On integrable classes of surfaces in the Euclidean space

Michal Marvan
Silesian University in Opava, Czech Republic

This poster concerns a joint project with H. Baran of classification of integrable PDEs describing immersed surfaces in \( \mathbb{R}^3 \). The integrability criterion we apply is the existence of an \( \mathfrak{sl}(n) \)-valued curvature representation depending on a non-removable parameter.

**Aims** of the project:

- obtaining lists of integrable classes of surfaces, as complete as possible
- identifying known cases
- finding mutual transformations
- obtaining new PDE integrable in the sense of soliton theory.

**Criterion of integrability**

In the case of two independent variables \( x, y \) and a matrix Lie algebra \( \mathfrak{g} \), a \( \mathfrak{g} \)-valued **zero curvature representation** (ZCR) for a system of PDE \( \mathcal{E} \) is defined to be a form \( \alpha = A \, dx + B \, dy \) with \( A, B \in \mathfrak{g} \) such that

\[
D_y A - D_x B + [A, B] = 0 \pmod{\mathcal{E}}.
\]

A ZCR is the compatibility condition for the linear system

\[
D_x \Psi = A \Psi, \quad D_y \Psi = B \Psi.
\]

A **gauge transformation** with respect to a gauge matrix \( S \) is \( \Psi \mapsto S \Psi \), i.e.,

\[
A \mapsto (D_x S)S^{-1} + SAS^{-1}, \quad B \mapsto (D_y S)S^{-1} + SBS^{-1}.
\]
The zero curvature representation (ZCR) involving a parameter not removable by a gauge transformation is a prerequisite to integrability.

The classification problem

How to tell whether a given nonlinear system has a zero curvature representation?


Description of the method

Solve the determining system
\[
(D_y A - D_x B + [A, B])|_\epsilon = 0,
\]
\[
\sum_{I,J} (-\hat{D}) I \left( \frac{\partial F^l}{\partial u^I} C_l \right) |_{\epsilon} = 0
\]

Properties of the determining system

– is a system of differential equations in total derivatives
– is quasilinear in $A, B$ and linear in $C_l$
– usually possible to solve using computer algebra
– solution algorithms are resource demanding
– computation splits into cases to avoid division by zero (a consequence of nonlinearity in $A, B$).

The spectral parameter problem

Given a parameterless ZCR, when a parameter can be incorporated?

A cohomological solution

To solve the spectral parameter problem in a given Lie algebra:
1. compute the cohomological obstructions, resulting from expanding the zero curvature representation in terms of the (prospective) spectral parameter

\[ A = \sum_i A_i \lambda^i, \quad B = \sum_i B_i \lambda^i \]

\[ D_y A_0 - D_x B_0 + [A_0, B_0] = 0 \quad \text{(the seed ZCR)}, \]
\[ D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0, \]
\[ D_y A_2 - D_x B_2 + [A_2, B_0] + [A_1, B_1] + [A_0, B_2] = 0, \]

etc.

2. compute the full zero curvature representation using the information obtained in the first step to cut off branches.


The classification project

We consider geometrically determined classes of surfaces, meaning classes determined by a single condition

\[ F(p_1, \ldots, p_k) = 0, \]

where \( p_i \) are differential invariants with respect to reparameterizations and euclidean motions (principal curvatures, their gradients, etc.).

We classify relations \( F = 0 \) such that

- the associated Gauss–Mainardi–Codazzi equations possess a ZCR depending on a nonremovable (spectral) parameter;
- the ZCR has a prescribed order \( r \) and takes values in a prescribed Lie algebra \( \mathfrak{sl}(n) \).

Weingarten surfaces

To start with, we focused on Weingarten surfaces, i.e., classes of immersed surfaces in \( \mathbb{E}^3 \) determined by a functional relation between the principal curvatures \( k_1, k_2 \).

Thus, the classification problem is: Which functional relations

\[ f(k_1, k_2) = 0 \]
determine an integrable class of Weingarten surfaces?

Examples

Classical integrable classes: $K = k_1k_2 = \text{const}$, $H = \frac{1}{2}(k_1 + k_2) = \text{const}$, more generally, $aK + bH + c = \text{const}$ (linear Weingarten surfaces).

Forgotten integrable classes: E.g., $1/k_1 - 1/k_2 = \text{const}$.


Preliminaries

Parameterized by the lines of curvature, surfaces $r(x, y)$ have the fundamental forms

$$I = u^2 \, dx^2 + v^2 \, dy^2, \quad II = \frac{u^2}{\rho} \, dx^2 + \frac{v^2}{\sigma} \, dy^2,$$

where $\rho, \sigma$ are the principal radii of curvature, $\rho = 1/k_1, \sigma = 1/k_2$.

In the Weingarten case, $\rho = \rho(\sigma)$, the Mainardi–Codazzi subsystem can be explicitly solved. The full GMC system then reduces to the Gauss equation alone, which can be written in the form

$$R_{xx} + S_{yy} + T = 0,$$

where $R, S, T$ are functions of $\sigma$.

Results of computation

Weingarten surfaces determined by an explicit dependence $\rho(\sigma)$ possess a one-parametric zero curvature representation if the determining equation

$$\rho'''' = \frac{3}{2\rho^2} \rho'' + \frac{\rho' - 1}{\rho - \sigma} \rho'' + 2 \frac{(\rho' - 1)\rho' (\rho' + 1)}{(\rho - \sigma)^2}$$

holds (the prime denotes $d/d\sigma$).


The determining equation has two geometric symmetries:

- scaling (changing the ruler) $\rho \mapsto e^T \rho$, $\sigma \mapsto e^T \sigma$;
− translation (offsetting, normal shift) \( \rho \mapsto \rho + T, \sigma \mapsto \sigma + T \).

These symmetries help us to reduce the order by two. The resulting 1st order ODE is separable.

The general solution \( \rho(\sigma) \) is given by the elliptic integral

\[
\rho + \sigma = \frac{1}{m} \int m^{(\rho - \sigma)} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, ds.
\]

Here \( m \) is a scaling parameter, the integration constant is an offsetting parameter, and \( c \) is a “true” parameter.

**Summary of the special cases**

Except \( \rho = \text{const} \) and \( \sigma = \text{const} \), the special cases when the above elliptic integral reduces to elementary functions are, up to scaling and offsetting,

<table>
<thead>
<tr>
<th>relation</th>
<th>integrable equation</th>
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<tbody>
<tr>
<td>( \rho + \sigma = 0 )</td>
<td>( z_{xx} + z_{yy} + c^2 = 0 )</td>
</tr>
<tr>
<td>( \rho \sigma = 1 )</td>
<td>( z_{xx} + z_{yy} - \sinh z = 0 )</td>
</tr>
<tr>
<td>( \rho \sigma = -1 )</td>
<td>( z_{xx} - z_{yy} + \sin z = 0 )</td>
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<tr>
<td>( \rho - \sigma = \sinh(\rho + \sigma) )</td>
<td>((\tanh z - z)<em>{xx} + (\coth z - z)</em>{yy} + \csch 2z = 0)</td>
</tr>
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</tr>
<tr>
<td>( \rho - \sigma = 1 )</td>
<td>( z_{xx} + (1/z)_{yy} + 2 = 0 )</td>
</tr>
<tr>
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<tr>
<td>( \rho - \sigma = -\cot \rho )</td>
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</table>

All the special cases were known in the XIX century.

The general case

\[
\rho + \sigma = \frac{1}{m} \int m^{(\rho - \sigma)} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, ds
\]

is also provably integrable through a link to deformations of quadrics of revolution.