On integrable classes of surfaces in the Euclidean space

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This poster concerns a joint project with H. Baran of classification of integrable PDEs describing immersed surfaces in $\mathbb{R}^3$. The integrability criterion we apply is the existence of an $\mathfrak{sl}(n)$-valued curvature representation depending on a non-removable parameter.

Aims of the project:

– obtaining lists of integrable classes of surfaces, as complete as possible
– identifying known cases
– finding mutual transformations
– obtaining new PDE integrable in the sense of soliton theory.

Criterion of integrability

In the case of two independent variables x, y and a matrix Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-valued *zero curvature representation* (ZCR) for a system of PDE $\mathcal{E}$ is defined to be a form $\alpha = A\,dx + B\,dy$ with $A, B \in \mathfrak{g}$ such that

$$D_y A - D_x B + [A, B] = 0 \quad \operatorname{mod} \mathcal{E}.$$

A ZCR is the compatibility condition for the linear system

$$D_x \Psi = A\Psi, \quad D_y \Psi = B\Psi.$$

A *gauge transformation* with respect to a gauge matrix S is $\Psi \longmapsto S\Psi$, i.e.,

$$A \longmapsto (D_x S)S^{-1} + SAS^{-1}, \quad B \longmapsto (D_y S)S^{-1} + SBS^{-1}.$$

The zero curvature representation (ZCR) involving a parameter not removable by a gauge transformation is a *prerequisite to integrability.*

The classification problem

How to tell whether a given nonlinear system has a zero curvature representation?

A solution: M.M., A direct procedure to compute zero-curvature representations. The case $\mathfrak{sl}_2$, in: *Secondary Calculus and Cohomological Physics,* Proc. Conf. Moscow, 1997 (ELibEMS, 1998) pp. 10.

Description of the method

Solve the *determining system*

$$(D_y A - D_x B + [A, B])|_{\mathcal{E}} = 0,$$

$$\sum_{I,l} (-\widehat{D})_I \left(\frac{\partial F^l}{\partial u_I^k} C_l \right) \Big|_{\mathcal{E}} = 0$$

with auxiliary variables $C_l \neq 0$, supposed to be in a normal form. For the normal forms see op. cit., P. Sebestyén, Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations, *Rep. Math. Phys.* 55 (2005) No. 3, 435–445 and P. Sebestyén, On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations, *Rep. Math. Phys.* 62 (2008) No. 1.

Properties of the determining system

– is a system of differential equations in total derivatives
– is quasilinear in A, B and linear in C_l
– usually possible to solve using computer algebra
– solution algorithms are resource demanding
– computation splits into cases to avoid division by zero (a consequence of nonlinearity in A, B).

The spectral parameter problem

Given a parameterless ZCR, when a parameter can be incorporated?

Example. Gauss–Weingarten equations = a parameterless zero curvature representation of the Gauss–Mainardi–Codazzi equations.

A cohomological solution

To solve the spectral parameter problem in a given Lie algebra:
1. compute the cohomological obstructions, resulting from expanding the zero curvature representation in terms of the (prospective) spectral parameter $A = \sum_i A_i \lambda^i$, $B = \sum_i B_i \lambda^i$

$$D_y A_0 - D_x B_0 + [A_0, B_0] = 0 \quad \text{(the seed ZCR)},$$
$$D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0,$$
$$D_y A_2 - D_x B_2 + [A_2, B_0] + [A_1, B_1] + [A_0, B_2] = 0,$$

etc.
2. compute the full zero curvature representation using the information obtained in the first step to cut off branches.

For details see M.M., On the spectral parameter problem, *Acta Appl. Math.* **109** (2010) 239–255.

The classification project

We consider *geometrically determined* classes of surfaces, meaning classes determined by a single condition

$$F(p_1, \dots, p_k) = 0,$$

where p_i are differential invariants with respect to reparameterizations and euclidean motions (principal curvatures, their gradients, etc.).

We classify relations $F = 0$ such that
– the associated Gauss–Mainardi–Codazzi equations possess a ZCR depending on a nonremovable (spectral) parameter;
– the ZCR has a prescribed order r and takes values in a prescribed Lie algebra $\mathfrak{sl}(n)$.

Weingarten surfaces

To start with, we focused on Weingarten surfaces, i.e., classes of immersed surfaces in $\mathbf{E}^3$ determined by a functional relation between the principal curvatures k_1, k_2.

Thus, the classification problem is: Which functional relations

$$f(k_1, k_2) = 0$$

determine an integrable class of Weingarten surfaces?

Examples

Classical integrable classes: $K = k_1 k_2 = \text{const}$, $H = \frac{1}{2}(k_1 + k_2) = \text{const}$, more generally, $aK + bH + c = \text{const}$ (linear Weingarten surfaces).

Forgotten integrable classes: E.g., $1/k_1 - 1/k_2 = \text{const}$.

See H. Baran and M. Marvan, On integrability of Weingarten surfaces: a forgotten class, *J. Phys. A: Math. Theor.* **42** (2009) 404007.

Preliminaries

Parameterized by the lines of curvature, surfaces $\mathbf{r}(x, y)$ have the fundamental forms

$$\mathrm{I} = u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2, \qquad \mathrm{II} = \frac{u^2}{\rho}\,\mathrm{d}x^2 + \frac{v^2}{\sigma}\,\mathrm{d}y^2.$$

where ρ, σ are the principal radii of curvature, $\rho = 1/k_1$, $\sigma = 1/k_2$.

In the Weingarten case, $\rho = \rho(\sigma)$, the Mainardi–Codazzi subsystem can be explicitly solved. The full GMC system then reduces to the Gauss equation alone, which can be written in the form

$$R_{xx} + S_{yy} + T = 0,$$

where R, S, T are functions of σ.

Results of computation

Weingarten surfaces determined by an explicit dependence $\rho(\sigma)$ possess a one-parametric zero curvature representation if the *determining equation*

$$\rho''' = \frac{3}{2\rho'}\rho''^2 + \frac{\rho' - 1}{\rho - \sigma}\rho'' + 2\frac{(\rho' - 1)\rho'(\rho' + 1)}{(\rho - \sigma)^2}$$

holds (the prime denotes $\mathrm{d}/\mathrm{d}\sigma$).

See H. Baran and M. Marvan, Classification of integrable Weingarten surfaces possessing an sl(2)-valued zero curvature representation, *Nonlinearity* **23** (2010) 2577–2597.

Thed determining equation has two geometric symmetries:

– scaling (changing the ruler) $\rho \longmapsto \mathrm{e}^T \rho$, $\sigma \longmapsto \mathrm{e}^T \sigma$;

– translation (offsetting, normal shift) $\rho \longmapsto \rho + T$, $\sigma \longmapsto \sigma + T$.

These symmetries help us to reduce the order by two. The resulting 1st order ODE is separable.

The general solution $\rho(\sigma)$ is given by the elliptic integral

$$\rho + \sigma = \frac{1}{m} \int^{m(\rho-\sigma)} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s.$$

Here m is a scaling parameter, the integration constant is an offsetting parameter, and c is a "true" parameter.

Summary of the special cases

Except $\rho = \text{const}$ and $\sigma = \text{const}$, the special cases when the above elliptic integral reduces to elementary functions are, up to scaling and offsetting,

relation	integrable equation
$\rho + \sigma = 0$	$z_{xx} + z_{yy} + \mathrm{e}^z = 0$
$\rho\sigma = 1$	$z_{xx} + z_{yy} - \sinh z = 0$
$\rho\sigma = -1$	$z_{xx} - z_{yy} + \sin z = 0$
$\rho - \sigma = \sinh(\rho + \sigma)$	$(\tanh z - z)_{xx} + (\coth z - z)_{yy} + \operatorname{csch} 2z = 0$
$\rho - \sigma = \sin(\rho + \sigma)$	$(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$
$\rho - \sigma = 1$	$z_{xx} + (1/z)_{yy} + 2 = 0$
$\rho - \sigma = \tanh \rho$	$\frac{1}{4}(\sinh z - z)_{xx} + (\coth \frac{1}{2} z)_{yy} + \coth \frac{1}{2} z = 0$
$\rho - \sigma = \tan \rho$	$\frac{1}{4}(\sin z - z)_{xx} + (\cot \frac{1}{2} z)_{yy} + \cot \frac{1}{2} z = 0$
$\rho - \sigma = \coth \rho$	$\frac{1}{4}(\sinh z + z)_{xx} - (\tanh \frac{1}{2} z)_{yy} + \tanh \frac{1}{2} z = 0$
$\rho - \sigma = -\cot \rho$	$\frac{1}{4}(\sin z + z)_{xx} + (\tan \frac{1}{2} z)_{yy} + \tan \frac{1}{2} z = 0$

All the special cases were known in the XIX century.

The general case

$$\rho + \sigma = \frac{1}{m} \int^{m(\rho-\sigma)} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s$$

is also provably integrable through a link to deformations of quadrics of revolution.