Lax representations, integrable hierarchies, and non-central extensions of symmetry algebras of differential equations

Oleg I. Morozov

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EXAMPLE 1. The hyper-CR equation for Einstein–Weyl structures

$$u_{yy} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy} \qquad (\mathcal{E}_1)$$

- G.M. Kuz′mina, 1967
- V.G. Mikhalev, 1992
- M.V. Pavlov, 2003
- M. Dunajski, 2004

Lax representation with a non-removable parameter:

$$\left\{ \begin{array}{lcl} v_t & = & (\lambda^2 - \lambda\, u_x - u_y)\, v_x, \\ v_y & = & (\lambda - u_x)\, v_x. \end{array} \right.$$

The hyper-CR equation

Algebra of contact symmetries $\mathrm{Sym}(\mathcal{E}_1)$

$$
\begin{aligned}
\phi_0(A) &= -A u_t - \tfrac{1}{2}\, y\,(y u_x - 2x) A'' - A'(x u_x + y u_y - u) + \tfrac{1}{6}\, y^3\, A''', \\
\phi_1(A) &= -y\, A'\, u_x - A\, u_y + x\, A' + \tfrac{1}{2}\, y^2\, A'', \\
\phi_2(A) &= -A\, u_x + y\, A', \\
\phi_3(A) &= A, \\
\psi_0 &= -2\, x\, u_x - y\, u_y + 3\, u, \\
\psi_1 &= -y\, u_x + 2\, x,
\end{aligned}
$$

where $A = A(t)$.

Commutators:

$$
\begin{aligned}
\{\phi_i(A), \phi_j(B)\} &= \phi_{i+j}(A\, B' - B\, A'), \\
\{\psi_i, \phi_k(A)\} &= -k\, \phi_{k+i}(A), \\
\{\psi_0, \psi_1\} &= -\psi_1.
\end{aligned}
$$

The hyper-CR equation

$$\mathrm{Sym}(\mathcal{E}_1) = \mathfrak{p}_{\diamond} \ltimes \mathfrak{p}_{4,\infty},$$

$$\mathfrak{p}_{\diamond} = \langle \psi_0, \psi_1 \rangle,$$

$$\mathfrak{p}_{4,\infty} = \langle \phi_0(A), \ldots \phi_3(A) \rangle = \mathbb{R}_3[h] \otimes \mathfrak{w}[t],$$

where $\mathfrak{w}[t] = \langle t^i \, \partial_t \mid i \in \mathbb{N}_0 \rangle$, $\quad \mathbb{R}_3[h] = \mathbb{R}[h]/(h^4 = 0)$.

The hyper-CR equation

Maurer–Cartan forms: α_0, α_1, $\theta_{k,n}$, $k \in \{0, ..., 3\}$, $n \geq 0$,

$$\alpha_i(\psi_j) = \delta_{ij}, \qquad \alpha_i(\phi_k(t^n)) = 0,$$

$$\theta_{k,n}(\psi_i) = 0, \qquad \tfrac{1}{m!}\,\theta_{k,n}(\phi_l(t^m)) = \delta_{kl}\delta_{nm}$$

Denote $\quad \Theta = \sum\limits_{k=0}^{3} h_0^k \sum\limits_{m=0}^{\infty} \frac{1}{m!}\, h_1^m\, \theta_{k,m}, \qquad \nabla_i = \frac{\partial}{\partial_{h_i}},$

then the structure equations of $\mathrm{Sym}(\mathcal{E}_1)$ take the form

$$\left\{ \begin{array}{lcl} d\alpha_0 & = & 0, \\ d\alpha_1 & = & \alpha_0 \wedge \alpha_1, \\ d\Theta & = & \nabla_1(\Theta) \wedge \Theta + (h_0\,\alpha_0 + h_0^2\,\alpha_1) \wedge \nabla_0(\Theta), \end{array} \right.$$

(recall that $h_0^k = 0$ when $k > 3$).

The hyper-CR equation

From the structure equations we have:

$$H^1(\mathrm{Sym}(\mathcal{E}_1)) = \mathbb{R}\,\alpha_0,$$

$$H^2_{c\alpha_0}(\mathfrak{p}_\diamond) = \begin{cases} \mathbb{R}\,[\alpha_0 \wedge \alpha_1], & c = 1, \\ \{[0]\}, & c \neq 1. \end{cases}$$

Moreover,

$$H^2_{\alpha_0}(\mathfrak{p}_\diamond) \subseteq H^2_{\alpha_0}(\mathrm{Sym}(\mathcal{E}_1)),$$

hence equation

$$d\sigma = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1$$

is compatible with the structure equations of $\mathrm{Sym}(\mathcal{E}_1)$ and defines a non-central extension of this Lie algebra.

The hyper-CR equation

We have ($r \neq 0$, $q, s \in \mathbb{R}$ are parameters):

$$\begin{aligned}
\alpha_0 &= dq, \\
\alpha_1 &= -\mathrm{e}^q\, ds, \\
\sigma &= \mathrm{e}^q\,(dv - q\, ds), \\
\theta_{0,0} &= r\, dt, \\
\theta_{1,0} &= r\,\mathrm{e}^q\,(dy + p_1\, dt), \\
\theta_{2,0} &= r\,\mathrm{e}^{2q}\,(dx + (p_1 - s)\, dy + p_2\, dt), \\
\theta_{3,0} &= p\,\mathrm{e}^{3q}\,(du + (p_1 - 2\,s)\, dx + (p_2 - s\, p_1 + s^2)\, dy + p_3\, dt).
\end{aligned}$$

We know

$$\theta_{3,0} = p\,\mathrm{e}^{3q}\,(du - u_x\, dx - u_y\, dy - u_t\, dt)$$

$$\Rightarrow \qquad \left\{ \begin{array}{lcl} p_1 - 2\, s & = & -u_x, \\ p_2 - s\, p_1 + s^2 & = & -u_y, \\ p_3 & = & -u_t. \end{array} \right.$$

The hyper-CR equation

Consider

$$\sigma - \theta_{2,0} =$$
$$\mathrm{e}^{q}\,\big(dv - q\,ds - r\,\mathrm{e}^{q}\,\big(dx + (s-u_x)\,dy + (s^2 - s\,u_x - u_y)\,dt\big)\big),$$

rename $\quad q = v_s, \quad r = v_x\,\exp(-v_s) \qquad \Rightarrow$

$$\sigma - \theta_{2,0} =$$
$$\mathrm{e}^{v_s}\,\big(dv - v_s\,ds - v_x\,(dx + (s-u_x)\,dy + (s^2 - s\,u_x - u_y)\,dt)\big).$$

Then $\sigma - \theta_{2,0} = 0 \qquad \Leftrightarrow$

$$\left\{\begin{array}{lcl} v_t & = & (s^2 - s\,u_x - u_y)\,v_x, \\ v_y & = & (s - u_x)\,v_x. \end{array}\right.$$

$\Rightarrow\;\; \sigma - \theta_{2,0}$ is the Wahlquist–Estabrook form for the Lax representation of the hyper-CR equation, $\lambda = s$.

Integrable hierarchy associated to the hyper-CR equation

Consider the Lie algebra

$\mathfrak{p}_\diamond \ltimes \mathfrak{p}_{n+1,\infty}$, where $\mathfrak{p}_{n+1,\infty} = \mathbb{R}_n[h] \otimes \mathfrak{w}[t]$.

The structure equations: the same system

$$\left\{ \begin{array}{lcl} d\alpha_0 & = & 0, \\ d\alpha_1 & = & \alpha_0 \wedge \alpha_1, \\ d\Theta & = & \nabla_1(\Theta) \wedge \Theta + (h_0\,\alpha_0 + h_0^2\,\alpha_1) \wedge \nabla_0(\Theta) \end{array} \right.$$

with

$$\Theta = \sum_{k=0}^{n} h_0^k \sum_{m=0}^{\infty} \tfrac{1}{m!}\, h_1^m\, \theta_{k,m}.$$

Integrable hierarchy associated to the hyper-CR equation

Rename: $t \mapsto t_{n-1}$, $y \mapsto t_{n-2}$, $x \mapsto t_{n-3}$, then

$$\theta_{n,0} = r\,\mathrm{e}^{nq}\,\left(du - \sum_{i=0}^{n-1} u_{t_i} dt_i\right)$$

$\Rightarrow$

$$\sigma - \theta_{n-1,0} =$$
$$\mathrm{e}^{v_s}\,\left(dv - v_s\,ds - v_{t_0}\,dt_0 - \sum_{i=1}^{n-1}\left(s^i - \sum_{j=0}^{i-1} s^{i-j-1}\,u_{t_j}\right) v_{t_0}\,dt_i\right).$$

Integrable hierarchy associated to the hyper-CR equation

$$\left\{\begin{array}{lcl} v_{t_1} & = & (s - u_{t_0})\, v_{t_0}, \\ v_{t_2} & = & (s^2 - s\, u_{t_0} - u_{t_1})\, v_{t_0}, \\ & & \dots \\ v_{t_i} & = & \left(s^i - \sum\limits_{j=0}^{i-1} s^{i-j-1}\, u_{t_j}\right) v_{t_0}, \\ & & \dots \\ v_{t_{n-1}} & = & \left(s^{n-1} - s^{n-2}\, u_{t_0} - s^{n-3}\, u_{t_1} - \dots - s\, u_{t_{n-3}} - u_{t_{n-2}}\right)\, v_{t_0}. \end{array}\right.$$

- M. Dunajski, 2004,
- M.V. Pavlov, 2003; L.V. Bogdanov, M.V. Pavlov, 2017

Integrable hierarchy associated to the hyper-CR equation

Denote by $\mathcal{H}_{n-1}$ the compatibility conditions of this system. Then $\mathcal{H}_2$ is the hyper-CR equation

$$u_{t_1t_1} = u_{t_0t_2} + u_{t_1}\, u_{t_0t_0} - u_{t_0}\, u_{t_0t_1},$$

$\mathcal{H}_3$ is the hyper-CR equation plus

$$u_{t_1t_2} = u_{t_0t_3} + u_{t_2}\, u_{t_0t_0} - u_{t_0}\, u_{t_0t_2},$$
$$u_{t_1t_3} = u_{t_2t_2} + u_{t_1}\, u_{t_0t_2} - u_{t_2}\, u_{t_0t_1},$$

$\mathcal{H}_4$ consists of $\mathcal{H}_3$ plus

$$u_{t_0t_4} = u_{t_2t_2} + u_{t_0}\, u_{t_0t_3} - u_{t_3}\, u_{t_0t_0} + u_{t_1}\, u_{t_0t_2} - u_{t_2}\, u_{t_0t_1},$$
$$u_{t_1t_4} = u_{t_2t_3} + u_{t_0}\, u_{t_0t_3} - u_{t_3}\, u_{t_0t_1},$$
$$u_{t_2t_4} = u_{t_3t_3} + u_{t_2}\, u_{t_0t_3} - u_{t_3}\, u_{t_0t_2},$$

etc., system $\mathcal{H}_{n-1}$ consists of equations from $\mathcal{H}_{n-2}$ supplemented by equations

$$u_{t_{i-1}t_n} = u_{t_it_{n-1}} + u_{t_0}u_{t_0t_{n-1}} - u_{t_{n-1}}u_{t_0t_{i-1}}, \quad 1 \le i \le n-2.$$

Integrable hierarchy associated to the hyper-CR equation

Denote by $\mathcal{H}_{n-1}$ the compatibility conditions of this system. Then $\mathcal{H}_2$ is the hyper-CR equation

$$u_{t_1 t_1} = u_{t_0 t_2} + u_{t_1}\, u_{t_0 t_0} - u_{t_0}\, u_{t_0 t_1},$$

$\mathcal{H}_3$ is the hyper-CR equation plus

$$\begin{aligned} u_{t_1 t_2} &= u_{t_0 t_3} + u_{t_2}\, u_{t_0 t_0} - u_{t_0}\, u_{t_0 t_2} \\ u_{t_1 t_3} &= u_{t_2 t_2} + u_{t_1}\, u_{t_0 t_2} - u_{t_2}\, u_{t_0 t_1}: \end{aligned}$$

$\mathcal{H}_4$ consists of $\mathcal{H}_3$ plus

$$\begin{aligned} u_{t_0 t_4} &= u_{t_2 t_2} + u_{t_0}\, u_{t_0 t_3} - u_{t_3}\, u_{t_0 t_0} + u_{t_1}\, u_{t_0 t_2} - u_{t_2}\, u_{t_0 t_1}, \\ u_{t_1 t_4} &= u_{t_2 t_3} + u_{t_0}\, u_{t_0 t_3} - u_{t_3}\, u_{t_0 t_1}, \\ u_{t_2 t_4} &= u_{t_3 t_3} + u_{t_2}\, u_{t_0 t_3} - u_{t_3}\, u_{t_0 t_2}, \end{aligned}$$

etc., system $\mathcal{H}_{n-1}$ consists of equations from $\mathcal{H}_{n-2}$ supplemented by equations

$$u_{t_{i-1} t_n} = u_{t_i t_{n-1}} + u_{t_0} u_{t_0 t_{n-1}} - u_{t_{n-1}} u_{t_0 t_{i-1}}, \quad 1 \le i \le n-2.$$

Reduced quasi-classical self-dual Yang–Mills equation

EXAMPLE 2: rqsdYM equation:

$$u_{yz} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy} \qquad (\mathcal{E}_2)$$

- E.V. Ferapontov, K.R. Khusnutdinova, 2004

$$\left\{ \begin{array}{lcl} v_t & = & \lambda\, v_y - u_y\, v_x, \\ v_z & = & (\lambda - u_x)\, v_x. \end{array} \right.$$

The symmetry algebra: $\mathrm{Sym}(\mathcal{E}_2) = \mathfrak{q}_\diamond \ltimes \mathfrak{q}_{3,\infty}$,

$\mathfrak{q}_\diamond = \mathfrak{a} \ltimes (\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathfrak{v})$, $\dim \mathfrak{a} = 1$, $\dim \mathfrak{v} = 2$,

$\mathfrak{q}_{3,\infty} = \mathbb{R}_2[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[z]$.

Reduced quasi-classical self-dual Yang–Mills equation

The structure equations for $\mathrm{Sym}(\mathcal{E}_2)$:

$$\left\{\begin{array}{lcl} d\alpha & = & 0, \\ d\mathrm{B} & = & \nabla_1(\mathrm{B}) \wedge \mathrm{B}, \\ d\Gamma & = & \alpha \wedge \Gamma + \nabla_1(\Gamma) \wedge \mathrm{B} + \frac{1}{2}\,\nabla_1(\mathrm{B}) \wedge \Gamma, \\ d\Theta & = & \nabla_2(\Theta) \wedge \Theta + \nabla_1(\Theta) \wedge (\mathrm{B} + h_0\,\Gamma) \\ & & +h_0\,\nabla_0(\Theta) \wedge \left(\frac{1}{2}\,\nabla_1(\mathrm{B}) + h_0\,\nabla_1(\Gamma) - \alpha\right), \end{array}\right.$$

$$\mathrm{B} = \beta_0 + h_1\,\beta_1 + \frac{1}{2}\,h_1\,\beta_2,$$

$$\Gamma = \gamma_0 + h_1\,\gamma_1,$$

$$\Theta = \sum_{k=0}^{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{h_0^k h_1^i h_2^j}{i!j!}\,\theta_{k,i,j}.$$

Reduced quasi-classical self-dual Yang–Mills equation

$H^1(\mathrm{Sym}(\mathcal{E}_2)) = \mathbb{R}\,\alpha,$

$$H^2_{c\alpha}(\mathfrak{q}_\diamond) = \left\{ \begin{array}{ll} \mathbb{R}\,[\gamma_0 \wedge \gamma_1], & c = 2, \\ \{[0]\}, & c \neq 2. \end{array} \right.$$

$H^2_{\alpha}(\mathfrak{q}_\diamond) \subseteq H^2_{\alpha}(\mathrm{Sym}(\mathcal{E}_2)),$

$\Rightarrow$

non-central extension for $\mathrm{Sym}(\mathcal{E}_2)$ with the additional structure equation

$$d\sigma = 2\,\alpha \wedge \sigma + \gamma_0 \wedge \gamma_1.$$

Integration: $\theta_{2,0,0}$ is a multiple of the contact form $\Rightarrow$

$\sigma - \gamma_1 - \theta_{1,0,0}$ defines the covering

$$\left\{ \begin{array}{lcl} v_t & = & s\,v_y - u_y\,v_x - \frac{1}{2}\,s^2, \\ v_z & = & (s - u_x)\,v_x \end{array} \right.$$

over the rqsdYM.

Integrable hierarchy associated with rqsdYM

Replace

$$\mathfrak{q}_\diamond \ltimes (\mathbb{R}_2[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[z]) \quad\mapsto\quad \mathfrak{q}_\diamond \ltimes (\mathbb{R}_n[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[z]),$$

integrate the same structure equations with

$$\Theta = \sum_{k=0}^{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{h_0^k h_1^i h_2^j}{i!j!}\, \theta_{k,i,j},$$

rename $t \mapsto y_1$, $x \mapsto t_0$, $y \mapsto y_0$, $z \mapsto t_1$ $\Rightarrow$

$$\left\{\begin{array}{lcl} v_{y_1} & = & s\, v_{y_0} - u_{y_0}\, v_{t_0} - \frac{1}{2}\, s^2 \\ v_{t_1} & = & (s - u_{t_0})\, v_{t_0}, \\ v_{t_2} & = & (s^2 - s\, u_{t_0} - u_{t_1})\, v_{t_0}, \\ & & \dots \\ v_{t_i} & = & \left(s^i - \sum\limits_{j=0}^{i-1} s^{i-j-1}\, u_{t_j}\right) v_{t_0}, \\ & & \dots \\ v_{t_{n-1}} & = & \left(s^{n-1} - s^{n-2}\, u_{t_0} - s^{n-3}\, u_{t_1} - \dots - s\, u_{t_{n-3}} - u_{t_{n-2}}\right) v_{t_0}. \end{array}\right.$$

Integrable hierarchy associated with rqsdYM

Compatibility conditions: $\mathcal{H}_{n-1}$ supplemented by the rqsdYM equation

$$u_{t_0 y_1} = u_{t_1 y_0} - u_{y_0}\, u_{t_0 t_0} + u_{t_0}\, u_{t_0 y_0}$$

and system

$$u_{t_i y_1} = u_{t_{i+1} y_0} - u_{y_0}\, u_{t_0 t_i} + u_{t_i} u_{t_0 y_0}, \quad 0 \le i \le n-2.$$

The 4D universal hierarchy equation

EXAMPLE 3. 4D UH equation :

$$u_{zz} = u_{tx} + u_z u_{xy} - u_x u_{yz} \qquad (\mathcal{E}_3)$$

- L.V. Bogdanov, M.V. Pavlov, 2017,
- M.V. Pavlov, N. Stoilov, 2017.

Lax representation with a non-removable parameter:

$$\left\{ \begin{array}{lcl} v_t & = & \lambda^2\, v_x - (\lambda\, u_x + u_z)\, v_y, \\ v_z & = & \lambda\, v_x - u_x\, v_y. \end{array} \right.$$

The 4D universal hierarchy equation

The symmetry algebra

$$\mathrm{Sym}(\mathcal{E}_3) = \mathfrak{r}_\diamond \ltimes (\mathbb{R}_1[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[y])$$

The structure equations:

$$\left\{ \begin{array}{rcl} d\beta_1 & = & 0, \\ d\beta_2 & = & \beta_1 \wedge \beta_2, \\ d\beta_3 & = & 0, \\ d\beta_4 & = & (\beta_3 - \beta_1) \wedge \beta_4, \\ d\beta_5 & = & \beta_3 \wedge \beta_5 - \beta_2 \wedge \beta_4, \\ d\beta_6 & = & (2\,\beta_3 - \beta_1) \wedge \beta_6 + \frac{1}{2}\,\beta_4 \wedge \beta_5. \\ d\Theta_0 & = & \nabla_2(\Theta_0) \wedge \Theta_0 + \nabla_1(\Theta_0) \wedge (\beta_2 + h_1\,\beta_1) \\ d\Theta_1 & = & \nabla_2(\Theta_1) \wedge \Theta_0 + \nabla_2(\Theta_0) \wedge \Theta_1 + (\beta_1 - \beta_3) \wedge \Theta_1 \\ & & +(\beta_2 + h_1\,\beta_1) \wedge \nabla_1(\Theta_1) + (\beta_5 + h_1\,\beta_4) \wedge \nabla_1(\Theta_0). \end{array} \right.$$

Integrable hierarchy associated to 4D UH equation

The second non-central extension of $\mathfrak{r}_\diamond$ provides the above covering over 4D UH equation.

Hierarchy: replace $\mathbb{R}_1[h]$ by $\mathbb{R}_n[h]$, extend the structure equations correctly, then integrate $\Rightarrow$

$$
\left\{
\begin{array}{lcl}
v_{y_1} & = & s\, v_{y_0} - u_{y_0}\, v_{t_0}\,, \\
v_{y_2} & = & s^2\, v_{y_0} - (u_{y_1} + s\, u_{y_0})\, v_{t_0}\,, \\
v_{t_1} & = & (s - u_{t_0})\, v_{t_0}\,, \\
v_{t_2} & = & (s^2 - s\, u_{t_0} - u_{t_1})\, v_{t_0}\,, \\
 & & \dots \\
v_{t_i} & = & \left(s^i - \sum\limits_{j=0}^{i-1} s^{i-j-1}\, u_{t_j}\right) v_{t_0}\,, \\
 & & \dots \\
v_{t_{n-1}} & = & \left(s^{n-1} - s^{n-2}\, u_{t_0} - s^{n-3}\, u_{t_1} - \dots - s\, u_{t_{n-3}} - u_{t_{n-2}}\right)\, v_{t_0}\,.
\end{array}
\right.
$$

Integrable hierarchy associated to 4D UH equation

Compatibility conditions: $\mathcal{H}_{n-1}$ supplemented by the 4D UH equation

$$u_{y_0 y_2} = u_{y_1 y_1} + u_{y_0}\, u_{t_0 y_1} - u_{y_1}\, u_{t_0 y_0}$$

and system

$$u_{t_k y_0} = u_{t_{k-2} y_2} + u_{y_1}\, u_{t_0 t_{k-2}} - u_{t_{k-2}}\, u_{t_0 y_1} + u_{y_0}\, u_{t_0 t_{k-1}} - u_{t_{k-1}}\, u_{t_0 y_0},$$

$$u_{t_m y_1} = u_{t_{m-1} y_2} + u_{y_1}\, u_{t_0 t_{m-1}} - u_{t_{m-1}}\, u_{t_0 y_1}$$

with $2 \leq k \leq n-1$, $1 \leq m \leq n-1$.