Cartan's Structure Theory of Symmetry Pseudo-Groups, Zero-Curvature Representations and Bäcklund Transformations of Differential Equations.

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Lie pseudo-groups

A pseudo-group $\mathfrak{G}$ on a manifold M is a set of local diffeomorphisms $\Phi \colon \mathcal{U} \to \hat{\mathcal{U}}$, $\Phi \colon x \mapsto \hat{x}$ such that

1) if $\Phi \in \mathfrak{G}$, $\Psi \in \mathfrak{G}$, and their composition $\Psi \circ \Phi$ is defined, then $\Psi \circ \Phi \in \mathfrak{G}$;
2) $\Phi \in \mathfrak{G} \Rightarrow \Phi^{-1} \in \mathfrak{G}$;
3) $\mathrm{id}_M \in \mathfrak{G}$.

A pseudo-group $\mathfrak{G}$ is called a Lie pseudo-group, if

4) the functions $\hat{x} = \Phi(x)$ are local analytic solutions of a system of PDEs (Lie equations of the pseudo-group $\mathfrak{G}$)

$$R\left(x, \Phi(x), \frac{\partial \Phi(x)}{\partial x}, ..., \frac{\partial^{\#I} \Phi(x)}{\partial x^I}\right) = 0.$$

Maurer–Cartan forms of the Lie pseudo-group $\mathfrak{G}$: a collection of 1-forms

$$\omega^i \in \Omega^1(M \times N \times H), \qquad i \in \{1, ..., \dim M + \dim N\},$$

where N is a manifold, H is a finite Lie group.

A local diffeomorphism Φ on M, $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$ belongs to $\mathfrak{G}$ whenever there exists a fibre-preserving diffeomorphism Ψ on $M \times N \times H$, $\Psi\colon \mathcal{W} \to \hat{\mathcal{W}}$ such that

- Φ is the projection of Ψ w.r.t. $M \times N \times H \to M$;
- $\Psi^* \left(\omega^i|_{\hat{\mathcal{W}}}\right) = \omega^i|_{\mathcal{W}}$.

Élie Cartan's method of equivalence

Structure equations of a Lie pseudo-group $\mathfrak{G}$:

$$d\omega^i = A^i_{\alpha j}(U^\sigma)\,\pi^\alpha \wedge \omega^j + B^i_{jk}(U^\sigma)\,\omega^j \wedge \omega^k, \qquad B^i_{jk} = -B^i_{kj},$$

$$dU^\kappa = C^\kappa_j(U^\sigma)\,\omega^j,$$

$U^\sigma \colon M \to \mathbb{R}$, $\sigma \in \{1, ..., s\}$, $s < \dim M$, — invariants of the pseudo-group $\mathfrak{G}$

- π^α — depend on differentials of coordinates on H;
- involutivity conditions are satisfied,
- compatibility conditions are satisfied.

Maurer–Cartan forms and structure equations of a Lie pseudo-group can be found from its Lie equations algorithmically.

Involutivity conditions:

$$r^{(1)} = n \dim H - \sum_{k=1}^{n-1} (n-k)\, \sigma_k,$$

where $n = \dim M + \dim N$, $r^{(1)}$ is the dimension of the linear space of coefficients z^α_j such that the replacement $\pi^\alpha \mapsto \pi^\alpha + z^\alpha_j\, \omega^j$ preserves the structure equations;

$$\sigma_k = \max_{u_1,\ldots,u_k} \operatorname{rank} \mathbb{A}_k(u_1, \ldots, u_k) - \sum_{j=1}^{k-1} \sigma_j,$$

$$\mathbb{A}_1(u_1) = \left(A^i_{\alpha j}\, u^j_1 \right),$$

$$\mathbb{A}_q(u_1, \ldots, u_q) = \left(\begin{array}{c} \mathbb{A}_{q-1}(u_1, \ldots, u_{q-1}) \\ A^i_{\alpha j}\, u^j_q \end{array} \right), \; q \in \{2, \ldots, n-1\}.$$

Compatibility conditions:

- $d(d\omega^i) = 0 = d\left(A^i_{\alpha j}\,\pi^\alpha \wedge \omega^j + B^i_{jk}\,\omega^j \wedge \omega^k\right)$
- $d(dU^\kappa) = 0 = d(C^\kappa_j\,\omega^j)$

$\Longrightarrow$

over-determined system for the coefficients $A^i_{\alpha j}$, B^i_{jk}, C^κ_j;

Élie Cartan's method of equivalence

THEOREM (*Third fundamental Lie's theorem in Cartan's form*): For a Lie pseudo-group there exists a collection of Maurer–Cartan forms with involutive and compatible structure equations.

THEOREM (*Third inverse fundamental Lie's theorem in Cartan's form*): For a given involutive and compatible system of structure equations there exists a collection of 1-forms ω^1, ..., ω^n and functions U^1, ..., U^s satisfying this system. The forms ω^1, ..., ω^m are Maurer–Cartan forms of a Lie pseudo-group, and the functions U^1, ..., U^s are invariants of this pseudo-group.

Élie Cartan's method of equivalence

- É. Cartan. Œuvres Complètes, Paris: Gauthier - Villars, Vol. 2, Part II, 1953
- Vasil'eva M.V. Structure of Infinite Lie Groups of Transformations. Moscow: MSPI, 1972 (in Russian)
- Gardner R.B. The method of equivalence and its applications. CBMS–NSF regional conference series in applied math., SIAM, Philadelphia, 1989.
- Olver P.J. Equivalence, Invariants, and Symmetry. Cambridge: Cambridge University Press, 1995
- Stormark O. Lie's Structural Approach to PDE Systems. Cambridge: CUP, 2000

Contact transformations

- Trivial bundle: $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\pi\colon (x^i, u) \mapsto (x^i)$
- Jets of the second order: $J^2(\pi)$, (x^i, u, u_i, u_{ij}), $u_{ij} = u_{ji}$
- Contact forms : $\vartheta_0 = du - u_j\, dx^j$, $\vartheta_i = du_i - u_{ij}\, dx^j$
- Pseudo-group of contact transformations $\mathrm{Cont}(J^2(\pi))$:

  $$\Psi\colon J^2(\pi) \to J^2(\pi),\ \Psi\colon (x^i, u, u_i, u_{ij}) \mapsto (\hat{x}^i, \hat{u}, \hat{u}_i, \hat{u}_{ij})$$

  such that Ψ preserves the algebraic ideal of contact forms:

  $$\Psi^*(d\hat{u} - \hat{u}_j\, d\hat{x}^j) = a\,(du - u_j\, dx^j),$$
  $$\Psi^*(d\hat{u}_i - \hat{u}_{ij}\, d\hat{x}^j) = P_i^j\,(du_j - u_{jk}\, dx^k) + Q_i\,(du - u_j\, dx^j),$$
  $$\Psi^* d\hat{x}^i = b_j^i\, dx^j + R^i\,(du - u_j\, dx^j) + S^{ij}\,(du_j - u_{jk}\, dx^k),$$
  $$a \neq 0,\ \det(b_j^i) \neq 0,\ \det(P_i^j) \neq 0$$

Contact transformations

Maurer–Cartan forms for $\mathrm{Cont}(J^2(\pi))$:

$\Theta_0 = a\,(du - u_i\,dx^i)$,
$\Theta_i = a\,B^j_i\,(du_j - u_{jk}\,dx^k) + g_i\,\Theta_0$,
$\Theta_{ij} = a\,B^k_i\,B^l_j\,(du_{kl} - u_{klm}\,dx^m) + s_{ij}\,\Theta_0 + w^k_{ij}\,\Theta_k$,
$\Xi^i = b^i_j\,dx^j + c^i\,\Theta_0 + f^{ij}\,\Theta_j$,

where $a \neq 0$, $\det(b^i_j) \neq 0$, $b^i_k\,B^k_j = \delta^i_j$, $f^{ik} = f^{ki}$,
$s_{ij} = s_{ji}$, $w^k_{ij} = w^k_{ji}$, $u_{klm} = u_{lkm} = u_{kml}$

Structure equations

$d\Theta_0 = \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i$,
$d\Theta_i = \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Theta_{ik}$,
$d\Theta_{ij} = \Phi^k_i \wedge \Theta_{kj} - \Phi^0_0 \wedge \Theta_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Xi^k \wedge \Theta_{ijk}$,
$d\Xi^i = \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k$

Symmetry pseudo-groups of PDEs

- PDE of the second order: $\iota : \mathcal{E} \to J^2(\pi)$
- Contact symmetries of $\mathcal{E}$ — contact transformations which map $\mathcal{E}$ into itself: $\mathrm{Cont}(\mathcal{E}) \subset \mathrm{Cont}(J^2(\pi))$,
- **Maurer–Cartan forms for** $\mathrm{Cont}(\mathcal{E})$ **can be found from the reduced forms** $\theta_0 = \iota^* \Theta_0$, $\theta_i = \iota^* \Theta_i$, $\theta_{ij} = \iota^* \Theta_{ij}$, $\xi^i = \iota^* \Xi^i$, **by procedures of Cartan's equivalence method**
- Details:
  - Fels M., Olver P.J. Moving coframes I. A practical algorithm. // Acta Appl. Math., 1998, Vol. 51, pp. 161–213
  - Morozov O.I. Moving coframes and symmetries of differential equations. // J. Phys. A: Math. Gen., 2002, Vol. 35, pp. 2965 – 2977

Coverings (Lax pairs, Bäcklund transformations, prolongation structures, zero - curvature representations, integrable extensions, ...):

- Lax P.D. // Comm. Pure Appl. Math., 1969, Vol. 21, pp. 467 – 490
- V.E. Zakharov, A.B. Shabat. // Funct. Analysis Appl. 1974, Vol. 6, No 6, pp. 43 – 54
- H.D. Wahlquist, F.B. Estabrook, 1975, // J. Math. Phys., 1975, Vol. 16, pp. 1 – 7
- I.S. Krasil'shchik, A.M. Vinogradov, // Acta Appl. Math., 1984, Vol. 2, pp. 79–86
- I.S. Krasil'shchik, A.M. Vinogradov // Acta Appl. Math., 1989, Vol. 15, pp. 161–209

Coverings of differential equations

- Infinite jet bundle $J^\infty(\pi)$,
- Coordinates $(x^i, u, u_i, u_{ij}, ..., u_I, ...)$, $I = (i_1, i_2, ..., i_m)$,
- Infinitely prolonged differential equation

$$\mathcal{E}^\infty \subset J^\infty(\pi),$$

- Total derivatives

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\#I \ge 0} u_{Ii} \frac{\partial}{\partial u_I}, \qquad \bar{D}_i = D_i|_{\mathcal{E}^\infty}.$$

Coverings of differential equations

- Covering over $\mathcal{E}^\infty$:

$$\tau : \widetilde{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \mathcal{V} \to \mathcal{E}^\infty, \qquad \mathcal{V} = \{(v^\kappa) \mid 0 \le \kappa \le \infty\},$$

- Extended total derivatives

$$\widetilde{D}_i = \bar{D}_i + \sum_\kappa T_i^\kappa(x^j, u_I, v^\rho)\,\frac{\partial}{\partial v^\kappa},$$

$$[\widetilde{D}_i, \widetilde{D}_j] = 0 \iff (x^i, u_I) \in \mathcal{E}^\infty$$

- Extended contact forms (Wahlquist-Estabrook forms)

$$\widetilde{\vartheta}^\kappa = dv^\kappa - T_i^\kappa(x^j, u_I, v^\rho)\,dx^i$$

Coverings of differential equations

The problem of recognizing whether a given differential equation has a covering is of great importance. Different techniques were proposed to solve it.

$n = 2$.

- H.D. Wahlquist, F.B. Estabrook, 1975
- R. Dodd, A. Fordy, 1983
- C. Hoenselaers, 1986
- S.Yu. Sakovich, 1995
- M. Marvan, 1997
- S. Igonin, 2006
- ...

Coverings of differential equations

The problem is much more difficult in the case of $n > 2$:

- G.M. Kuz'mina, 1967
- H.C. Morris, 1976
- V.E. Zakharov, 1982
- G.S. Tondo, 1985
- M. Marvan, 1992
- B.K. Harrison, 2002
- ...

Coverings of differential equations

G.M. Kuz'mina. On a possibility to reduce a system of two partial differential equations of the first order to a single equation of the second order. // Proc. Moscow State Pedagogical Institute, 1967, Vol. 271, 67–76 (in Russian)

$$u_{yy} = u_{tx} + u\,u_{xx} + u_x^2 \qquad \text{(dispersionless KP)}$$

Covering

$$\begin{cases} v_t = (v^2 - u)\,v_x - u_y - v\,u_x, \\ v_y = v\,v_x - u_x \end{cases}$$

Excluding u: define w such that $w_x = v$ and $w_y = \frac{1}{2}\,v^2 - u$, then

$$w_{yy} = w_{tx} + \left(\tfrac{1}{2}\,w_x^2 - w_y\right)\,w_{xx} \qquad \text{(modified dKP)}$$

The central idea: to apply Cartan's structure theory of Lie pseudo-groups

Integrable extensions

Bryant R.L., Griffiths P.A. Characteristic Cohomology of Differential Systems (II): Conservation Laws for a Class of Parabolic Equations. Duke Math. J., 1995, Vol. 78, pp. 531–676:

$\mathbf{n = 2}$, finite-dimensional coverings

Definition 1. Let

$$d\omega^i = A^i_{\alpha j}\,\pi^\alpha \wedge \omega^j + B^i_{jk}\,\omega^j \wedge \omega^k, \tag{1}$$

$$dU^\kappa = C^\kappa_j\,\omega^j \tag{2}$$

be structure equations of a Lie pseudo-group $\mathfrak{G}$. Its coeffi- cients are supposed to be functions of the invariants U^σ of $\mathfrak{G}$. Consider the system

$$d\tau^q = D^q_{\rho r}\,\eta^\rho\wedge\tau^r + E^q_{rs}\,\tau^r\wedge\tau^s + F^q_{r\beta}\,\tau^r\wedge\pi^\beta$$

$$+G^q_{rj}\,\tau^r \wedge \omega^j + H^q_{\beta j}\,\pi^\beta \wedge \omega^j + I^q_{jk}\,\omega^j \wedge \omega^k, \tag{3}$$

$$dV^\epsilon = J^\epsilon_j\,\omega^j + K^\epsilon_q\,\tau^q, \tag{4}$$

with unknown 1-forms τ^q, $q \in \{1, ..., Q\}$, η^ρ, $\rho \in \{1, ..., R\}$, and unknown functions V^ϵ, $\epsilon \in \{1, ..., S\}$, $Q, R, S \in \mathbb{N}$. The coefficients of this system are supposed to be functions of U^σ and V^ϵ). System (3), (4) is called an integrable exten- sion of system (1), (2), if equations (1) – (4) are simultane- ously compatible and involutive.

Suppose system (3), (4) is an integrable extension of system (1), (2). Then, in accordance with the third inverse fundamental theorem of Lie, system (1)–(4) defines a Lie pseudo-group $\mathfrak{H}$.

Definition 2. The integrable extension (3), (4) is called trivial , if there exists a change of variables on the manifold of action of the pseudo-group $\mathfrak{H}$ such that in the new variables equations (3), (4) do not contain the forms ω^j, π^β, and the coefficients of (3), (4) do not depend on U^q. Otherwise, the integrable extension is called non-trivial .

Let θ^α_K, ξ^j be Maurer–Cartan forms of the pseudo-group $\mathrm{Cont}(\mathcal{E})$ of symmetries for a PDE $\mathcal{E}$ such that θ^α_K are contact forms (their restrictions on each solution of the equation $\mathcal{E}$ are equal to 0), and ξ^j are horizontal forms ($\xi^1 \wedge \ldots \wedge \xi^n \neq 0$ on each solution).

Definition 3. Nontrivial integrable extension of the structure equations of the pseudo-group $\mathrm{Cont}(\mathcal{E})$

$$d\omega^q = \Pi^q_r \wedge \omega^r + \xi^j \wedge \Omega^q_j$$

is called contact integrable extension when

- $\Omega^q_j \equiv 0 \pmod{\theta^\alpha_K, \omega^q_j}$ for a set of additional forms ω^q_j;
- $\Omega^q_j \not\equiv 0 \pmod{\omega^q_j}$
- coefficients of expansions of Ω^q_j w.r.t. $\{\theta^\alpha_I, \omega^r_i\}$ and Π^q_r w.r.t. $\{\theta^\alpha_I, \xi^j, \omega^r, \omega^r_i\}$ depend on the invariants of $\mathrm{Cont}(\mathcal{E})$ and, maybe, on a set of additional functions W^ρ, $\rho \in \{1, \dots, \Lambda\}$, $\Lambda \geqslant 1$.
- In the latter case there exist functions $P^{I\rho}_\alpha$, Q^ρ_q, $R^{j\rho}_q$, S^ρ_j such that

  $$dW^\rho = P^{I\rho}_\alpha\,\theta^\alpha_I + Q^\rho_q\,\omega^q + R^{j\rho}_q\,\omega^q_j + S^\rho_j\,\xi^j.$$

  These equations are required to satisfy the compatibility conditions.

Plebañski's second heavenly equation

The second heavenly equation (J.F. Plebañski, J. Math. Phys., 1975, Vol. 16, pp. 2395 – 2402):

$$u_{xz} = u_{ty} + u_{yy}\, u_{zz} - u_{yz}^2$$

Covering:

$$\begin{cases} v_t = (u_{yz} + \lambda)\, v_z - u_{zz}\, v_y, \\ v_x = u_{yy}\, v_z - (u_{yz} - \lambda)\, v_y \end{cases}$$

- J.F. Plebañski, ibid
- Viquar Husain, Phys. Rev. Lett., 1994, Vol. 72, pp. 800–803
- L.V. Bogdanov, B.G. Konopelchenko, Phys. Lett. A, 2005, Vol. 345, pp. 137–143

Plebañski's second heavenly equation

THEOREM. The symmetry pseudo-group of the second heavenly eqution has two contact integrable extensions with the following Wahlquist–Estabrook forms:

$$\begin{aligned}\omega_1 &= q_1\,(dv + (v_{zz}\,v_y - (u_{yz}+\lambda)\,v_z)\,dt \\ &\quad +((u_{yz}-\lambda)\,v_y - u_{yy}\,v_z)\,dx - v_y\,dy - v_z\,dz)\,,\end{aligned}$$

with $\lambda = \mathrm{const}$ and

$$\begin{aligned}\omega_2 &= q_2\,(dv + (v_{zz}\,v_y - (u_{yz}+v)\,v_z)\,dt \\ &\quad +((u_{yz}-v)\,v_y - u_{yy}\,v_z)\,dx - v_y\,dy - v_z\,dz)\,,\end{aligned}$$

Plebañski's second heavenly equation

The first form corresponds to the known covering of the second heavenly equation, while the second form gives its new covering

$$\left\{ \begin{array}{lcl} v_t & = & (u_{yz} + v)\, v_z - u_{zz}\, v_y, \\ v_x & = & u_{yy}\, v_z - (u_{yz} - v)\, v_y \end{array} \right.$$

Details:

- O.I. Morozov, Global and Stochastic Analysis, 2011, Vol.1, pp. 89 – 102 (arXiv : 1104.3011)

Generalized (2+1)-dDym equation

$$u_{ty} = u_y\, u_{xx} + 2\,(2\,\kappa + 1)\, u_x\, u_{xy} + u_y^{8\kappa+5} u_{yy}$$

- $\kappa = -\frac{1}{2}$:
  - E.V. Ferapontov, K.R. Khusnutdinova,
    Comm. Math. Phys., 2004, Vol. 248, pp. 187 – 206
  - V.S. Dryuma, 2007
  - E.V. Ferapontov, A. Moro, V.V. Sokolov,
    Comm. Math. Phys., 2009, Vol. 285, pp. 31 – 65
- $\kappa = 0$:
  - E.V. Ferapontov, A.V. Odesskii, N.M. Stoilov,
    `arXiv:1007.3782`
- $\kappa = -\frac{5}{8}$:
  - O.I. Morozov, J. Geom. Phys., 2009, Vol. 59, pp. 1461 – 1475

Generalized (2+1)-dDym equation

THEOREM. When $\kappa \notin \{-\frac{5}{8}, -\frac{3}{4}, -\frac{1}{2}\}$, the symmetry pseudo-group of the generalized (2+1)-dDym equation has two contact integrable extensions with the Wahlquist–Estabrook forms

$$\omega_0 = \frac{u_{xy}}{u_y^{4\kappa+3} v_y} \Big(dv - \lambda\, u_y^{4\,\kappa+2}\, v_y\, dx - v_y\, dy - 2\,(2\,\kappa+1)\, u_y^{4\,\kappa+2}\, v_y\, (\lambda\, u_x - (4\,\kappa+3)^{-1}\, u_y^{4\,\kappa+3})\, dt \Big)$$

and

$$\omega_0 = \frac{u_{xy}}{u_y^{4\kappa+3} H^{2\,\kappa+1}} \Big(dw - u_y^{4\kappa+2}\, H^{2\kappa+1}\, dx - w_y\, dy - H^{2\,\kappa+1}\, u_y^{4\,\kappa+2}\, (\alpha\, u_x + \beta\, u_y^{4\,\kappa+3}\, H^{2\,\kappa}\, H')\, dt \Big),$$

where the function $H = H(w_y)$ is a solution of the ODE

$$H' = (2\,\kappa+1)^{-1}\,H^{-2\,\kappa}\,\sqrt{H+\lambda^2},$$

while $\alpha = 2\,(2\,\kappa+1)$, $\beta = 2\,(2\,\kappa+1)^2(8\,\kappa+5)^{-1}$, and $\lambda^2 = -(8,\kappa+5)(4\,\kappa+3)^{-1}$.

When $\kappa = -\frac{3}{4}$, the symmetry pseudo- group of the generalized (2+1)-dDym equation has a contact integrable extension with the Wahlquist–Estabrook form

$$\omega_0 = u_{xy}\,G'\,\left(dw - \frac{1}{u_y\,G'}\,(dx + (G-u_x)\,dt) - w_y\,dy\right),$$

where the function $G = G(w_y)$ is a solution of the following ODE:

$$G' = \exp\left(\tfrac{1}{2}\,G^2\right).$$

Generalized (2+1)-dDym equation

The corresponding coverings are defined by the systems

$$\left\{\begin{array}{lcl} v_t & = & 2\,(2\,\kappa+1)\,u_y^{4\,\kappa+2}\,v_y\,(\lambda\,u_x-(4\,\kappa+3)^{-1}\,u_y^{4\,\kappa+3}), \\ v_x & = & \lambda\,u_y^{4\,\kappa+2}\,v_y, \end{array}\right.$$

$$\left\{\begin{array}{lcl} w_t & = & H^{2\,\kappa+1}\,u_y^{4\,\kappa+2}\,(\alpha\,u_x+\beta\,u_y^{4\,\kappa+3}\,H^{2\,\kappa}\,H'), \\ w_x & = & u_y^{4\,\kappa+2}\,H^{2\,\kappa+1}, \end{array}\right.$$

when $\kappa \notin \{-\frac{5}{8},-\frac{3}{4},-\frac{1}{2}\}$ and

$$\left\{\begin{array}{lcl} w_t & = & \dfrac{G-u_x}{u_y\,G'}, \\ w_x & = & \dfrac{1}{u_y\,G'}, \end{array}\right.$$

when $\kappa = -\frac{3}{4}$.

These systems define Bäcklund transformations from the generalized (2+1)-dDym equation to the equations

$$
\begin{aligned}
v_{ty} &= \left(\frac{v_x}{\lambda\, v_y}\right)^{\frac{1}{4\kappa+2}} v_{xx} + \left(\frac{v_x}{\lambda\, v_y}\right)^{\frac{8\kappa+5}{4\kappa+2}} v_{yy} \\
&\quad + \left(\frac{v_t}{v_x} + \lambda^{-\frac{6\kappa+4}{2\kappa+1}}\, \frac{4\kappa-2}{4\kappa+3} \left(\frac{v_x}{\lambda\, v_y}\right)^{\frac{4\kappa+3}{4\kappa+2}}\right) v_{xy}, \\
w_{ty} &= H^{-\frac{1}{2}}\, w_x^{\frac{1}{4\kappa+2}}\, w_{xx} + w_x^{\frac{8\kappa+5}{4\kappa+2}}\, H^{-\frac{8\kappa+5}{2}}\, w_{yy} \\
&\quad + \left(\frac{w_t}{w_x} - \frac{4\kappa+2}{8\kappa+5}\, w_x^{\frac{4\kappa+3}{4\kappa+2}}\, H^{-\frac{4\kappa+3}{2}}\, (H+\lambda^2)^{\frac{1}{2}}\right) w_{xy},
\end{aligned}
$$

and

$$
w_{ty} = \frac{1}{w_x \exp\left(\frac{1}{2}\, G^2\right)}\, w_{xx} + \frac{w_t + w_x^2}{w_x}\, w_{xy} + w_x \exp\left(\frac{1}{2}\, G^2\right)\, w_{yy}.
$$