Deformed cohomologies of symmetry pseudogroups and coverings of differential equations

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Deformed cohomologies

Deformed (exotic, twisted, covariant, ...) cohomologies:
the Morse theory for smooth multi-valued functionals,
symplectic geometry, algebraic topology, theory of Lie algebras

- S.P. Novikov, 1986, 2002, 2005
- L.A. Alaniya, 1997
- D.V. Millionshchikov, 2002,
- ...

Deformed cohomologies

$\mathfrak{g}$ – a Lie algebra over $\mathbb{R}$, $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ – its representation, $C^k(\mathfrak{g}, V)$ – the space of all k–linear skew-symmetric mappings from $\mathfrak{g}$ to V, $k \geq 1$, differential

$$d\theta(X_1, ..., X_{k+1}) = \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q)\left(\theta(X_1, ..., \hat{X}_q, ..., X_{k+1})\right) + \sum_{1 \leq p < q \leq k+1} (-1)^{p+q} \theta([X_p, X_q], X_1, ..., \hat{X}_p, ..., \hat{X}_q, ..., X_{k+1})$$

Chevalley–Eilenberg differential complex

$$\dots \xrightarrow{d} C^k(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \dots$$

Its cohomologies

$$H^k(\mathfrak{g}, V) = \frac{\ker d\colon C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)}{\mathrm{im}\, d\colon C^{k-1}(\mathfrak{g}, V) \longrightarrow C^k(\mathfrak{g}, V)}$$

$\rho_0\colon X \mapsto 0 \Rightarrow$ cohomologies with trivial coefficients $H^*(\mathfrak{g})$.

Deformed cohomologies

Let $d\omega = 0$, for $\lambda \in \mathbb{R}$ define the deformed differential

$$d_{\lambda\omega}\theta = d\theta + \lambda\,\omega \wedge \theta$$

Then $d\omega = 0 \Longrightarrow d_{\lambda\omega}^2 = 0$, so we have differential complex

$$\cdots \xrightarrow{d_{\lambda\omega}} C^k(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} C^{k+1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} \ldots$$

Its cohomologies are referred to as deformed cohomologies of $\mathfrak{g}$ and denoted by $H^*_{\lambda\omega}(\mathfrak{g})$

Deformed cohomologies

EXAMPLE. Consider the Lie algebra $\mathfrak{h}$ with generators X_1, ... , X_5 and non-zero commutators

$$\begin{aligned}
[X_1, X_2] &= \quad X_2, \\
[X_1, X_3] &= \ -X_3, \\
[X_1, X_4] &= -2\, X_4, \\
[X_2, X_3] &= \ -X_5.
\end{aligned}$$

Then for the dual 1-forms θ^i such that $\theta^i(X_j) = \delta^i_j$ we have

$$\begin{aligned}
d\theta^1 &= 0, \\
d\theta^2 &= -\theta^1 \wedge \theta^2, \\
d\theta^3 &= \quad \theta^1 \wedge \theta^3, \\
d\theta^4 &= 2\, \theta^1 \wedge \theta^4, \\
d\theta^5 &= \quad \theta^2 \wedge \theta^3.
\end{aligned}$$

We have $\quad H^1(\mathfrak{h}) = \mathbb{R}\, [\theta^1] = \mathbb{R}\, \theta^1 \quad$ and $\quad H^2(\mathfrak{h}) = \{0\} \Rightarrow$ there are no central extensions.

Deformed cohomologies

Then

$$H^2_{\lambda\,\theta^1}(\mathfrak{h}) = \begin{cases} \mathbb{R}\,[\theta^3 \wedge \theta^4], & \lambda = -3, \\ \mathbb{R}\,[\theta^1 \wedge \theta^4], & \lambda = -2, \\ \mathbb{R}\,[\theta^1 \wedge \theta^3] \oplus \mathbb{R}\,[\theta^2 \wedge \theta^4] \oplus \mathbb{R}\,[\theta^3 \wedge \theta^5], & \lambda = -1, \\ \mathbb{R}\,[\theta^1 \wedge \theta^2] \oplus \mathbb{R}\,[\theta^2 \wedge \theta^5], & \lambda = 1, \\ 0, & \text{otherwise} \end{cases}$$

Deformed cohomologies

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Deformed cohomologies

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$$H^2_{\lambda\,\theta^1}(\mathfrak{h}) = \begin{cases} \mathbb{R}\,[\theta^3 \wedge \theta^4], & \lambda = -3, \\ \mathbb{R}\,[\theta^1 \wedge \theta^4], & \lambda = -2, \\ \mathbb{R}\,[\theta^1 \wedge \theta^3] \oplus \mathbb{R}\,[\theta^2 \wedge \theta^4] \oplus \mathbb{R}\,[\theta^3 \wedge \theta^5], & \lambda = -1, \\ \mathbb{R}\,[\theta^1 \wedge \theta^2] \oplus \mathbb{R}\,[\theta^2 \wedge \theta^5], & \lambda = 1, \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$

$$d\omega - 3\,\theta^1 \wedge \omega = \theta^3 \wedge \theta^4$$

compatible with the structure equations of $\mathfrak{h}$.

Deformed cohomologies

Then

$$H^2_{\lambda\,\theta^1}(\mathfrak{h}) = \begin{cases} \mathbb{R}\,[\theta^3 \wedge \theta^4], & \lambda = -3, \\ \mathbb{R}\,[\theta^1 \wedge \theta^4], & \lambda = -2, \\ \mathbb{R}\,[\theta^1 \wedge \theta^3] \oplus \mathbb{R}\,[\theta^2 \wedge \theta^4] \oplus \mathbb{R}\,[\theta^3 \wedge \theta^5], & \lambda = -1, \\ \mathbb{R}\,[\theta^1 \wedge \theta^2] \oplus \mathbb{R}\,[\theta^2 \wedge \theta^5], & \lambda = 1, \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ non-central extension

$$\begin{aligned} d\omega^1 &= 3\,\theta^1 \wedge \omega^1 + \theta^3 \wedge \theta^4, \\ d\omega^2 &= 2\,\theta^1 \wedge \omega^2 + \theta^1 \wedge \theta^4, \\ d\omega^3 &= \theta^1 \wedge \omega^3 + \theta^1 \wedge \theta^3, \\ d\omega^4 &= \theta^1 \wedge \omega^4 + \theta^2 \wedge \theta^4, \\ d\omega^5 &= \theta^1 \wedge \omega^5 + \theta^3 \wedge \theta^5, \\ d\omega^6 &= -\theta^1 \wedge \omega^6 + \theta^1 \wedge \theta^2, \\ d\omega^7 &= -\theta^1 \wedge \omega^7 + \theta^2 \wedge \theta^5. \end{aligned}$$

Coverings of differential equations

A differential covering for a differential equation $\mathcal{E}$

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, ...) = 0, \quad i, j \in \{1, ..., n\}$$

is defined by an over-determined system

$$w_{\alpha, x^j} = T_{\alpha j}(x^i, u, u_{x^i}, u_{x^i x^j}, ..., w_\beta), \qquad \alpha, \beta \in \mathbb{N}$$

such that its compatibility conditions coincide with $\mathcal{E}$.

Geometrically, the covering is defined by the flat connection

$$D_{x^j} \mapsto \widetilde{D}_{x^j} = D_{x^j} + \sum_\alpha T_{\alpha j}(x^i, u, u_{x^i}, u_{x^i x^j}, ..., w_\beta) \frac{\partial}{\partial w_\alpha}$$

on $\mathcal{E} \times \mathcal{W} \to \mathcal{E}$,

$$[\widetilde{D}_{x^i}, \widetilde{D}_{x^j}] = 0 \quad \Leftrightarrow \quad F(x^i, u, u_{x^i}, u_{x^i x^j}, ...) = 0,$$

or by the Wahlquist–Estarbrook forms

$$\vartheta_\alpha = dw_\alpha - T_{\alpha k}(x^i, u, u_{x^i}, u_{x^i x^j}, ..., w_\beta)\, dx^k.$$

Coverings of differential equations

The general definition, details:

- I.S. Krasil′shchik, A.M. Vinogradov, Acta Appl. Math., 1984, Vol. 2, 79–86, Acta Appl. Math., 1989, Vol. 15, 161–209.
- Krasil′shchik I.S., Lychagin V.V., Vinogradov A.M. Geometry of jet spaces and nonlinear partial differential equations. N.Y.: Gordon and Breach, 1986
- Krasil′shchik I.S., Vinogradov A.V. (eds) Symmetries and Conservation Laws for Differential Equations of Mathematical Physics. Transl. Math. Monographs 182, Amer. Math. Soc., Providence, 1999

EXAMPLE. The potential Khokhlov–Zabolotskaya equation (Lin–Reissner–Tsien equation):

$$u_{yy} = u_{tx} + u_x\, u_{xx}$$

Covering equations (G.M. Kuz′mina 1967):

$$\begin{cases} v_t &= \frac{1}{3}\, v_x^3 - u_x\, v_x - u_y, \\ v_y &= \frac{1}{2}\, v_x^2 - u_x \end{cases}$$

Coverings of differential equations

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Put $v = w_0$, $v_x = w_1$, $v_{xx} = w_2$, etc., $\Rightarrow$ the extended total derivatives:

$$\widetilde{D}_t = D_t + \sum_{s=0}^{\infty} \tilde{D}_x^s \left(\tfrac{1}{3}\, w_1^3 - u_x\, w_1 - u_y\right)\, \frac{\partial}{\partial w_s},$$

$$\widetilde{D}_x = D_x + \sum_{s=0}^{\infty} w_{s+1}\, \frac{\partial}{\partial w_s},$$

$$\widetilde{D}_y = D_y + \sum_{s=0}^{\infty} \tilde{D}_x^s \left(\tfrac{1}{2}\, w_1^2 - u_x\right)\, \frac{\partial}{\partial w_s}.$$

Coverings of differential equations

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The Wahlquist–Estabrook forms

$$\vartheta_0 = dv - \left(\tfrac{1}{3}\, v_x^3 - u_x\, v_x - u_y\right)\, dt - v_x\, dx - \left(\tfrac{1}{2}\, v_x^2 - u_x\right)\, dy,$$

$$\vartheta_k = \widetilde{D}_x^k(\vartheta_0).$$

Lie pseudogroups & Cartan's method

A pseudogroup $\mathfrak{G}$ on a manifold M is a set of local diffeomorphisms $\Phi \colon \mathcal{U} \to \hat{\mathcal{U}}$, $\Phi \colon x \mapsto \hat{x}$, such that

1) if $\Phi \in \mathfrak{G}$, $\Psi \in \mathfrak{G}$, and if their composition $\Psi \circ \Phi$ is defined, then $\Psi \circ \Phi \in \mathfrak{G}$;
2) $\Phi \in \mathfrak{G} \Rightarrow \Phi^{-1} \in \mathfrak{G}$;
3) $\mathrm{id}_M \in \mathfrak{G}$;

A pseudogroup $\mathfrak{G}$ is called a Lie pseudogroup, if

4) the functions $\hat{x} = \Phi(x)$ are local analytic solutions of a system of PDEs (Lie equations of the pseudogroup $\mathfrak{G}$)

$$R\left(x, \Phi(x), \frac{\partial \Phi(x)}{\partial x}, ..., \frac{\partial^{\# I} \Phi(x)}{\partial x^I}\right) = 0.$$

Diffeomorphisms from a Lie pseudo-group can be characterized by the requirement to preserve a collection of 1-forms called invariant forms or Maurer-Cartan forms of this pseudo-group.

THEOREM (3^{rd} fundamental Lie's theorem in Cartan's form): For a Lie pseudo-group there exists a collection of Maurer– Cartan forms with involutive and compatible structure equations.

THEOREM (3^{rd} inverse fundamental Lie's theorem in Cartan's form): For a given involutive and compatible system of equations

$$d\omega^i = A^i_{\alpha j}\pi^\alpha \wedge \omega^j + \frac{1}{2}\, B^i_{jk}\, \omega^j \wedge \omega^k,$$

there exists a collection of 1-forms ω^1, ..., ω^m satisfying this system. The forms ω^1, ..., ω^m are Maurer–Cartan forms of a Lie pseudo-group.

- É. Cartan. Œuvres Complètes. Paris: Gauthier - Villars, 1953
- M.V. Vasil′eva. Structure of Infinite Lie Groups of Transformations. Moscow: MSPI, 1972 (in Russian)
- O. Stormark. Lie's Structural Approach to PDE Systems. Cambridge: CUP, 2000

Symmetry pseudogroups & Cartan's method

EXAMPLE. Maurer-Cartan forms of the symmetry pseudogroup of a differential equation can be found algorithmically by means of Cartan's method of equivalence.

Details, examples:

- P.J. Olver. Equivalence, invariants, and symmetry. Cambridge: CUP, 1995
- M. Fels, P.J. Olver. Acta Appl. Math., 1998, Vol. 51, 161–213
- O.M. J. Phys. A: Math. Gen., 2002, Vol. 35, 2965–2977
- O.M. J. Math. Sci., 2006, Vol. 135, 2680–2694

EXAMPLE. The pKhZ equation $\mathcal{E}_1$: $u_{yy} = u_{tx} + u_x\, u_{xx}$

Structure equations of the symmetry pseudogroup:

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3,\\
d\theta_1 &= (\eta_1 - \eta_2) \wedge \theta_1 - 2\,\eta_3 \wedge \theta_3 - \theta_0 \wedge (\xi^2 + \sigma_{22}) + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12} + \xi^3 \wedge \sigma_{13},\\
d\theta_2 &= \tfrac{1}{2}\,(\eta_1 - \eta_2) \wedge \theta_2 + \xi^1 \wedge \sigma_{12} + \xi^2 \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23},\\
d\theta_3 &= \tfrac{3}{4}\,(\eta_1 - \eta_2) \wedge 3\,\theta_3 - \eta_3 \wedge \theta_2 + \xi^1 \wedge \sigma_{13} + \xi^2 \wedge \sigma_{23} + \xi^3 \wedge \sigma_{12},\\
d\xi^1 &= \eta_2 \wedge \xi^1,\\
d\xi^2 &= \tfrac{1}{2}\,(\eta_1 + \eta_2) \wedge \xi^2 + \eta_3 \wedge \xi^3 - \theta_2 \wedge \xi^1,\\
d\xi^3 &= \tfrac{1}{4}\,(\eta_1 + 3\,\eta_2) \wedge \xi^3 + 2\,\eta_3 \wedge \xi^1,\\
d\sigma_{11} &= (\eta_1 - 2\,\eta_2) \wedge \sigma_{11} - 4\,\eta_3 \wedge \sigma_{13} - (\eta_4 - \theta_2) \wedge \theta_0 + \eta_6 \wedge \xi^2 + \eta_7 \wedge \xi^3 + \eta_8 \wedge \xi^1\\
&\qquad - 5\,\theta_1 \wedge (\xi^2 + \sigma_{22}) + \theta_2 \wedge \sigma_{12} - 2\,\theta_3 \wedge \sigma_{23},\\
d\sigma_{12} &= \tfrac{1}{2}\,(\eta_1 - 3\,\eta_2) \wedge \sigma_{12} - 2\,\eta_3 \wedge \sigma_{23} + \eta_4 \wedge \xi^2 + \eta_5 \wedge \xi^3 + \eta_6 \wedge \xi^1\\
&\qquad - 2\,\theta_2 \wedge (\xi^2 + \sigma_{22}),\\
d\sigma_{13} &= \tfrac{1}{4}\,(3\,\eta_1 - 7\,\eta_2) \wedge 3\,\sigma_{13} - 3\,\eta_3 \wedge \sigma_{12} + \eta_5 \wedge \xi^2 + \eta_6 \wedge \xi^3 + \eta_7 \wedge \xi^1\\
&\qquad - 3\,\theta_3 \wedge (\xi^2 + \sigma_{22}),\\
&\qquad \dots\\
d\eta_1 &= \xi^1 \wedge (\xi^2 + \sigma_{22}),\\
d\eta_2 &= -3\,\xi^1 \wedge (\xi^2 + \sigma_{22}),\\
&\qquad \dots
\end{aligned}
$$

THEOREM. $H^1(\mathrm{Sym}(\mathcal{E}_1)) = \mathbb{R}\,[\omega]$, $H^2_{-\omega}(\mathrm{Sym}(\mathcal{E}_1)) \ni \mathbb{R}\,[\Omega]$,

where $\omega = \frac{1}{4}\,(3\,\eta_1 + \eta_2)$ and

$\Omega = \frac{1}{4}\,(\eta_1 - \eta_2 - 4\,\eta_3) \wedge (\xi^1 + \xi^2 + \xi^3) + \theta_2 \wedge (\xi^1 + \xi^3) + \theta_3 \wedge \xi^1.$

CONJECTURE.

$$H^2_{\lambda\,\omega}(\mathrm{Sym}(\mathcal{E}_1)) = \begin{cases} \mathbb{R}\,[\Omega], & \lambda = -1, \\ \{0\}, & \lambda \neq -1. \end{cases}$$

COROLLARY. Equation

$$d\vartheta = \omega \wedge \vartheta + \Omega$$

is compatible with the structure equations of $\mathrm{Sym}(\mathcal{E}_1)$. We get

$\vartheta = a\,\left(dv - \left(\frac{1}{3}\,v_x^3 - u_x\,v_x - u_y\right)\,dt - v_x\,dx - \left(\frac{1}{2}\,v_x^2 - u_x\right)\,dy\right)$

with $a = u_{xxx}^{5/2}\,(u_{xxx}v_x - u_{xxy})^{-2}$.

This is the Wahlquist–Estabrook form of Kuz′mina's covering.

EXAMPLE. The Boyer–Finley equation $\mathcal{E}_2$

$$u_{tx} = \mathrm{e}^{u_y}\, u_{yy}.$$

Structure equations of the symmetry pseudogroup:

$$
\begin{aligned}
d\theta_0 &= \theta_0 \wedge (\theta_3 - \sigma_{33}) + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3,\\
d\theta_1 &= \eta_1 \wedge \theta_1 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{33} + \xi^3 \wedge \sigma_{13},\\
d\theta_2 &= \theta_2 \wedge (\eta_1 + \theta_3 + \xi^3) + \xi^1 \wedge \sigma_{33} + \xi^2 \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23},\\
d\theta_3 &= \xi^1 \wedge \sigma_{13} + \xi^2 \wedge \sigma_{23} + (\theta_3 + \sigma_{33}) \wedge \xi^3,\\
d\xi^1 &= (\sigma_{33} - \theta_3 - \eta_1) \wedge \xi^1,\\
d\xi^2 &= (\eta_1 + \sigma_{33} + \xi^3) \wedge \xi^2,\\
d\xi^3 &= (\sigma_{33} - \theta_3) \wedge \xi^3,\\
d\sigma_{11} &= (2\,\eta_1 + \theta_3 - \sigma_{33}) \wedge \sigma_{11} + \eta_2 \wedge \xi^3 + \eta_3 \wedge \xi^1 - \sigma_{13} \wedge \xi^2 + \theta_1 \wedge (\xi^2 + \sigma_{13}),\\
d\sigma_{13} &= (\eta_1 + \theta_3 - \sigma_{33}) \wedge \sigma_{13} + \eta_2 \wedge \xi^1 + (\theta_3 + 2\,\sigma_{33} - \xi^3) \wedge \xi^2,\\
d\sigma_{22} &= \sigma_{22} \wedge (2\,\eta_1 + \theta_3 + 2\,\xi^3 + \sigma_{33}) + \eta_4 \wedge \xi^3 + \eta_5 \wedge \xi^2 + \theta_2 \wedge (\xi^1 + \sigma_{23})\\
&\qquad - \sigma_{23} \wedge \xi^1,\\
d\sigma_{23} &= \sigma_{23} \wedge (\eta_1 + \xi^3 + \sigma_{33}) + \eta_4 \wedge \xi^2 + (\theta_3 + 2\,\sigma_{33} - \xi^3) \wedge \xi^1,\\
d\sigma_{33} &= \xi^1 \wedge \sigma_{13} + \xi^2 \wedge \sigma_{23} + \xi^3 \wedge (\sigma_{33} - \theta_3),\\
&\qquad \ldots
\end{aligned}
$$

Deformed cohomology and integrable extensions

THEOREM.

$$H^1(\mathrm{Sym}(\mathcal{E}_2)) = \mathbb{R}\,[\omega],$$

$$H^2_{-\omega}(\mathrm{Sym}(\mathcal{E}_2)) \ni \mathbb{R}\,[\Omega_1] \oplus \mathbb{R}\,[\Omega_2],$$

where

$$\omega = \sigma_{33} - \theta_3,$$

$$\Omega_1 = \eta_1 \wedge (\xi^1 + \xi^2 + \xi^3) - (\theta_1 + \xi^2) \wedge \xi^1 + (\theta_3 + \xi^3) \wedge \xi^2,$$

$$\Omega_2 = (\sigma_{33} - \theta_3) \wedge \xi^3.$$

CONJECTURE.

$$H^2_{\lambda\,\omega}(\mathrm{Sym}(\mathcal{E}_2)) = \begin{cases} \mathbb{R}\,[\Omega_1] \oplus \mathbb{R}\,[\Omega_2], & \lambda = -1, \\ \{0\}, & \lambda \neq -1. \end{cases}$$

COROLLARY. Equation

$$d\vartheta = \omega \wedge \vartheta + \Omega_1$$

is compatible with the structure equations of $\mathrm{Sym}(\mathcal{E}_2)$. Its solution

$$\vartheta = u_{yy}\,\left(dv - \left(u_t + e^{v_y}\right)\,dt + e^{u_y - v_y}\,dx - v_y\,dy\right)$$

is the Wahlquist–Estabrook form of the covering defined by the following system

$$\left\{\begin{array}{rcl} v_t & = & u_t + e^{v_y}, \\ v_x & = & -e^{u_y - v_y} \end{array}\right.$$

(V.E. Zakharov 1982; M.V. Saveliev, A.M. Vershik 1989; A.A. Malykh, Y. Nutku, M.B. Sheftel, P. Winternitz 1998).