An integrable class of Chebyshev nets

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Abstract: We study surfaces equipped with a Chebyshev net such that the Gauss curvature K and a naturally defined curvature G of the net satisfy a linear condition $\alpha K + \beta G + \gamma = 0$, where α, β, γ are constants. These surfaces form an integrable class. We point out some of its noteworthy peculiarities.

WORKSHOP ON

INTEGRABLE NONLINEAR EQUATIONS

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Chebyshev nets

Surfaces of interest in geometry and applications often bear a special net of curves (two 1-parametric families of curves).

Chebyshev coordinate nets are characterized by the first fundamental form

$$\mathrm{I} = \mathrm{d}x^2 + 2\cos\varphi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2.$$

They were originally introduced by P.L. Chebyshev in the context of clothing. When a fabric moulds to the body, the warp and weft fibers do not slide across each other at the points of intersection. Instead, they slant and the warp-weft angle can take arbitrary values $\varphi(x, y) \neq k\pi$.

A Chebyshev net locally exists on every smooth surface and is determined given a pair of transversal lines on the surface.

Example: String bags

Photo by Morlawmina

In this example there is a prescribed supporting surface.

Different settings

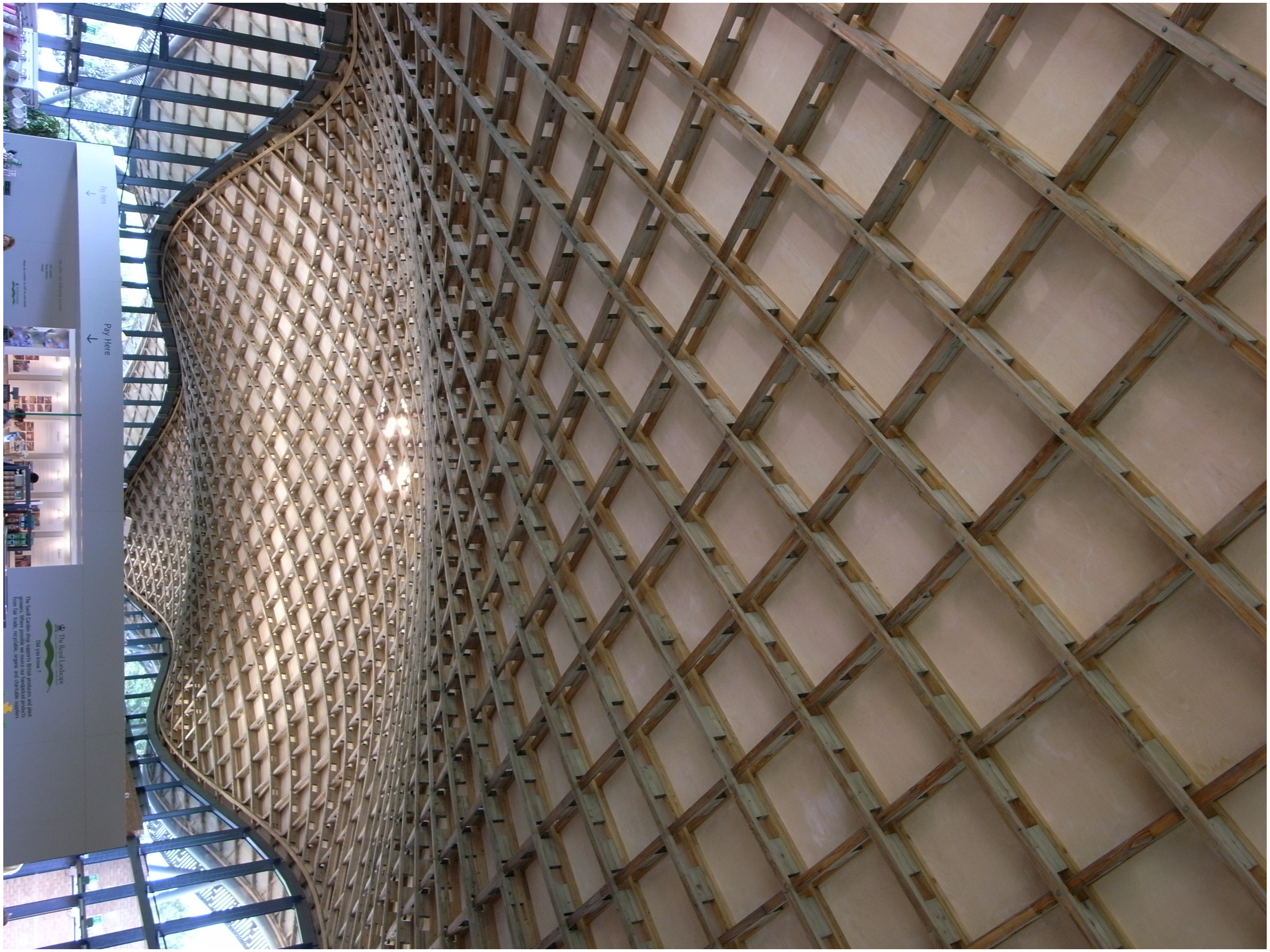
Quite often there is no predefined supporting surface. For instance, the Chebyshev net can be a mathematical model of a physical net subject to certain PDE (equilibrium conditions) and boundary conditions.

Heinz Thomas, Zur Frage des Gleichgewichts von Tschebyscheff-Netzen aus verknoteten und gespannten Fäden, *Math. Z.* **47** (1942) 66–77.

This idea is used in free form construction. Timber grid shells are equilibrium states of Chebyshev nets under elastic forces combined with gravity. The actual shape is determined by boundary conditions.

Photo by Chinar2011

Example. A timber roof



← gravity

A picture by Oosoom

Integrability

Several soliton-theoretic integration methods rely on a 1-parametric zero curvature representation

$$D_y A - D_x B + [A, B] = 0$$

with A, B depending on a parameter.

A classification result

I.S. Krasil'shchik and M. Marvan, Coverings and integrability of the Gauss–Mainardi–Codazzi equations, *Acta Appl. Math.* **56** (1999) 217–230.

We looked for 1-parametric $\mathfrak{sl}(2)$-valued zero curvature representations for the Gauss–Mainardi–Codazzi equations in Chebyshev coordinates.

The classification result (incomplete), included five classes. One of them will be considered below.

Notation

Consider a surface $\mathbf{r}(x, y)$ immersed in the three-dimensional Euclidean space. Let $\mathbf{n}(x, y)$ be the unit normal. Up to rigid motions, $\mathbf{r}$ is determined by its first and second fundamental forms.

We assume the first fundamental form

$$\mathrm{I} = \mathrm{d}x^2 + 2\cos\varphi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2.$$

The second fundamental form is arbitrary, written as

$$\mathrm{II} = b_{11}\,\mathrm{d}x^2 + 2b_{12}\,\mathrm{d}x\,\mathrm{d}y + b_{22}\,\mathrm{d}y^2.$$

Convenient variables are h_{ij} defined by

$$b_{ij} = h_{ij}\sin\varphi.$$

Under this notation, the Gauss curvature is simply

$$K = h_{11}h_{22} - h_{12}^2 = \det h.$$

Curvature linear nets

The Gauss–Mainardi–Codazzi equations are

$$\varphi_{xy} + (h_{11}h_{22} - h_{12}^2)\sin\varphi = 0,$$

$$h_{11,y} = h_{12,x} - (\cot\varphi)\varphi_y h_{11} + (\sin\varphi)^{-1}\varphi_x h_{22},$$

$$h_{12,y} = h_{22,x} - (\sin\varphi)^{-1}\varphi_y h_{11} + (\cot\varphi)\varphi_x h_{22}.$$

With three equations on four unknowns $\varphi, h_{11}, h_{12}, h_{22}$, the GMC system is obviously underdetermined. According to the classification result mentioned above, the system possesses a 1-parametric zero curvature representation if we add the extra equation

$$\alpha K + \beta G + \gamma = 0,$$

where α, β, γ are arbitrary constants, not all being equal to zero, and $K = h_{11}h_{22} - h_{12}^2$, $G = h_{12}$.

The meaning of G.

While K is the usual Gaussian curvature of the surface, the geometric meaning of $G = h_{12}$ is not obvious. Nets have been studied extensively, yet there seems to be no earlier reference to G than by

W.K. Schief, Discrete Chebyshev nets and a universal permutability theorem, *J. Phys. A: Math. Theor.* **40** (2007) 4775–4801.

In the spirit of R. Sauer, a net is a smooth limit of a quadrilateral mesh. Every quadrilateral segment is a tetrahedron. Define

$$G = \frac{3}{2} \lim_{h \to 0} \frac{V_h}{A_h^2} = \frac{6 \lim\limits_{h \to 0} \dfrac{V_h}{h^4}}{4 \lim\limits_{h \to 0} \dfrac{A_h^2}{h^4}} = 6 \frac{[\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_{xy}]}{[\mathbf{r}_x, \mathbf{r}_y, \mathbf{n}]^2} = h_{12}.$$

where V_h is the volume of the tetrahedron and A_h is the area of its triangular face. Schief's definition is equivalent to this one.

Well understood special cases of curvature linear nets

Several special cases of the condition $\alpha K + \beta G + \gamma = 0$ are well known.

The Lund–Regge system. The case of $\alpha = 0$, $\beta\gamma \neq 0$ is well understood due to Schief [op. cit.]. Here $G = \sigma = h_{12}$ is a constant. If $\gamma \neq 0$, then σ can be reduced to 1 by rescaling. The Gauss–Weingarten equations imply the Lund–Regge equation $\mathbf{r}_{xy} = \mathbf{r}_x \times \mathbf{r}_y$.

Translation surfaces. If $\alpha = \gamma = 0$, $\beta \neq 0$, then condition $\alpha K + \beta G + \gamma = 0$ yields $G = h_{12} = 0$ and $\mathbf{r}_{xy} = 0$. Consequently, we recover the well-known class of translation surfaces.

Surfaces of constant Gaussian curvature.

If $\beta = 0$, then condition $\alpha K + \beta G + \gamma = 0$ means that K is a constant. An example is provided by an arbitrary surface of constant Gaussian curvature equipped with an arbitrary (local) Chebyshev net.

Double constant curvature case. Following Hazzidakis, the Chebyshev net on a pseudospherical surface can be chosen to be asymptotic, i.e., $h_{11} = h_{22} = 0$. Then, by GMC equations, $G = h_{12}$ is constant and φ satisfies the sine–Gordon equation. Both K and G are constant. Moreover, $K = -G^2$.

Here is the converse statement:

Proposition 1. *Let K and G be constant, let $K + G^2 = 0$. Then either $K = G = 0$ or $h_{11} = h_{22} = 0$. Consequently, the Chebyshev net is either planar or asymptotic, according to whether $K = 0$ or not.*

The zero curvature representation

As we have seen in the previous section, the cases of $\alpha = 0$ or $\beta = 0$ have been understood rather well. Therefore, we assume that $\alpha \neq 0 \neq \beta$ in what follows. Moreover, we set

$$\alpha = 1$$

for the sake of simplicity.

The Gauss–Weingarten system induces an $\mathfrak{so}(3)$-valued zero curvature representation of the Gauss–Mainardi–Codazzi system. Choose the orthonormal frame $\mathbf{p}, \mathbf{q}, \mathbf{n}$, where $\mathbf{n}$ is the unit normal vector and $\mathbf{p}, \mathbf{q}$ are the unit vectors along the bisector lines $x + y = \text{const}$, $x - y = \text{const}$, respectively. Then

$$\mathbf{p} = \frac{1}{2}\,\frac{\mathbf{r}_x + \mathbf{r}_y}{\cos\frac{1}{2}\,\varphi}, \qquad \mathbf{q} = \frac{1}{2}\,\frac{\mathbf{r}_x - \mathbf{r}_y}{\sin\frac{1}{2}\,\varphi}.$$

From

$$\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\\ \mathbf{n}\end{pmatrix}_x = \begin{pmatrix} 0 & A_{12} & A_{13}\\ -A_{12} & 0 & A_{23}\\ -A_{13} & -A_{23} & 0\end{pmatrix}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\\ \mathbf{n}\end{pmatrix},$$

$$\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\\ \mathbf{n}\end{pmatrix}_y = \begin{pmatrix} 0 & B_{12} & B_{13}\\ -B_{12} & 0 & B_{23}\\ -B_{13} & -B_{23} & 0\end{pmatrix}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\\ \mathbf{n}\end{pmatrix}$$

we easily get

$$\begin{aligned} A_{12} &= \tfrac{1}{2}\,\varphi_x, & B_{12} &= -\tfrac{1}{2}\,\varphi_y,\\ A_{13} &= (h_{11}+h_{12})\sin\tfrac{1}{2}\,\varphi, & B_{13} &= (h_{12}+h_{22})\sin\tfrac{1}{2}\,\varphi, \qquad (1)\\ A_{23} &= (h_{11}-h_{12})\cos\tfrac{1}{2}\,\varphi, & B_{23} &= (h_{12}-h_{22})\cos\tfrac{1}{2}\,\varphi. \end{aligned}$$

It follows that A, B take values in $\mathfrak{so}(3)$ and the zero curvature condition $D_yA - D_xB + [A,B] = 0$ holds as a consequence of the Gauss–Mainardi–Codazzi system.

Let λ be a parameter, let $\Lambda_\pm$denote the two solutions of the system

$$\Lambda_+ + \Lambda_- = \frac{\beta}{\lambda}(1 - \lambda^2), \qquad \Lambda_+\Lambda_- = \gamma(1 - \lambda^2).$$

Then $\mathfrak{so}(3)$ matrices A, B such that

$$\begin{aligned} A_{12} &= \tfrac{1}{2}\,\varphi_x, & B_{12} &= -\tfrac{1}{2}\,\varphi_y, \\ A_{13} &= (\lambda h_{11} + \lambda h_{12} + \Lambda_-)\sin\frac{\varphi}{2}, & B_{13} &= (\lambda h_{12} + \lambda h_{22} + \Lambda_+)\sin\frac{\varphi}{2}, \\ A_{23} &= (\lambda h_{11} - \lambda h_{12} - \Lambda_-)\cos\frac{\varphi}{2}, & B_{23} &= (\lambda h_{12} - \lambda h_{22} + \Lambda_+)\cos\frac{\varphi}{2} \end{aligned}$$

satisfy $D_y A - D_x B + [A, B] = 0$ and, therefore, constitute a zero curvature representation depending on the spectral parameterλ. When $\lambda = \pm 1$, we obtain the matrices A, B we started with.

Everything is easily translated to $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$.

It is easy to see that the parameter is not removable by a gauge transformation.

The spectral curve

The spectral parameter runs over the elliptic curve

$$w^2 = (\beta^2 + 4\gamma)z^4 - 2(\beta^2 + 2\gamma)z^2 + \beta^2.$$

In two special cases the spectral curve is of genus 0.

When $\gamma = 0$, the curve degenerates to the pair of quadrics $w^2 = \beta^2(z^2 - 1)^2$ and

$$\Lambda_\pm = 0, \qquad \Lambda_\mp = \frac{\beta}{\lambda}(1 - \lambda^2),$$

where either upper signs or lower signs are to be used.

When $\beta^2 + 4\gamma = 0$, the curve degenerates to the quadric $w^2 = -2(\beta^2 + 2\gamma)z^2 + \beta^2$ and the substitution

$$\lambda = -\frac{2\beta\mu}{\mu^2 + \beta^2}, \qquad \Lambda_- = \frac{\beta^2}{2\mu}\,\frac{\mu^2 - \beta^2}{\mu^2 + \beta^2}, \qquad \Lambda_+ = -\frac{\mu}{2}\,\frac{\mu^2 - \beta^2}{\mu^2 + \beta^2}$$

provides a rational parameterization by μ.

Curvature proportional nets

Consider the case of $\gamma = 0$, $\alpha\beta \neq 0$. The two curvatures $K$and G are proportional. The nets satisfying $K = \beta G$ will be called *curvature proportional nets*.

As we have seen, the spectral curve degenerates to genus 0.

Incidentally, $\gamma = 0$ is the only case when there exists a vectorial potential. Define $\mathbf{m}$ by the compatible equations

$$\mathbf{m}_x = (h_{12} - \beta)\,\mathbf{r}_x - h_{11}\,\mathbf{r}_y,$$
$$\mathbf{m}_y = h_{22}\,\mathbf{r}_x + (\beta - h_{12})\,\mathbf{r}_y.$$

Remark. By a straightforward computation, $\mathbf{m}_{xy} \cdot \mathbf{n} = 0$. Therefore, $\mathbf{m}(x, y)$ is a conjugate net. This provides us with a link to another extensively studied class of nets, but $\mathbf{m}(x, y)$ does not seem to fall into any class that have been actually studied.

The associated pseudospherical surfaces

Define surfaces S^+, S^- by the position vectors

$$\mathbf{r}^+ = \mathbf{m} + \beta\mathbf{r}, \quad \mathbf{r}^- = \mathbf{m} - \beta\mathbf{r}.$$

Proposition 2. *The associated surfaces S^+, S^- are of the Gauss curvature -1, i.e., pseudospherical.*

As the next step, we equip the surfaces S^+, S^- with the asymptotic Chebyshev coordinates $\xi^\pm, \eta^\pm$. The angle $\phi^\pm(\xi^\pm, \eta^\pm)$ between the coordinate lines satisfies the sine-Gordon equation.

Proposition 3. *Every solution φ, h_{ij} of the GMC system with $\gamma = 0$ induces two sine-Gordon solutions, i.e., functions $\phi^\pm(\xi^\pm, \eta^\pm)$ such that*

$$\phi^\pm_{\xi^\pm \eta^\pm} = \sin\phi^\pm.$$

Explicit formulas are rather huge, hence omitted.

Peculiarities

We obtain a scheme of coverings

$$\begin{array}{ccc} & \mathrm{GMC} & \\ \swarrow & & \searrow \\ \mathrm{sG} & & \mathrm{sG} \end{array}$$

resembling that of the Bäcklund transformation between pseudospherical surfaces. However, the coverings are infinite-dimensional and cannot be reduced to finite-dimensional ones. Hence, the BT is different from the classical one and, moreover, useless for breeding the sine-Gordon solutions.

Given a solution of the sine-Gordon equation (or a pseudospherical surface), we cannot recover the covering solution of the GMC system (or the corresponding curvature proportional net) unless by solving a complicated non-overdetermined nonlinear system of partial differential equations.

Remark.

The two coverings are not completely useless. For instance, we can lift the conservation laws from the sine-Gordon to the GMC system.

The standard zero curvature representation of the sine-Gordon equation is lifted to the zero curvature representation of the GMC system shown above, as expected.

Moreover, the covering allows us to transform easily curvature proportional nets, if we knew any, to pseudospherical surfaces. So far we have only been able to obtain rotational curvature proportional nets in terms of hyperelliptic functions (jointly with P. Blaschke). Had we have more time, we would have written down the corresponding sine-Gordon solutions.