left(x,\varphi\right),c\left(x,t\right)=C\left(x,\varphi\right)$$

$$\frac{\partial a\left(c\right)}{\partial \varphi}+\frac{\partial c}{\partial x}=0$$

The solution c(x,ϕ) contains shocks only if

$$t=0:\ c=0,\ \ x=0:\ c=1,\ \ a^{//}\left(c\right)<0$$

Conclusions

For any initial-boundary value problem of f(s,x), the flux is Riemann invariant; the characteristics allow for 1st integral yielding implicit formulae for the characteristics. First integrals for front trajectories are obtained by integration of differential mass balance form $f(s,x)dt-sdx$ over the closed contours in plane (x,t) that comprise two arriving characteristics f^- and f^+ and the intervals of axes x and t where the initial-boundary values are given.

Saturation S that corresponds to upscaled value $F=F(S)$ is an average in x of the "microscale" inverse function $s=f^{-1}(F,x)$.

The numerical solution obtained by an explicit finite difference method with advance over Δx for micro scale model, coincides with the solution for the large-scale system obtained by history-based upscaling, in the points on numerical cell boundaries x_0, x_1, …, x_n.

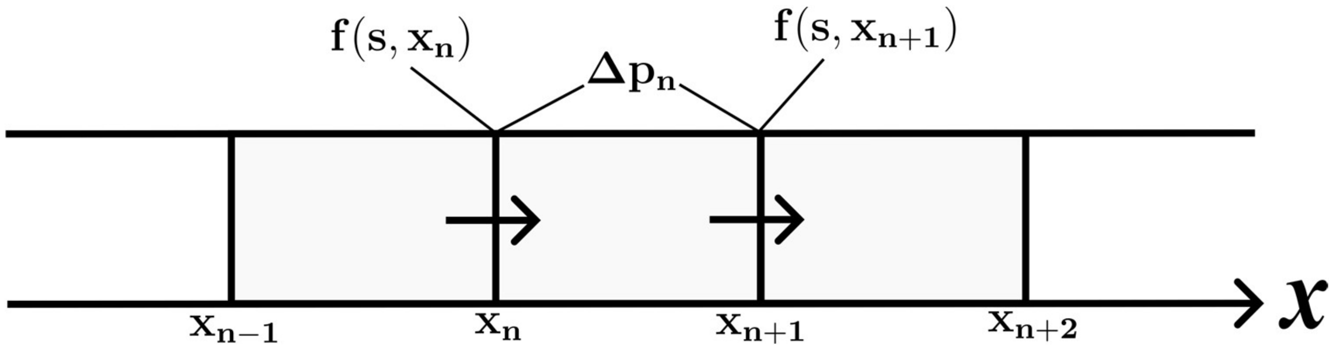
There are two challenges:

1 - Lab determining of fractional flow function f(s) from breakthrough water-cut history is a common method. It is valid for the case of micro-heterogeneous media f=f(s,x)?

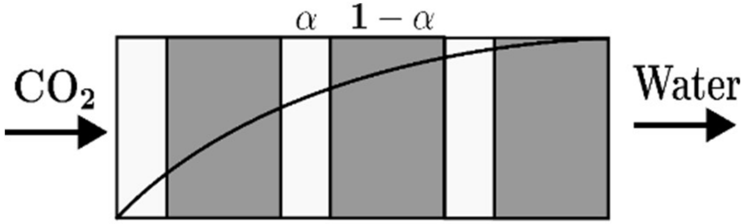
In lab, upscaling must give the same values at the end of core x=1. Upscaling of Riemann problem

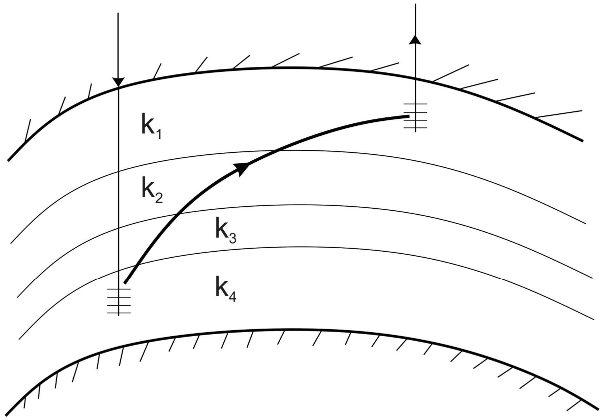
2 - Resolution of measurements are significantly higher than the minimum grid size of the numerical model, i.e. f=f(s,x). How to calculate f(s) that will present the same results at the course grid?

In numerical model, the same numerical finite-difference solution. Upscaling of initial-boundary value problem

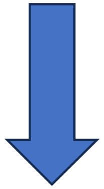


Upscaling in micro heterogeneous formations

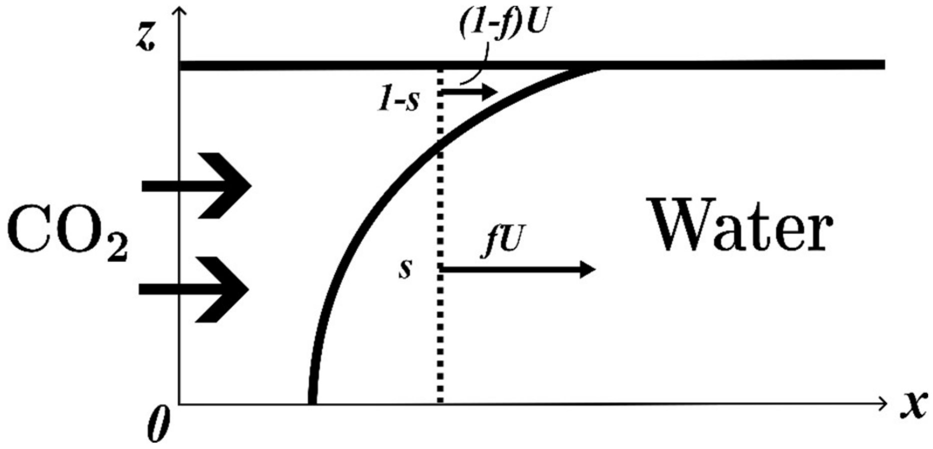




$$\frac{\partial s}{\partial t}+\frac{\partial f\left(s,x\right)}{\partial x}=0$$



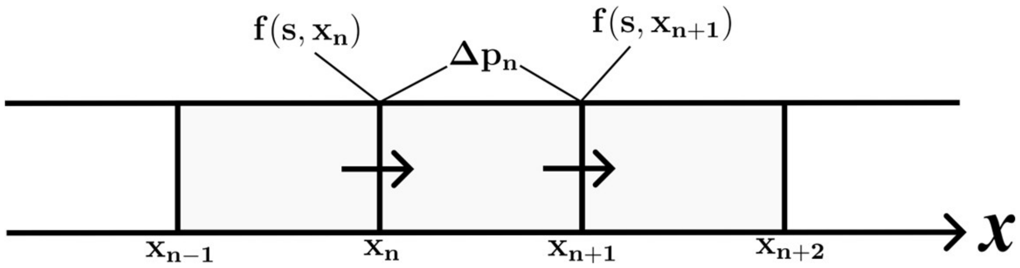
$$\frac{\partial S}{\partial t}+\frac{\partial F\left(S\right)}{\partial x}=0$$



Schematic for displacement of water by gas

$$\frac{\partial s}{\partial t} + \frac{\partial f\left(s, x\right)}{\partial x} = 0$$

$$\frac{\partial S}{\partial t} + \frac{\partial F\left(S\right)}{\partial x} = 0$$



Numerical schema for characteristic finite-difference solution of 1D transport equation

Discontinuous solutions – shocks, jumps:

$$S\left(D-0\right)=S^{-},\ S\left(D+0\right)=S^{+},\ \ v=D$$

Shock occurs near to v=D

$$S\left(v\right)=\begin{cases} S^{+}, & v>D \\ S^{-}, & v<D \end{cases}$$

Mass balance on the shock

$$D=\frac{f\left(S^{+}\right)-f\left(S^{-}\right)}{S^{+}-S^{-}}$$

Stability of solution with respect to small perturbations in linearised equation

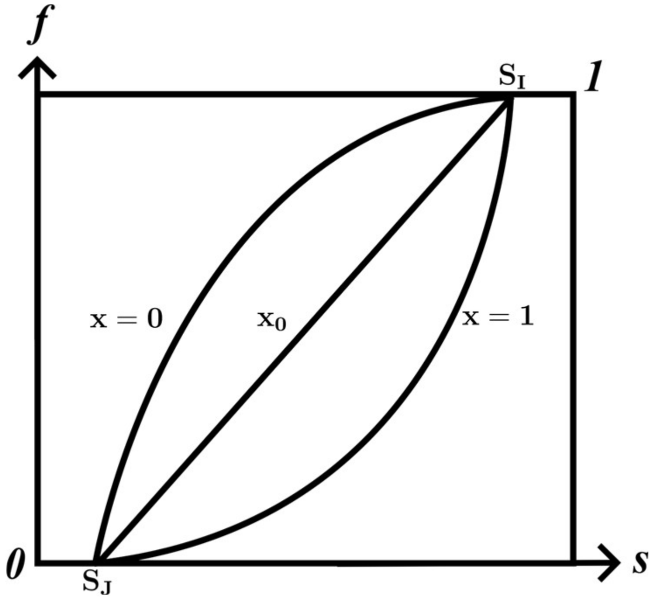
$$f'\left(S^{-}\right)<D<f'\left(S^{+}\right)$$

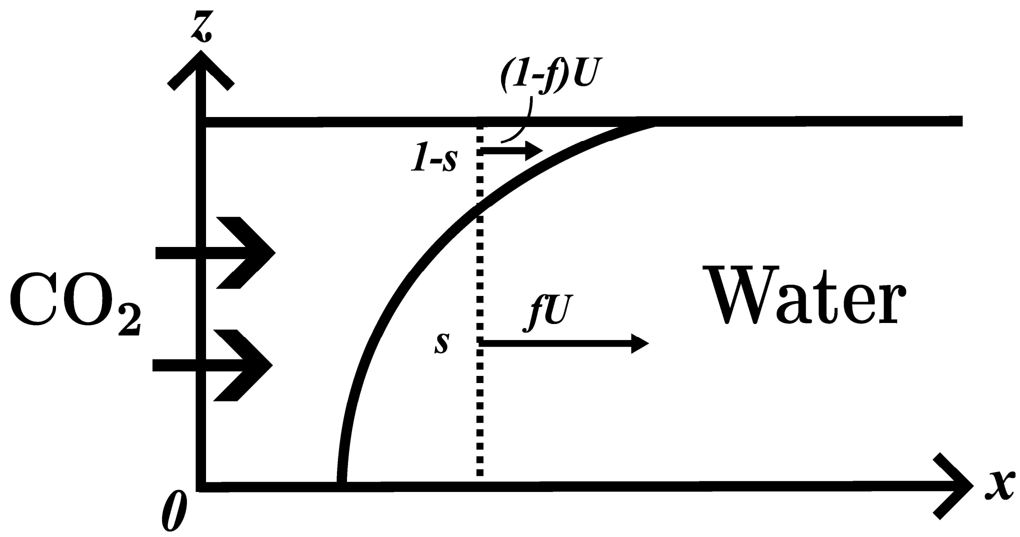
Stability of solution with respect to small perturbations in original equation

$$D>\frac{f\left(s\left(v\right)\right)-f\left(S^{-}\right)}{s\left(v\right)-S^{-}}$$

2. Mass balance for 1D flow in micro heterogeneous flow

$$\frac{\partial s}{\partial t} + \frac{\partial f(s, x)}{\partial x} = 0$$





IBCs:

$$t = 0 : \ s = S_I = 1$$

$$x = 0 : \ f = 0 (s = S_{wi})$$