# Lie pseudo-groups and zero-curvature representations of differential equations

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# Lie pseudo-groups

A pseudo-group $\mathfrak{G}$ on a manifold $M$ is a set of local diffeomorphisms $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$, $\Phi\colon x \mapsto \hat{x}$ such that

1) if $\Phi \in \mathfrak{G}$, $\Psi \in \mathfrak{G}$, and their composition $\Psi \circ \Phi$ is defined, then $\Psi \circ \Phi \in \mathfrak{G}$;
2) $\Phi \in \mathfrak{G} \Rightarrow \Phi^{-1} \in \mathfrak{G}$;
3) $\mathrm{id}_M \in \mathfrak{G}$.

A pseudo-group $\mathfrak{G}$ is called a Lie pseudo-group, if

4) the functions $\hat{x} = \Phi(x)$ are local analytic solutions of a system of PDEs (Lie equations of the pseudo-group $\mathfrak{G}$)

$$R\left(x, \Phi(x), \frac{\partial \Phi(x)}{\partial x}, \dots, \frac{\partial^{\#I} \Phi(x)}{\partial x^I}\right) = 0.$$

EXAMPLE: Lie groups = finite Lie pseudo-groups

EXAMPLE: conformal transformations in $\mathbb{R}^2$:

$$\Phi\colon (x,y) \mapsto (\hat{x},\hat{y}),$$

$$(d\hat{x})^2 + (d\hat{y})^2 = \lambda(x,y)\left((dx)^2+(dy)^2\right), \qquad \lambda(x,y) \neq 0$$

Lie equations = Cauchy – Riemann equations

$$\frac{\partial \hat{x}}{\partial x} = \frac{\partial \hat{y}}{\partial y}, \qquad \frac{\partial \hat{x}}{\partial y} = -\frac{\partial \hat{y}}{\partial x}$$

# Lie's infinitesimal method

- Infinitesiaml generators of the Lie pseudo-group $\mathfrak{G}$:

$$\hat{x} = \Phi(x) = x + \varepsilon\,\varphi(x) + \dots$$

- Infinitesimal defining system (linearized Lie equations):

$$\Phi \in \mathfrak{G} \Longleftrightarrow L\left(x, \varphi(x), \frac{\partial \varphi(x)}{\partial x}, \dots, \frac{\partial^{\#I} \varphi(x)}{\partial x^I}\right) = 0$$

- **Integration**

# Cartan's method

Maurer–Cartan forms of the Lie pseudo-group $\mathfrak{G}$: a collection of 1-forms

$$\omega^i \in \Omega^1(M \times N \times H), \qquad i \in \{1, ..., \dim M + \dim N\},$$

where $N$ is a manifold, $H$ is a finite Lie group.

A local diffeomorphism $\Phi$ on $M$, $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$ belongs to $\mathfrak{G}$ whenever there exists a fibre-preserving diffeomorphism $\Psi$ on $M \times N \times H$, $\Psi\colon \mathcal{W} \to \hat{\mathcal{W}}$ such that

- $\Phi$ is the projection of $\Psi$ w.r.t. $M \times N \times H \to M$;
- $\Psi^* \left(\omega^i|_{\hat{\mathcal{W}}}\right) = \omega^i|_{\mathcal{W}}$.

EXAMPLE: finite-dimensional Lie group $G$,

- $V_1$, ..., $V_n$ — an invariant base of $TG$,
- $\omega^1$, ..., $\omega^n$ — the dual base, $\omega^i(V_j) = \delta^i_j$;
- then $\omega^i \in \Omega^1(G)$ are invariant forms on $G$ = Maurer–Cartan forms of $G$.
- A local diffeomorphism $\Phi$ on $G$ preserves $\omega^i$ iff $\Phi(h) = g \cdot h$ for some $g \in G$.

EXAMPLE: Maurer–Cartan forms of the pseudo-groups of conformal transformations

$$\omega_1,\ \omega_2 \in \Omega^1(\mathbb{R}^2 \times H), \qquad M = \mathbb{R}^2,$$

$$H = \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \ \middle|\ a^2 + b^2 \neq 0 \right\},$$

$$\omega_1 = a\,dx + b\,dy, \qquad \omega_2 = -b\,dx + a\,dy$$

## Cartan's method

Structure equations of a Lie pseudo-group $\mathfrak{G}$:

$$d\omega^i = A^i_{\alpha j}(U^\sigma)\,\pi^\alpha \wedge \omega^j + B^i_{jk}(U^\sigma)\,\omega^j \wedge \omega^k, \qquad B^i_{jk} = -B^i_{kj},$$

$$dU^\kappa = C^\kappa_j(U^\sigma)\,\omega^j,$$

$U^\sigma \colon M \to \mathbb{R}$, $\sigma \in \{1, ..., s\}$, $s < \dim M$, — invariants of the pseudo-group $\mathfrak{G}$,

$$\Phi^* \left(U^\kappa|_{\tilde{\mathfrak{U}}}\right) = U^\kappa|_{\mathfrak{U}},$$

- $\pi^\alpha$ — depend on differentials of coordinates on $H$;
- involutivity conditions are satisfied,
- compatibility conditions are satisfied.

Maurer–Cartan forms and structure equations of a Lie pseudo-group can be found from its Lie equations algorithmically.

Involutivity conditions:

$$r^{(1)} = n \,\dim\, H - \sum_{k=1}^{n-1} (n-k)\,\sigma_k,$$

where $n = \dim M + \dim N$, $r^{(1)}$ is the dimension of the linear space of coefficients $z^{\alpha}_{j}$ such that the replacement $\pi^{\alpha} \mapsto \pi^{\alpha} + z^{\alpha}_{j}\,\omega^{j}$ preserves the structure equations;

$$\sigma_k = \max_{u_1,\ldots,u_k} \,\mathrm{rank}\; \mathbb{A}_k(u_1,\ldots,u_k) - \sum_{j=1}^{k-1} \sigma_j,$$

$$\mathbb{A}_1(u_1) = \left( A^{i}_{\alpha j}\, u^{j}_{1} \right),$$

$$\mathbb{A}_q(u_1,\ldots,u_q) = \begin{pmatrix} \mathbb{A}_{q-1}(u_1,\ldots,u_{q-1}) \\ A^{i}_{\alpha j}\, u^{j}_{q} \end{pmatrix}, \; q \in \{2,\ldots,n-1\}.$$

Compatibility conditions:

- $d(d\omega^i) = 0 = d\left(A^i_{\alpha j}\,\pi^\alpha \wedge \omega^j + B^i_{jk}\,\omega^j \wedge \omega^k\right)$
- $d(dU^\kappa) = 0 = d(C^\kappa_j\,\omega^j)$

$\Longrightarrow$

- over-determined system for the coefficients $A^i_{\alpha j}$, $B^i_{jk}$, $C^\kappa_j$;
- $d\pi^\alpha = W^\alpha_{\lambda j}\,\chi^\lambda \wedge \omega^j + X^\alpha_{\beta\gamma}\,\pi^\beta \wedge \pi^\gamma + Y^\alpha_{\beta j}\,\pi^\beta \wedge \omega^j + Z^\alpha_{jk}\,\omega^j \wedge \omega^k.$

THEOREM (*Third fundamental Lie's theorem in Cartan's form*): For a Lie pseudo-group there exists a collection of Maurer–Cartan forms with involutive and compatible structure equations.

THEOREM (*Third inverse fundamental Lie's theorem in Cartan's form*): For a given involutive and compatible system of structure equations there exists a collection of 1-forms $\omega^1$, ..., $\omega^n$ and functions $U^1$, ...., $U^s$ satisfying this system. The forms $\omega^1$, ..., $\omega^m$ are Maurer–Cartan forms of a Lie pseudo-group, and the functions $U^1$, ...., $U^s$ are invariants of this pseudo-group.


- Cartan É. Œuvres Complètes. Paris: Gauthier - Villars, 1953
- Vasil'eva M.V. Structure of Infinite Lie Groups of Transformations. Moscow: MSPI, 1972 (in Russian)
- Stormark O. Lie's Structural Approach to PDE Systems. Cambridge: CUP, 2000

EXAMPLE: G – a finite Lie group

- $V_i$ — an invariant base in $TG$
- $[V_i, V_j] = c^k_{ij}\, V_k$, $\quad c^k_{ij}$ — structure constants of $G$,
- skew- symmetry: $\quad c^k_{ij} = -c^k_{ji}$,
- Jacobi's identity: $\quad c^q_{mk}\, c^k_{ij} + c^q_{jk}\, c^k_{mi} + c^q_{ik}\, c^k_{jm} = 0$,
- structure equations:

$$d\omega^i = -\tfrac{1}{2}\, c^i_{jk}\, \omega^j \wedge \omega^k,$$

- compatibility = Jacobi's identity

EXAMPLE: conformal transformations on $\mathbb{R}^2$:

$$\omega_1 = a\,dx + b\,dy, \qquad \omega_2 = -b\,dx + a\,dy \qquad \Longrightarrow$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} d\omega_1 \\ d\omega_2 \end{pmatrix} = \begin{pmatrix} da & db \\ -db & da \end{pmatrix} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= \begin{pmatrix} da & db \\ -db & da \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1 & \pi_2 \\ -\pi_2 & \pi_1 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

# Contact transformations

- Trivial bundle: $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\pi\colon (x^i, u) \mapsto (x^i)$
- the second order jets: $J^2(\pi)$, $(x^i, u, u_i, u_{ij})$, $u_{ij} = u_{ji}$
- contact forms : $\vartheta_0 = du - u_j\,dx^j$, $\vartheta_i = du_i - u_{ij}\,dx^j$.
- The pseudo-group of contact transformations $\mathrm{Cont}(J^2(\pi))$:
$$\Psi\colon J^2(\pi) \to J^2(\pi),\ \Psi\colon (x^i, u, u_i, u_{ij}) \mapsto (\hat{x}^i, \hat{u}, \hat{u}_i, \hat{u}_{ij})$$
such that
$$\Psi^*(d\hat{u} - \hat{u}_j\,d\hat{x}^j) = a\,(du - u_j\,dx^j),$$
$$\Psi^*(d\hat{u}_i - \hat{u}_{ij}\,d\hat{x}^j) = P_i^j\,(du_j - u_{jk}\,dx^k) + Q_i\,(du - u_j\,dx^j),$$
$$\Psi^* d\hat{x}^i = b_j^i\,dx^j + R^i\,(du - u_j\,dx^j) + S^{ij}\,(du_j - u_{jk}\,dx^k),$$
$$a \neq 0,\ \det(b_j^i) \neq 0,\ \det(P_i^j) \neq 0$$

Maurer–Cartan forms of the pseudo-group $\mathrm{Cont}(J^2(\pi))$:

$$\Theta_0 = a\,(du - u_i\,dx^i),$$
$$\Theta_i = a\,B^j_i\,(du_j - u_{jk}\,dx^k) + g_i\,\Theta_0,$$
$$\Theta_{ij} = a\,B^k_i\,B^l_j\,(du_{kl} - u_{klm}\,dx^m) + s_{ij}\,\Theta_0 + w^k_{ij}\,\Theta_k,$$
$$\Xi^i = b^i_j\,dx^j + c^i\,\Theta_0 + f^{ij}\,\Theta_j,$$

where $\quad b^i_k\,B^k_j = \delta^i_j, \quad f^{ik} = f^{ki},\; s_{ij} = s_{ji}, \quad w^k_{ij} = w^k_{ji},$

$u_{klm} = u_{lkm} = u_{kml}$.

Structure equations

$$d\Theta_0 = \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,$$
$$d\Theta_i = \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Theta_{ik},$$
$$d\Theta_{ij} = \Phi^k_i \wedge \Theta_{kj} - \Phi^0_0 \wedge \Theta_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Xi^k \wedge \Theta_{ijk},$$
$$d\Xi^i = \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k$$

# Symmetry pseudo-groups of differential equations

- A PDE of the second order: $\iota \colon \mathcal{E} \to J^2(\pi)$
- Contact symmtries of $\mathcal{E}$ — contact transformations that map $\mathcal{E}$ into itself: $\mathrm{Cont}(\mathcal{E}) \subset \mathrm{Cont}(J^2(\pi))$,
- **Maurer-Cartan forms of the pseudo-group $\mathrm{Cont}(\mathcal{E})$ can be found from the restrictions $\theta_0 = \iota^* \, \Theta_0$, $\theta_i = \iota^* \, \Theta_i$, $\theta_{ij} = \iota^* \, \Theta_{ij}$, $\xi^i = \iota^* \, \Xi^i$ of Maurer-Cartan forms of the pseudo-group $\mathrm{Cont}(J^2(\pi))$ on $\mathcal{E}$ algorithmically by means of Cartan's method of equivalence**
- Details:
  - M. Fels, P.J. Olver. Moving coframes I. A practical algorithm. // Acta Appl. Math., 1998, Vol. 51, pp. 161–213
  - O.I. Morozov. Moving coframes and symmetries of differential equations. // J. Phys. A: Math. Gen., 2002, Vol. 35, pp. 2965 – 2977

EXAMPLE: Liouville's equation $u_{xy} = e^u$

Maurer–Cartan forms of the symmetry pseudo-group:

$$\theta_0 = du - u_x dx - u_y dy,$$
$$\theta_1 = q^{-1}\,(du_x - u_{xx} dx - e^u dy),$$
$$\theta_2 = q\,e^{-u}\,(du_y - e^u dx - u_{yy} dy),$$
$$\theta_{11} = q^{-2}\,\left(du_{xx} - u_x du_x + (u_x u_{xx} + q^3 r_1)\,dx\right),$$
$$\theta_{22} = q^2\,e^{-2u}\,\left(du_{yy} - u_y du_y + (u_y u_{yy} + e^{3u} q^{-3} r_2) dy\right),$$
$$\xi^1 = q\,dx,$$
$$\xi^2 = q^{-1} e^u dy,$$
$$\eta_1 = q^{-1}\,(dq - u_x\,\xi^1),$$
$$\eta_2 = dr_1 - 3\,r_1\,\eta_1 + q^{-2}(u_{xx} + u_x^2)\,(\theta_1 + \xi^2) + 3\,q^{-1} u_x\,\theta_{11} + r_3\,\xi^1,$$
$$\eta_3 = dr_2 + 3r_2(\eta_1 + \theta_0) + \frac{q^2}{e^{2u}}(u_{yy} + u_y^2)(\theta_2 + \xi^1) + \frac{3q}{e^u} u_y \theta_{22} + r_4\,\xi^2.$$

# Applications

- equivalence problems for DEs;
- symmetry analysis of classes of DEs;
- differential invariants, automorphic systems of DEs, Vessiot group splittings of DEs;
- Darboux integrability;
- coverings of DEs.

# Coverings of differential equations

Coverings (Lax pairs, Bäcklund transformations, prolongation structures, zero - curvature representations, integrable extensions, ...):

- Lax P.D. // Comm. Pure Appl. Math., 1969, **21**, 467 – 490
- M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur. // Stud. Appl. Math., 1974, **53**, 249 – 315
- V.E. Zakharov, A.B. Shabat. // Funct. Analysis Appl. 1974, **6**, No 6, 43 – 54
- H.D. Wahlquist, F.B. Estabrook, 1975, // J. Math. Phys., 1975, **16**, 1 – 7
- ...
- I.S. Krasil'shchik, A.M. Vinogradov, // Acta Appl. Math., 1984, **2**, 79–86
- I.S. Krasil'shchik, A.M. Vinogradov // Acta Appl. Math., 1989, **15**, 161–209
- ...

# Coverings of differential equations

- Trivial bundle
  $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad \pi\colon (x^i, u) \mapsto (x^i), \quad i \in \{1, ..., n\}$
- The bundle of infinite jets of sections of $\pi$: $J^\infty(\pi)$,
  coordinates $(x^i, u, u_i, u_{ij}, ..., u_I, ...)$, $I = (i_1, i_2, ..., i_m)$
- Total derivatives

$$D_k = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} u_{Ik} \frac{\partial}{\partial u_I}, \qquad [D_i, D_j] = 0$$

- Contact 1-forms

$$\vartheta_I = du_I - u_{Ij}\, dx^j, \qquad d\vartheta_I = dx^j \wedge \vartheta_{Ij}$$

- Differential equation $\mathcal{E}$: $\qquad F(x^i, u, u_I) = 0$
- Infinitely prolonged differential equation $\mathcal{E}^\infty \subset J^\infty(\pi)$:
  $D_{k_1} \circ D_{k_2} \circ \cdots \circ D_{k_m}(F) = 0$
- Restricted total derivatives

$$\bar{D}_k = D_k|_{\mathcal{E}^\infty}, \qquad [\bar{D}_i, \bar{D}_j] = 0$$

# Coverings of differential equations

- Covering over $\mathcal{E}^\infty$:

$$\tau \colon \widetilde{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \mathcal{V} \to \mathcal{E}^\infty, \qquad \mathcal{V} = \{(v^\kappa) \mid 0 \le \kappa \le \infty\}$$

- Extended total derivatives

$$\widetilde{D}_i = \bar{D}_i + \sum_\kappa P_i^\kappa(x^j, u_I, v^\rho)\,\frac{\partial}{\partial v^\kappa},$$

$$[\widetilde{D}_i, \widetilde{D}_j] = 0 \iff (x^i, u_I) \in \mathcal{E}^\infty$$

- Extended contact forms (Wahlquist-Estabrook forms)

$$\widetilde{\vartheta}_0^\kappa = dv^\kappa - P_i^\kappa(x^j, u_I, v^\rho)\,dx^i,$$

$$\widetilde{\vartheta}_I^\kappa = \widetilde{D}_I(\widetilde{\vartheta}_0^\kappa),$$

$$d\widetilde{\vartheta}_I^\kappa \equiv 0 \mod \widetilde{\vartheta}_K^\rho, \vartheta_K, \qquad d\widetilde{\vartheta}_I^\kappa \not\equiv 0 \mod \widetilde{\vartheta}_K^\rho$$

Example: Liouville's equation

$$u_{xy} = e^u$$

- Covering: the fibre coordinate $v$, extended total derivatives:

$$\widetilde{D}_x = \bar{D}_x + (u_x + e^v)\,\frac{\partial}{\partial v}, \quad \widetilde{D}_y = \bar{D}_y - \frac{1}{2}\,e^{u-v}\,\frac{\partial}{\partial v}$$

- Bäcklund transformation:

$$\begin{cases} v_x = u_x + e^v, \\ v_y = -\frac{1}{2}\,e^{u-v} \end{cases}$$

- Excluding $u$: Clairin's equation

$$v_{xy} + e^v\,v_y = 0.$$

- Wahlquist–Estabrook form:

$$\widetilde{\vartheta} = dv - (u_x + e^v)\,dx + \frac{1}{2}\,e^{u-v}\,dy$$

# Coverings of differential equations

The problem of recognizing whether a given differential equation has a covering is of great importance. Different techniques were proposed to solve it.
$n = 2$.

- H.D. Wahlquist, F.B. Estabrook, 1975
- R. Dodd, A. Fordy, 1983
- C. Hoenselaers, 1986
- S.Yu. Sakovich, 1995
- M. Marvan, 1997
- S. Igonin, 2006
- ...

Of the special interest are equations with coverings that have a nonremovable (spectral) parameter.

The problem is much more difficult in the case of $n > 2$:

- G.M. Kuz'mina, 1967
- H.C. Morris, 1976
- V.E. Zakharov, 1982
- G.S. Tondo, 1985
- M. Marvan, 1992
- B.K. Harrison, 2002
- ...

G.M. Kuz'mina. On a possibility to reduce a system of two partial differential equations of the first order to a single equation of the second order. // Proc. Moscow State Pedagogical Institute, 1967, Vol. 271, 67–76 (in Russian)

$$u_{yy} = u_{tx} + u\,u_{xx} + u_x^2 \qquad \text{(dispersionless KP)}$$

Covering

$$\begin{cases} v_t = (v^2 - u)\,v_x - u_y - v\,u_x, \\ v_y = v\,v_x - u_x \end{cases}$$

Excluding $u$: define $w$ such that $w_x = v$ and $w_y = \frac{1}{2}\,v^2 - u$, then

$$w_{yy} = w_{tx} + \left(\tfrac{1}{2}\,w_x^2 - w_y\right)\,w_{xx} \qquad \text{(modified dKP)}$$

The central idea: to apply Cartan's structure theory of Lie pseudo-groups

# Coverings of differential equations

Liouville equation $u_{xy} = e^u$,

- the emphasized Maurer-Cartan forms:

$$\eta_1 = \frac{dq}{q} - u_x\,dx, \quad \xi^1 = q\,dx, \quad \xi^2 = \frac{e^u}{q}\,dy,$$

- denote $q = e^v$, take the linear combination:

$$\widetilde{\vartheta} = \eta_1 - \xi^1 + \frac{1}{2}\,\xi^2 = dv - (u_x + e^v)\,dx + \frac{1}{2}\,e^{u-v}\,dy$$

- **This is the Wahlquist–Estabrook form of the aforementioned covering**

EXAMPLE: Plebañski's second heavenly equation

$$u_{xz} = u_{ty} + u_{xx}\,u_{yy} - u_{xy}^2$$

Covering: J.F. Plebañski, 1975

$$\left\{ \begin{array}{l} q_t = (u_{xy} - \lambda)\,q_x - u_{xx}\,q_y, \\ q_z = u_{yy}\,q_x - (u_{xy} + \lambda)\,q_y \end{array} \right.$$

Maurer–Cartan forms of the symmetry pseudo-group

$$\begin{array}{l}
\xi^1 = b_{11}\,dt + b_{14}\,dz, \\
\xi^2 = v^{-1}\,(b_{11}\,dx + b_{14}\,dy - (b_{11}\,(w-1)\,u_{xy} + b_{14}\,u_{xx} + b_{41}\,v)\,dt \\
\qquad -(b_{14}\,(w+1)\,u_{xy} - b_{11}\,u_{yy} + b_{44}\,v)\,dz), \\
\xi^3 = v^{-1}\,(b_{41}\,dx + b_{44}\,dy + (b_{11}\,v - b_{41}\,(w-1)\,u_{xy} - b_{44}\,u_{xx})\,dt \\
\qquad +(b_{14}\,v - b_{44}\,(w+1)\,u_{xy} + b_{41}\,u_{yy})\,dz), \\
\xi^4 = b_{41}\,dt + b_{44}\,dz, \\
\eta_1 = (b_{44}\,db_{11} - b_{41}\,db_{14})\,(b_{11}b_{44} - b_{14}b_{41})^{-1} + r_1\,\xi^1 + r_2\,\xi^4, \\
\eta_4 = (b_{11}\,db_{44} - b_{14}\,db_{41})\,(b_{11}b_{44} - b_{14}b_{41})^{-1} - r_1\,\xi^1 - r_2\,\xi^4, \\
\eta_5 = -3\,v^{-1}\,dv + \eta_1 + \eta_4,
\end{array}$$

Substituting for

$$v = q,\ b_{11} = q_x,\ b_{14} = q_y,\ w = \lambda\, u_{xy}^{-1}$$

into the linear combination

$$\tfrac{1}{3}\,(\eta_1 + \eta_4 - \eta_5) - \xi^2 - \xi^4,$$

yields the Wahlquist–Estabrook form of the covering

$$\widetilde{\vartheta}_0 = q^{-1}\,(dq - ((u_{xy} - \lambda)\,q_x - u_{xx}\,q_y)\,dt - q_x\,dx - q_y\,dy$$
$$-(u_{yy}\,q_x - (u_{xy} + \lambda)\,q_y)\,dz)$$

Bryant R.L., Griffiths P.A. Characteristic Cohomology of Differential Systems (II): Conservation Laws for a Class of Parabolic Equations // Duke Math. J., **78**, 531–676 (1995):

$n = 2$, finite-dimensional coverings

**Definition 1**. Let

$$d\omega^i = A^i_{\alpha j}(U^\sigma)\,\pi^\alpha\wedge\omega^j + B^i_{jk}(U^\sigma)\,\omega^j\wedge\omega^k, \qquad (1)$$

$$dU^\kappa = C^\kappa_j(U^\sigma)\,\omega^j \qquad (2)$$

be structure equations of a Lie pseudo-group $\mathfrak{G}$. Then the system

$$d\tau^q = D^q_{\rho r}(U^\sigma, V^\epsilon)\,\eta^\rho\wedge\tau^r + E^q_{rs}(U^\sigma, V^\epsilon)\,\tau^r\wedge\tau^s + F^q_{r\beta}(U^\sigma, V^\epsilon)\,\tau^r\wedge\pi^\beta$$

$$+G^q_{rj}(U^\sigma, V^\epsilon)\,\tau^r\wedge\omega^j + H^q_{\beta j}(U^\sigma, V^\epsilon)\,\pi^\beta\wedge\omega^j + I^q_{jk}(U^\sigma, V^\epsilon)\,\omega^j\wedge\omega^k, \qquad (3)$$

$$dV^\epsilon = J^\epsilon_j(U^\sigma, V^\epsilon)\,\omega^j + K^\epsilon_q(U^\sigma, V^\epsilon)\,\tau^q, \qquad (4)$$

with unknown 1-forms $\tau^q$, $q \in \{1, \ldots, Q\}$, $\eta^\rho$, $\rho \in \{1, \ldots, R\}$, and unknown functions $V^\epsilon$, $\epsilon \in \{1, \ldots, S\}$, $Q, R, S \in \mathbb{N}$, is called an integrable extension of system (1), (2), if equations (1) – (4) are simultaneously compatible and involutive.

Suppose system (3), (4) is an integrable extension of system (1), (2). Then, in accordance with the third inverse fundamental theorem of Lie, system (1)–(4) defines a Lie pseudo-group $\mathfrak{H}$.

**Definition 2**. The integrable extension (3), (4) is called trivial , if there exists a change of variables on the manifold of action of the pseudo-group $\mathfrak{H}$ such that in the new variables equations (3), (4) do not contain the forms $\omega^j$, $\pi^\beta$, and the coefficients of (3), (4) do not depend on $U^q$. Otherwise, the integrable extension is called non-trivial .

Let $\theta^\alpha_K$, $\xi^j$ be Maurer–Cartan forms of the pseudo-group $\mathrm{Cont}(\mathcal{E})$ of symmetries for a PDE $\mathcal{E}$ such that $\theta^\alpha_K$ are contact forms (their restrictions on each solution of the equation $\mathcal{E}$ are equal to 0), and $\xi^j$ are horizontal forms ($\xi^1 \wedge ... \wedge \xi^n \neq 0$ on each solution).

**Definition 3**. Nontrivial integrable extension of the structure equations of the pseudo-group $\mathrm{Cont}(\mathcal{E})$

$$d\omega^q = \Pi^q_r \wedge \omega^r + \xi^j \wedge \Omega^q_j \tag{5}$$

is called contact integrable extension when

- $\Omega^q_j \equiv 0 \pmod{\theta^\alpha_K, \omega^q_j}$ for a set of additional forms $\omega^q_j$;
- $\Omega^q_j \not\equiv 0 \pmod{\omega^q_j}$
- coefficients of expansions of $\Omega^q_j$ w.r.t. $\{\theta^\alpha_I,\, \omega^r_i\}$ and $\Pi^q_r$ w.r.t. $\{\theta^\alpha_I,\, \xi^j,\, \omega^r,\, \omega^r_i\}$ depend on the invariants of $\mathrm{Cont}(\mathcal{E})$ and, maybe, on a set of additional functions $W^\rho$, $\rho \in \{1, \dots, \Lambda\}$, $\Lambda \geqslant 1$. In the latter case case there exist functions $P^{I\rho}_\alpha$, $Q^\rho_q$, $R^{j\rho}_q$, $S^\rho_j$ such that

$$dW^\rho = P^{I\rho}_\alpha\, \theta^\alpha_I + Q^\rho_q\, \omega^q + R^{j\rho}_q\, \omega^q_j + S^\rho_j\, \xi^j.$$

These equations satisfy the compatibility conditions.

r-dispersionless (2+1)-dimensional Dym equation
(M. Błaszak, 2002):

$$u_{ty} = u_x\,u_{xy} + \kappa\,u_y\,u_{xx}, \qquad \kappa \neq 0$$

special cases:

- $\kappa = 1$: dispersionless Novikov-Veselov equation
  B.G. Konopelchenko, A. Moro, 2004
- $\kappa = \frac{1}{2}$:
  E.V. Ferapontov, K.R. Khusnutdinova, S.P. Tsarev, 2004
- $\kappa = 2$:
  E.V. Ferapontov, A. Moro, V.V. Sokolov, 2007
- $\kappa \neq -2$:
  M.V. Pavlov, 2006
- $\kappa = -1$:
  V.Yu. Ovsienko, 2008, Hamiltonian system on an extension of the Virasoro group

Structure equations for the symmetry pseudo-group

$$d\theta_0 = \eta_1 \wedge \theta_0 + \xi_1 \wedge \theta_1 + \xi_2 \wedge \theta_2 + \xi_3 \wedge \theta_3,$$

$$d\theta_1 = 2\,\eta_1 \wedge \theta_1 - \left(\theta_{22} + \tfrac{1}{2}\,U\,(2\,\kappa - 1)\,\kappa^{-1}\,\xi_3\right) \wedge \theta_0 - (2\,\theta_{23} - (U+2)\,\theta_3 + \tfrac{1}{2}\,(3\,U - 2)\,\xi_2 - 2\,V\,\xi_3\big) \wedge \theta_1 + \xi_1 \wedge \theta_{11} + \xi_2 \wedge \theta_{12} + \xi_3 \wedge \theta_{22},$$

$$d\theta_2 = \eta_1 \wedge \theta_2 + \eta_2 \wedge \theta_0 - \left(\theta_{23} - \tfrac{1}{2}\,(U+2)\,\theta_3 + \xi_1 - \xi_2 - V\,\xi_3\right) \wedge \theta_2 + \xi_1 \wedge \theta_{12} + \xi_2 \wedge \theta_{22} + \xi_3 \wedge \theta_{23},$$

$$d\theta_3 = \xi_1 \wedge (\kappa^{-1}\,\theta_2 + \theta_{22}) + \xi_2 \wedge \left(\theta_{23} - \tfrac{1}{2}\,(U+2)\,\theta_3\right) + \xi_3 \wedge \theta_{33},$$

$$d\xi_1 = \xi_1 \wedge \left(\eta_1 + (U+2)\,\theta_3 - 2\,\theta_{23} - \tfrac{1}{2}\,(3\,U - 4)\,\xi_2 + 2\,V\,\xi_3\right),$$

$$d\xi_2 = \kappa^{-1}\,\xi_1 \wedge \left(\tfrac{1}{2}\,U\,\theta_0 - \theta_2 + \kappa\,\xi_2 - \xi_3\right) + \xi_2 \wedge \left(\tfrac{1}{2}\,(U+2)\,\theta_3 - \theta_{23} + V\,\xi_3\right)$$

$$d\xi_3 = \left(\eta_1 + \theta_3 + \tfrac{1}{2}\,(U+2)\,\xi_2\right) \wedge \xi_3,$$

$$\ldots$$

Invariants and their differentials:

$$U = \frac{u_y\,u_{xxy}}{u_{xy}^2}, \qquad V = \frac{u_y\,u_{xxx}}{u_{xy}^4}\,(u_{xy}\,u_{yy} - u_y\,u_{xyy}),$$

$$\begin{aligned} dU &= 2\,\eta_2 - \tfrac{1}{2}\,\kappa^{-1}\,U\,V\,\theta_0 + \tfrac{1}{2}\,U\,(U+2)\,\theta_3 - (U+2)\,\xi_1 \\ &\quad - \tfrac{1}{2}\,\kappa^{-1}\,U\,(\kappa\,U - 2\,\kappa + 2)\,\xi_2 + U\,V\,\xi_3 - U\,\theta_{23}, \\ dV &= \eta_6 - V\,\eta_1 + \tfrac{1}{2}\,V\,(U-2)\,\theta_3 + \kappa^{-1}\,(2\,\kappa\,V - 2\,U + 1)\,\xi_1 \\ &\quad - \tfrac{1}{2}\,V\,(U+4)\,\xi_2 + \tfrac{1}{2}\,(U+2)\,\theta_{33}. \end{aligned}$$

## r-dDym equation

Contact integrable extensions of the structure equations for the symmetry pseudo-groups of r-dDym equation of the simpliest form

$$d\omega_0 = \sum_{j=1}^{3} \left( \sum_{k=0}^{3} F_{jk}\,\theta_k + G_j\,\omega_1 \right) \wedge \xi^j +$$

$$\left( \sum_{i=0}^{3} A_i\,\theta_i + \sum_{i,j=1}^{3} B_{ij}\,\theta_{ij} + \sum_{s=1}^{7} C_s\,\eta_s + \sum_{j=1}^{3} D_j\,\xi^j + E\,\omega_1 \right) \wedge \omega_0$$

- Type 1: the coefficients depend on $U$, $V$;
- Type 2: the coefficients depend also on one additional invariant $W$,

$$dW = \sum_{i=0}^{3} H_i\,\theta_i + \sum_{i,j=1}^{3} I_{ij}\,\theta_{ij} + \sum_{s=1}^{7} J_s\,\eta_s + \sum_{j=1}^{3} K_j\,\xi^j + \sum_{q=0}^{1} L_q\,\omega_q.$$

There are no CIEs of the first type. Every CIE of the second type is contact-equivalent to the following:

$$
\begin{aligned}
d\omega_0 = & \Big(\theta_{23} - \tfrac{1}{2}\,(U+2)\,\theta_3 - \kappa^{-1}\,W^{-1}(\kappa\,(U+V\,W-1)-1)\,\xi^3 + \omega_1 \\
& + (1 - W\,(U-1))\,\xi^1\Big)\wedge\omega_0 + \tfrac{1}{2}\,\kappa^{-1}\,(U\theta_0 - 2\,\theta_2 + 2\,\kappa\,W\,\omega_1)\wedge\xi^1 \\
& + \omega_1\wedge\xi^2 + W^{-1}\,(\omega_1 + \kappa^{-1}\,\theta_3)\wedge\xi^3, \\
dW = & -(\kappa+1)\,W\,\omega_1 + \Big(\kappa\,W\,(1-U) - \tfrac{1}{2}\,U\,W + Z\Big)\,\omega_0 + W\,(\eta_1 - \theta_{23}) \\
& + \tfrac{1}{2}\,W\,(U+2)\,\theta_3 + Z\,\xi^2 + \Big(V\,W + \tfrac{1}{2}\,U + W^{-1}\,Z - 2) - \kappa^{-1}\Big)\,\xi^3 \\
& + W\,\Big(Z - W - 1 + \tfrac{1}{2}\,U\,W\Big)\,\xi^1.
\end{aligned}
$$

**REMARK**: $\kappa = -1,\ Z = -\frac{1}{2}\,W\,(U-2) \Longrightarrow$ nonremovable parameter

Third inverse fundamental theorem $\Longrightarrow$

$\kappa \notin \{-2,-1,0\}$:

$$\omega_0 = \frac{u_{xy}}{u_y\,v_x}\left(dv - \left(u_x\,v_x + \frac{\kappa}{\kappa+2}\,v_x^{\kappa+2}\right)\,dt - v_x\,dx + \frac{1}{\kappa}\,u_y\,v_x^{-\kappa}\,dy\right),$$

$$W = u_{xy}^2\,u_y^{-2}\,u_{xxx}^{-1}\,v_x^{\kappa+1}.$$

$\kappa = -2$:

$$\omega_0 = \frac{u_{xy}}{u_y\,v_x}\left(dv - (u_x\,v_x - 2\ln|\,v_x\,|)\;dt - v_x\,dx - \tfrac{1}{2}\,u_y\,v_x^2\,dy\right),$$

$$W = u_{xy}^2\,u_y^{-2}\,u_{xxx}^{-1}\,v_x^{-1}.$$

# r-dDym equation

$\kappa = -1,\ Z \neq -\frac{1}{2}\,W\,(U-2)$:

$$\omega_0 = \frac{u_{xy}}{u_y\,v_x}\,\left(dv - (u_x - v)\,v_x\,dt - v_x\,dx - u_y\,v^{-1}\,v_x\,dy\right),$$

$$W = u_{xy}^2\,u_y^{-2}\,u_{xxx}^{-1}\,v.$$

$\kappa = -1,\ Z = -\frac{1}{2}\,W\,(U-2)$:

$$\omega_0 = \frac{u_{xy}}{u_y\,v_x}\,\left(dv - (u_x - \lambda)\,v_x\,dt - v_x\,dx - \lambda^{-1}\,u_y\,v_x\,dy\right),$$

$$W = \lambda\,u_{xy}^2\,u_y^{-2}\,u_{xxx}^{-1}.$$

# r-dDym equation

- $\kappa \notin \{-2,-1,0\}$:

$$\begin{cases} v_t = u_x\, v_x + \frac{1}{\kappa+2}\, v_x^{\kappa+2}, \\ v_y = -u_y\, v_x^{-\kappa}. \end{cases}$$

  - M.V. Pavlov, 2006

$$v_{ty} = \tfrac{1}{\kappa+2}\left((\kappa+1)\, v_t - v_x^{\kappa+2}\right) v_x^{-1}\, v_{xy} - \kappa\, v_y\, v_x^{\kappa}\, v_{xx}$$

- $\kappa = -2$:

$$\begin{cases} v_t = u_x\, v_x + \ln|v_x|, \\ v_y = -u_y v_x^2. \end{cases}$$

$$v_{ty} = (v_t - \ln|v_x|)\, v_x^{-1}\, v_{xy} + 2\, v_y\, v_x^{-2}\, v_{xx}$$

- $\kappa = -1$:

$$\begin{cases} v_t = (u_x - v)\, v_x, \\ v_y = u_y\, v^{-1}\, v_x. \end{cases}$$

$$v_{ty} = (v_t\, v_x^{-1} + v)\, v_{xy} - v\, v_y\, v_x^{-1}\, v_{xx}$$

- $\kappa = -1$:

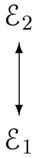
$$\begin{cases} v_t = (u_x + \lambda)\, v_x, \\ v_y = -\lambda^{-1}\, u_y v_x, \end{cases}$$

$$v_{ty} = (v_t\, v_x^{-1} - \lambda)\, v_{xy} + \lambda\, v_y\, v_x^{-1}\, v_{xx}$$

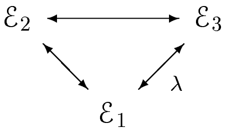
$\lambda \neq 0$ — nonremovable parameter
unliftable symmetry of the r-dDym equation with $\kappa = -1$:
$x\,\frac{\partial}{\partial x} + 2\,u\,\frac{\partial}{\partial u}$.

# r-dDym equation

$\kappa \notin \{-1, 0\}$:




$\kappa = -1$:



# r-mdKP equation

$r^{th}$ modified dispersionless KP equation (M. Błaszak, 2002):

$$u_{yy} = u_{tx} + \left(\tfrac{1}{2}\,(\kappa + 1)\,u_x^2 + u_y\right)\,u_{xx} + \kappa\,u_x\,u_{xy}$$

Special cases:

- $\kappa = 0$: mdKP,
  G.M. Kuz'mina, 1967; I.M. Krichever, 1988;
  B.A. Kupershmidt, 1990.
- $\kappa = 1$: dBKP,
  N. Dasgupta, R. Chowdhury, 1992; K. Takasaki, 1993;
  B.G. Konopelchenko, L. Martinez Alonso, 2003.
- $\kappa = -1$:
  M.V. Pavlov, 2003; M. Dunajski, 2004; E.V. Ferapontov, K.R. Khusnutdinova, 2004; V.Yu. Ovsienko, C. Roger, 2006

Contact integrable extensions:

$\kappa \notin \{-3, -1\}$:

First type:

$$d\omega_0 = \Big(\omega_1 + \tfrac{1}{2}(\eta_1 + \theta_{22}) + \tfrac{1}{16}(8\,V + \kappa^2 + 13\,\kappa + 12)\,(\kappa+1)^{-1}\,\xi^3$$
$$+\tfrac{1}{2}\,((\kappa+1)^2\,U - V^2 - 2\,(\kappa^2 + 3\,\kappa + 2)\,V)\,(\kappa+1)^{-2}\,\xi^1\Big)\wedge\omega_0$$
$$+\Big(\theta_3 - V\,(\kappa+1)^{-1}\,\theta_2 - \tfrac{1}{8}\,(\kappa - 4)\,\theta_0 + (\kappa+1)^{-2}\,V^2\,\omega_1\Big)\wedge\xi^1$$
$$+\omega_1\wedge\xi^2 + (\theta_2 - V\,(\kappa+1)^{-1}\,\omega_1)\wedge\xi^3.$$

Second type

$$d\omega_0 = \Big(\omega_1 + \tfrac{1}{2}\,(\theta_{22} + \eta_1 + (U - W^2 + 2\,(W - V))\,\xi^1) - \tfrac{1}{16}\,(8\,W - \kappa + 12)\,\xi^3\Big)$$
$$+\Big(W^2\,\omega_1 + W\,\theta_2 - \tfrac{1}{8}\,(\kappa - 4)\,\theta_0 + \theta_3\Big)\wedge\xi^1 + \omega_1\wedge\xi^2 + (W\,\omega_1 + \theta_2)\wedge\xi^3,$$
$$dW = \tfrac{1}{2}\,(2\,(Z_1 + 1) - \kappa\,(W - 1) - V)\,\omega_0 - (V + (\kappa+1)\,W)\,\omega_1 + \eta_2 - \theta_2$$
$$+\tfrac{1}{2}\,W\,(\eta_1 - \theta_{22}) + \tfrac{1}{2}\,W\,(2\,W\,Z_1 - U + 4\,V + W\,(W - \kappa))\,\xi^1 + Z_1\,\xi^2$$
$$+\tfrac{1}{16}\,(W\,(16\,Z_1 + 8\,W - 7\,\kappa - 4) + 16\,V)\,\xi^3.$$

$\kappa = -3$: additional CIE

$$
\begin{aligned}
d\omega_0 = & \left(\tfrac{1}{2}\,(\theta_{22}+\eta_1)+\tfrac{1}{8}\,(4\,U-V\,(V+8\,W-11))\,\xi^1+\omega_1\right.\\
& \left.-\tfrac{1}{16}\,(4\,V+16\,W+5)\,\xi^3\right)\wedge\omega_0+\left(\theta_3-W\,\theta_0+\tfrac{1}{2}\,V\,\theta_2+\tfrac{1}{4}\,V^2\,\omega_1\right)\wedge\xi^1\\
& +\omega_1\wedge\xi^2+\left(\theta_2+\tfrac{1}{2}\,V\,\omega_1\right)\wedge\xi^3\\
dW = & \left(Z+\tfrac{3}{2}\,W+\tfrac{21}{16}\right)\,\omega_0+\left(W+\tfrac{7}{8}\right)\,(\omega_1-\theta_{22})+Z\,\xi^2\\
& -\left(U\,\left(W+\tfrac{7}{8}\right)+V\,\left(W^2-\tfrac{203}{64}\right)-\tfrac{1}{32}\,V^2\,(8\,Z+7)\right)\,\xi^1\\
& +\tfrac{1}{64}\,(32\,V\,(Z+W)-16\,W\,(4\,W+3)+7\,(4\,V+1))\,\xi^3.
\end{aligned}
$$

# r-mdKP equation

$\kappa = -1$: the only CIE of the second type

$$d\omega_0 = \left(\tfrac{1}{2}\left(\theta_{22} + \eta_1 + (U - W^2 + 2\,W)\,\xi^1\right) - \tfrac{1}{16}\,(8\,W - 11)\,\xi^3 + \omega_1\right)\wedge\omega_0$$
$$+ \left(\theta_3 + \tfrac{5}{8}\,\theta_0 + W\,\theta_2 + W^2\,\omega_1\right)\wedge\xi^1 + \omega_1\wedge\xi^2 + (\theta_2 + W\,\omega_1)\wedge\xi^3),$$
$$dW = \tfrac{1}{2}(2\,Z + W + 1)\,\omega_0 - \theta_2 - \tfrac{1}{2}\,W\,(\theta_{22} + \eta_1) + \eta_2 + \tfrac{1}{16}\,W\,(16\,Z + 8\,W + 3)\,\xi^3$$
$$+ Z\,\xi^2 + \tfrac{1}{2}\,(W^2\,(2\,Z + 1) + W\,(W^2 - U))\,\xi^1$$

**REMARK**: $2\,Z + W + 1 = 0 \Longrightarrow$ nonremovable parameter

# r-mdKP equation

- $\kappa \notin \{-2, -\frac{3}{2}, -1\}$:

$$\begin{cases} v_t = \left(\frac{1}{2\kappa+3}\, v_x^{2\,(\kappa+1)} - u_x\, v_x^{\kappa+1} + \left(\frac{1}{2}\,(\kappa+1)\, u_x^2 - u_y\right)\right)\, v_x, \\ v_y = \left(\frac{1}{\kappa+2}\, v_x^{\kappa+1} - u_x\right)\, v_x. \end{cases}$$

  - $\kappa = 0$: J.-H. Chang, M.-H. Tu, 2000;
  - $\kappa = 1$: B.G. Konopelchenko, L. Martinez Alonso, 2003.
  - M.V. Pavlov, 2006

$$v_{yy} = v_{tx} - \kappa\,(v_y\, v_x^{-1} + v_x^{\kappa})\, v_{xy}$$
$$+((\kappa+1)\, v_y^2\, v_x^{-2} - v_t\, v_x^{-1} + \kappa\, v_x^{\kappa}\, v_y + \frac{(\kappa+1)^2}{2\kappa+3}\, v_x^{2(\kappa+1)})\, v_{xx}$$

- $\kappa \in \mathbb{R}$:

$$\begin{cases} v_t = \left(\frac{1}{2}\,(\kappa+1)\, u_x^2 - u_y\right)\, v_x, \\ v_y = -u_x\, v_x. \end{cases}$$

$$v_{yy} = v_{tx} + ((\kappa+1)\, v_y^2\, v_x^{-2} - v_t\, v_x^{-1})\, v_{xx} - \kappa\, v_y\, v_x^{-1}\, v_{xy},$$

# r-mdKP equation

- $\kappa = -2$:

$$\begin{cases} v_t = -v_x^{-1} - u_x - \left(\frac{1}{2}\,u_x^2 + u_y\right)\,v_x, \\ v_y = \ln|v_x| - u_x\,v_x \end{cases}$$

$$v_{yy} = v_{tx} + 2\,(\ln|v_x| + v_y)\,v_x^{-1}\,v_{xy}$$
$$-(v_t v_x + \ln|v_x|\,(\ln|v_x| - 2\,v_y + 1) + v_y^2 - v_y + 1)\,v_x^{-2}\,v_{xx}$$

- $\kappa = -\frac{3}{2}$:

$$\begin{cases} v_t = -\left(\frac{1}{4}\,u_x^2 + u_y\right)\,v_x - u_x\,\sqrt{|v_x|} + \ln|v_x|, \\ v_y = -u_x\,v_x + 2\,\sqrt{|v_x|} \end{cases}$$

$$v_{yy} = v_{tx} + \frac{3}{2}\,(v_y - 2\,|v_x|^{1/2})\,v_x^{-1}\,v_{xy}$$
$$+(v_x\,(\ln|v_x| - v_t - \frac{1}{2}\,v_y^2) + 3\,v_y\,|v_x|^{1/2} - 4)\,v_x^{-2}\,v_{xx}$$

- $\kappa = -3$:

$$\begin{cases} v_t = -u - (u_x^2 + u_y)\, v_x, \\ v_y = -v_x\, u_x - x \end{cases}$$

$$u = (v_y + v_x\, v_{tx} + 3\,(v_y + x)\, v_{xy} + x - v_x\, v_{yy}) v_{xx}^{-1} - v_t \\ -2\,(v_y + x)^2\, v_x^{-1}$$

# r-mdKP equation

- $\kappa = -1$:

$$\begin{cases} v_t = -\left(u_y - \lambda\, u_x - \lambda^2\right)\, v_x, \\ v_y = -(u_x - \lambda)\, v_x, \end{cases}$$

$\lambda \in \mathbb{R}$ — nonremovable parameter
M. Dunajski, 2004

$$v_{yy} = v_{tx} - (v_t + \lambda\, v_y)\, v_x^{-1}\, v_{xx} + (v_y + \lambda\, v_x)\, v_x^{-1}\, v_{xy}$$

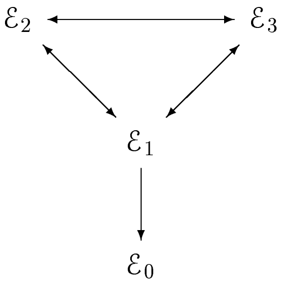
- $\kappa = -1$:

$$\begin{cases} v_t = -\left(u_y - v\, u_x - v^2\right)\, v_x, \\ v_y = -(u_x - v)\, v_x, \end{cases}$$

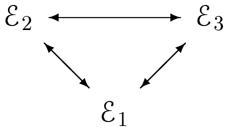
$$v_{yy} = v_{tx} - (v_t + v\, v_y)\, v_x^{-1}\, v_{xx} + (v_y + v\, v_x)\, v_x^{-1}\, v_{xy}$$

# r-mdKP equation

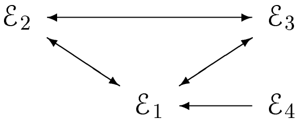
$\kappa = 0$:



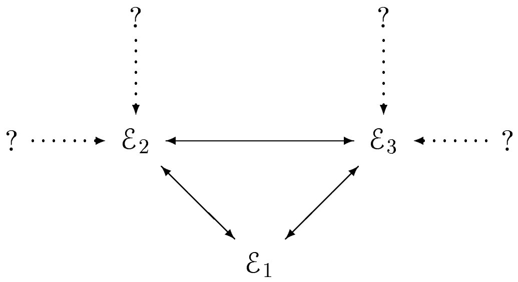

$\kappa \notin \{-3, 0\}$ :




$\kappa = -3$:



# r-mdKP equation




$$\mathcal{E}_1: \quad u_{yy} = u_{tx} + \left(\tfrac{1}{2}\,(\kappa+1)\,u_x^2 + u_y\right)\,u_{xx} + \kappa\,u_x\,u_{xy}$$

$$\mathcal{E}_2: \quad w_{yy} = w_{tx} + ((\kappa+1)\,w_y^2\,w_x^{-2} - w_t\,w_x^{-1})\,w_{xx} - \kappa\,w_y\,w_x^{-1}\,w_{xy},$$

$$\mathcal{E}_3: \quad v_{yy} = v_{tx} - \kappa\,(v_y\,v_x^{-1} + v_x^{\kappa})\,v_{xy} + ((\kappa+1)\,v_y^2\,v_x^{-2} - v_t\,v_x^{-1} + \kappa\,v_x^{\kappa}\,v_y + \tfrac{(\kappa+1)^2}{2\kappa+3}\,v_x^{2(\kappa+1)})\,v_{xx}$$

$\mathcal{E}_3$: double-modified dKP equation, M.V. Pavlov, 2006, 2010

$\kappa \notin \{-2, -3/2, -1\}$

$$u_{yy} = u_{tx} - \kappa \left(\frac{u_y}{u_x} + u_x^{\kappa}\right) u_{xy} + \left((\kappa+1)\,\frac{u_y^2}{u_x^2} - \frac{u_t}{u_x} + \kappa\, u_x^{\kappa}\, u_y + \frac{(\kappa+1)^2}{2\kappa+3}\, u_x^{2(\kappa+1)}\right) u_{xx}$$

Three contact integrable extensions:

1) $\mathcal{E}_2 \to \mathcal{E}_3$

2)

$$v_t = \frac{(\kappa+2)^2}{2\,\kappa+3}\,v_x^{2\,\kappa+3} - (\kappa+2)\,\left(\frac{u_y}{u_x} + u_x^{\kappa+1}\right)\,v_x^{\kappa+2}$$
$$+\left(\frac{u_t}{u_x} + (\kappa+2)\,u_x^{\kappa}u_y + \frac{(\kappa+1)(\kappa+2)}{2\,\kappa+3}\,u_x^{2\,\kappa+2}\right)\,v_x$$
$$v_y = -v_x^{\kappa+2} + \left(\frac{u_y}{u_x} + u_x^{\kappa+1}\right)\,v_x$$

Exclude $u \Longrightarrow$ the same equation for $v$.

3)

$$w_t = \left( \frac{u_t}{u_x} - (\kappa+1)(\kappa+2)\, u_x^{\kappa} \left( u_y - \frac{u_x^{\kappa+2}}{2\,\kappa+3} \right) \right) w_x$$

$$w_y = \left( \frac{u_y}{u_x} - (\kappa+1)\, u_x^{\kappa+1} \right) w_x$$

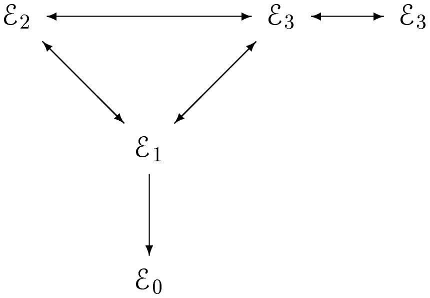
$\kappa \notin \{-2, -3/2, -1, 0\}$:

Exclude $u \Longrightarrow$ an equation of the third order for $w$

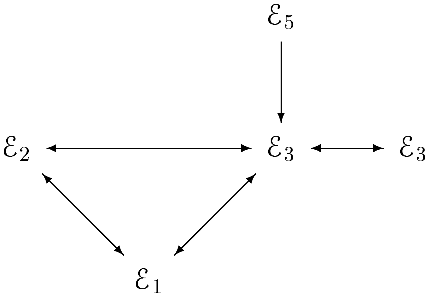
$\kappa = 0 \Longrightarrow \mathcal{E}_2 \to \mathcal{E}_3$

Details: `arXiv:1010.1828`

$\kappa = 0$:



$\kappa \notin \{-3, -2, -3/2, -1, 0\}$:



# r-mdKP equation

$\kappa = -3$: