Lie algebras of maximal class

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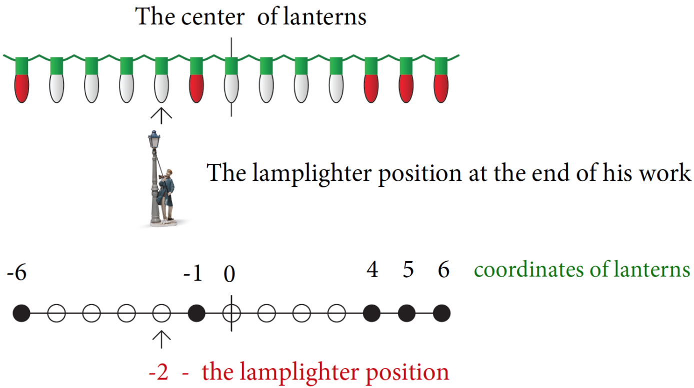
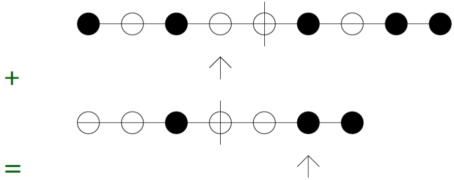
- Lamplighter groups
- Lamplighter Lie algebra
- Filiform Lie algebras and Lie algebras of maximal class
- Parafree Lie algebras
- The theorem by Ivanov, Mikhailov and Zaikovsky on the pro-nilpotent completion of a free Lie algebra
- Answer to Felix and Murillo's question

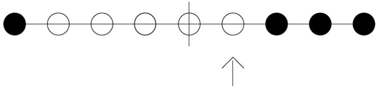
Lamplighter group by Jim Cannon



System state data: ({-6,-1,4,5,6}, -2)

The lamplighter product





the result

As a set it is $\mathbb{Z} \times \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$ with the multiplication

$$(k, \{a_n\}) \star (l, \{b_m\}) = (k+l, \{c_n\}), \quad c_n = a_n + b_{n-k}, n \in \mathbb{Z}.$$

It is more convenient to consider the Laurent polynomials $\sum_{n\in\mathbb{Z}} a_n x^n, a_n \in \mathbb{Z}_2$:

$$\left(k, \sum_{n\in\mathbb{Z}} a_n x^n\right) \star \left(l, \sum_{n\in\mathbb{Z}} b_n x^n\right) = \left(k+l, \sum_{n\in\mathbb{Z}} a_n x^n + x^k \sum_{n\in\mathbb{Z}} b_n x^n\right).$$

The integer lamplighter group $L(\mathbb{Z})$ and matrices

Consider 2×2-matrices

$$\begin{pmatrix} x^k & p(x) \\ 0 & 1 \end{pmatrix}, k \in \mathbb{Z},\ p(x) \in \mathbb{Z}_2[x, x^{-1}].$$

With the standard matrix multiplication.

The integer lamplighter group $L(\mathbb{Z})$, defined by Sergey Ivanov and Roman Mikhailov (2021) – they considered Laurent polynomials $p(t)$ with integer coefficients $p(x) \in \mathbb{Z}[x, x^{-1}]$.

The lamplighter Lie algebra

Definition (Ivanov, Mikhailov, Zaikovskii 2021)

The rational lamplighter Lie algebra is defined as the semidirect product $\mathfrak{l} = \mathbb{Q}t \ltimes \mathbb{Q}[x]$ of one-dimensional $\mathbb{Q}t$ and infinite-dimensional abelian $\mathbb{Q}[x]$ with relations

$$[t, p(x)] = xp(x), \quad p(x) \in \mathbb{Q}[x].$$

Filiform Lie algebras

A Lie algebra $\mathfrak{g}$ is nilpotent, if it exists $s > 0$

$$\mathfrak{g}^1 = \mathfrak{g} \supset \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}^s \supset \mathfrak{g}^{s+1} = [\mathfrak{g}, \mathfrak{g}^s] = 0, \ \mathfrak{g}^s \neq 0.$$

There is an estimate $s \leq \dim \mathfrak{g} - 1$.

Definition (Vergne, 1970)

A finite-dimensional Lie algebra $\mathfrak{g}$ is called filiform, if $s = \dim \mathfrak{g} - 1$.

Example

$\mathfrak{m}_0(n)$ is defined by a basis $e_1, e_2, \dots, e_n$ and relations

$$[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, \quad [e_i, e_k] = 0, \ i, k \neq 1, i+k > n.$$

Pro-nilpotent Lie algebras

Definition

A Lie algebra $\mathfrak{g}$ is called pro-nilpotent, if

$$\cap_{k=1}^{\infty}\mathfrak{g}^k = \{0\}, \quad \dim \mathfrak{g}/\mathfrak{g}^k < \infty, \ k = 1, 2, \dots$$

Consider an inverse spectrum of nilpotent Lie algebras

$$\dots \xrightarrow{p_{k+2,k+1}} \mathfrak{g}/\mathfrak{g}^{k+1} \xrightarrow{p_{k+1,k}} \mathfrak{g}/\mathfrak{g}^k \xrightarrow{p_{k,k-1}} \dots \xrightarrow{p_{3,2}} \mathfrak{g}/\mathfrak{g}^2 \xrightarrow{p_{2,1}} \mathfrak{g}/\mathfrak{g}^1,$$

Its inverse limit $\widehat{\mathfrak{g}} = \varprojlim_k \mathfrak{g}/\mathfrak{g}^k$ is called *completion* of $\mathfrak{g}$.

A pro-nilpotent Lie algebra $\mathfrak{g}$ is called *complete* if the inclusion $\mathfrak{g} \subset \widehat{\mathfrak{g}}$ is an isomorphism $\mathfrak{g} \cong \widehat{\mathfrak{g}}$.

Example

A Lie algebra $\mathfrak{m}_0$ is defined by its infinite basis $e_1, e_2, e_3, \dots$ and relations

$$[e_1, e_i] = e_{i+1}, i \geq 2, \quad [e_i, e_k] = 0,\ i, k \neq 1.$$

The Lie algebra $\mathfrak{m}_0 = \oplus_{i=1}^{+\infty} \langle e_i \rangle$ is not complete.

Its copletetion $\widehat{\mathfrak{m}}_0$ is the Lie algebra $\widehat{\mathfrak{m}}_0 = \prod_{i=1}^{+\infty} \langle e_i \rangle$ of formal power series $\sum_{i=1}^{+\infty} \alpha_i e_i$.

Remark

Lie algebras $\mathfrak{m}_0$ and the lamplighter algebra $\mathfrak{l}$ are isomorphic.

Denote by $t = e_1$ and $e_2 = 1, e_3 = x, e_4 = x^2, \dots$

Lie algebras of maximal class

Definition (Shalev, Zelmanov, 1990)

The co-class $cc(\mathfrak{g})$ of the pro-nilpotent Lie algebra $\mathfrak{g}$ is a number, possibly equal to $+\infty$, defined by the formula

$$cc(\mathfrak{g}) = \sum_{i=1}^{+\infty} \left(\dim(\mathfrak{g}^i/\mathfrak{g}^{i+1}) - 1 \right).$$

Example

$$cc(\mathfrak{m}_0) = (2-1) + (1-1) + (1-1) + \cdots = 1$$

Definition

The Lie algebra of coclass one $cc(\mathfrak{g}) = 1$ is called the Lie algebra **of maximal class**.

A classification

Theorem (Shalev and Zelmanov 1997, Fialowski 1983)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *be an infinite-dimensional* $\mathbb{N}$*-graded Lie algebra*

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, i, j \in \mathbb{N},$$

of maximal class $cc(\mathfrak{g}) = 1$*. Then* $\mathfrak{g}$ *is isomorphic to one Lie algebra of three given ones*

$$\mathfrak{m}_0,\ \mathfrak{m}_2,\ W^+,$$

where W^+ *is the positive part of the Witt algebra and the Lie algebra* $\mathfrak{m}_2$ *is defined by relations*

$$[e_1, e_i] = e_{i+1}, i \geq 2, \quad [e_2, e_j] = e_{j+2}, j \geq 3.$$

Definition

A Lie algebra $\mathfrak{g}$ is called parafree if
1) $\cap_{k=1}^{\infty}\mathfrak{g}^k = \{0\}$;
2) there is a free Lie algebra $\mathcal{L}_{\mathfrak{g}}$ and a homomorphism $\mathcal{L}_{\mathfrak{g}} \to \mathfrak{g}$, which induces the isomorphisms $\mathcal{L}_{\mathfrak{g}}/\mathcal{L}_{\mathfrak{g}}{}^k \to \mathfrak{g}/\mathfrak{g}^k$ for all $k = 1, 2, \dots$.

Example (Baur, Stammbach, 1980)

$\mathfrak{g} = \langle x \rangle \oplus \mathcal{L}(z, y_0, y_1, y_2, \dots)$ the semidirect sum of the one-dimensional $\langle x \rangle$ and the free Lie algebra generated by $z, y_0, y_1, \dots$ with relations

$$[x, y_i] = y_{i+1}, i \geq 0,$$
$$[x, z] = z + [y_1, y_0].$$

For a parafree Lie algebra $\mathfrak{g}$ there exists an isomorphism $\widehat{\mathcal{L}_{\mathfrak{g}}} \cong \widehat{\mathfrak{g}}$.

The famous **Stallings-Swan theorem** states that **a group of cohomological dimension one** is **free**.

Bourbaki in the book "Lie Groups and Algebras Chapter 2, §2, asked: "Is a similar statement true for Lie algebras, **is a Lie algebra of cohomological dimension one free?**"

Feldman (UMN, 1983) answered this question in the affirmative for two-generated Lie algebras.

Mikhalev (Jr.), Umirbaev and Zolotykh in 1996 constructed a **example of a non-free Lie algebra of cohomological dimension one** over a field of characteristic $p > 2$.

The question in the case **of characteristic zero** and characteristic $p = 2$ is still **open**.

One of the goals of Ivanov, Mikhailov, and Zaikovskii in the framework of the Baumslag program is to construct examples of countable (generally speaking, not finitely generated) **parafree groups with nonzero H_2** and (or) cohomological dimension greater than two. In 2021 they built countable parafree Lie algebras with nonzero H_2, as well as an example of a parafree Lie algebra of cohomological length greater than two.
During the proof, they made an interesting observation: **the second homology $H_2(\widehat{\mathcal{L}}(2), \mathbb{Z})$ of the pro-nilpotent completion of the free Lie algebra $\mathcal{L}(2)$ in two generators have uncountable dimension** moreover, for the free Lie algebra $\mathcal{L}(2)$ they are trivial $H_2(\mathcal{L}(2), \mathbb{Z}) = 0$.

Ivanov, Mikhailov, and Zaikovskii used the homology properties **of the lamplighter Lie algebra** $\mathfrak{l}$ over the ring $\mathbb{Z}$. In particular, they presented the homomorphism

$$\varphi : \mathcal{L}(2) \to \mathfrak{l},$$

initiating the mapping

$$H_2(\widehat{\mathcal{L}}(2), \mathbb{Z}) \to H_2(\widehat{\mathfrak{l}}, \mathbb{Z})$$

in homology with the image

$$Im\varphi \cong \mathbb{Z}[[u]],$$

isomorphic to the ring of integral formal series in the variable u.

Homology of chain algebra $(\Lambda^*(\mathfrak{g}), \delta)$ with coefficients in the ring R with boundary operator δ

$$\delta(x_1 \wedge \cdots \wedge x_p) = \sum_{1 \le i < j \le p} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \dots \widehat{x_i} \dots \widehat{x_i} \cdots \wedge x_p,$$

for all $p \geq 2$. In particular $x, y, z \in \mathfrak{g}$,

$$\begin{cases} \delta(x \wedge y) = -[x, y], \\ \delta(x \wedge y \wedge z) = -[x, y] \wedge z + x \wedge [y, z] - y \wedge [x, z]. \end{cases}$$

Define the homology $H_*(\mathfrak{g}, R)$ as homology $(\Lambda^*(\mathfrak{g}), \delta)$.

Félix, Murillo (2021) showed explicitly how much the two chain complexes associated with completions be different

$$\Lambda^*(\widehat{\mathcal{L}}(2)) \subset \widehat{\Lambda^*}(\mathcal{L}(2)).$$

Two-dimensional homology H_2 of the complex $\widehat{\Lambda}(\mathcal{L}(2))$ are trivial, but we can find an uncountable set of chains of the form $c \in \widehat{\Lambda^3}(\mathcal{L}(2))$, whose differentials δc lie in a smaller subcomplex $\Lambda^2(\widehat{\mathcal{L}}(2))$ and there they define non-trivial two-dimensional homology classes.

Félix, Murillo (2021) showed that in the case of the lamplighter Lie algebra $\mathfrak{l}$ all the q-homology subspaces $H_q(\widehat{\mathfrak{l}}, \mathbb{Q})$ are uncountable.

Remark

This fact is a consequence of the theorem by M. and Fialowski (2008), in which the bigraded cohomology $H^p_{(q)}(\mathfrak{m}_0, \mathbb{Q})$ Lie algebra of maximal class $\mathfrak{m}_0 \cong \mathfrak{l}$.

Question Félix, Murillo, 2021, TAMS.

Let $\mathfrak{g}$ be a finitely generated Lie algebra over $\mathbb{Q}$ такая, что $\cap_{n=1}^{+\infty}\mathfrak{g}^k = 0$. Suppose $\dim \widehat{\mathfrak{g}} = \infty$, does $H_2(\widehat{\mathfrak{g}}, \mathbb{Q})$ always have uncountable dimensions? The answer is **No**. Example. $\mathfrak{m}_2$ is a

Lie algebra of maximal class, which is given by the basis $e_1, e_2, e_3, \dots$ and relations

$$[e_1, e_i] = e_{i+1}, [e_2, e_j] = e_{j+2}, i \geq 2, j \geq 3.$$

It followas from the computations in M., Fialowski, J. of Algebra (2008) that

$$\dim H_2(\widehat{\mathfrak{m}_2}, \mathbb{Q}) = 2.$$

The little prince and the lamplighter

The little prince and the lamplighter

The fifth planet was very strange. It was the smallest of all. There was just enough room on it for a street lamp and a lamplighter. The little prince was not able to reach any explanation of the use of a street lamp and a lamplighter, somewhere in the heavens, on a planet which had no people, and not one house. But he said to himself, nevertheless:

"It may well be that this man is absurd. But he is not so absurd as the king, the conceited man, the businessman, and the tippler. For at least his work has some meaning. When he lights his street lamp, it is as if he brought one more star to life, or one flower. When he puts out his lamp, he sends the flower, or the star, to sleep. That is a beautiful occupation. And since it is beautiful, it is truly useful."