On symmetries of the Gibbons–Tsarev equation

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The Gibbons–Tsarev equation

$$\text{GT}: \qquad z_{yy} + z_x z_{xy} - z_y z_{xx} + 1 = 0.$$

J. Gibbons and S.P. Tsarev, Reductions of the Benney equations, *Physics Letters A* **211** (1996) 19–24.

The Benney chain:

$$a^{(n)}_\tau + a^{(n+1)}_\xi + n a^{(n-1)} a^{(0)}_\xi, \qquad n = 0, 1, 2, \ldots$$

with unknowns $a^{(n)}(\tau, \xi)$.

Gibbons–Tsarev reductions:
$a^{(n)}$, $n \geq N$, are functions of $a^{(0)}, a^{(1)}, \ldots a^{(N-1)}$.

We obtain the above Gibbons–Tsarev equation $\mathcal{E}$, if we set $N = 2$ and

$$x = a^{(1)}, \qquad y = a^{(0)}, \qquad z(x, y) = -a^{(2)} + \tfrac{1}{2}\, a^{(0)\,2}.$$

What is known?

– A nonlinear and a linear Lax pair.

– Infinitely many nonlocal conservation laws.

– Infinite dimensional algebra of nonlocal infinitesimal symmetries.

P. Holba, I.S. Krasil'shchik, O.I. Morozov and P. Vojčák, 2D reductions of the equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$ and their nonlocal symmetries, *J. Nonlin. Math. Phys.* **24** (2017) Suppl. 1, 36–47.

What is not known?

– No recursion operator known.

– No matrix zero-curvature representation known.

What are our new results?

– Explicit formulas for infinitely many nonlocal conservation laws.

– Remarkable and explicit formulas for infinitely many nonlocal infinitesimal symmetries.

– A uniqueness theorem for nonlocal symmetries.

Janet monomial notation for jet coordinates

$$z_X \leftrightsquigarrow \frac{\partial^{i+j} z}{\partial x^i \, \partial y^j} \quad \text{if} \quad X = x^i y^j,$$

where $x^i y^j$ is an arbitrary commutative monomial.

Diffiety $\mathcal{E}$

Internal coordinates: x, y, and z_X such that $X = x^k$ or $X = x^k y$.

If not stated otherwise, sums are over all internal coordinates z_X.

The total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum z_{xX} \frac{\partial}{\partial z_X}, \qquad D_y = \frac{\partial}{\partial y} + \sum z_{yX} \frac{\partial}{\partial z_X},$$

where

$$z_{yyX} = D_X(z_y z_{xx} - z_x z_{xy} - 1)$$

for every X, i.e., the non-internal jet coordinates become functions of the internal ones.

Weights

The Gibbons–Tsarev equation is homogeneous under the weights

$$|x| = 3, \quad |y| = 2, \quad |z| = 4.$$

Then

$$|z_{x^k}| = 4 - 3k, \quad |z_{x^k y}| = 2 - 3k.$$

The weights are due to existence of the scaling symmetry, see below.

The operators of total derivatives have the weights

$$|D_x| = -3, \quad |D_y| = -2.$$

Local symmetries

are

$$\mathcal{S}^{(i)} = \sum D_X Z^{(i)} \frac{\partial}{\partial z_X}, \qquad i = -4, \ldots, 0,$$

(sum over all internal coordinates), where

$$Z^{(-4)} = 1, \qquad z\text{-translation},$$

$$Z^{(-3)} = z_x, \qquad x\text{-translation},$$

$$Z^{(-2)} = z_y, \qquad y\text{-translation},$$

$$Z^{(-1)} = y z_x - 2x, \qquad \text{generalized Galilean boost},$$

$$Z^{(0)} = 2z - \tfrac{3}{2}\, x z_x - y z_y, \qquad \text{scaling}.$$

Provably, this is the complete set of local symmetries.

The vector field $\mathcal{S}^{(i)}$ is of weight i (as an operator).

Commutators.

The commutators of symmetries satisfy

$$[\mathcal{S}^{(i)}, \mathcal{S}^{(j)}] = (j - i)\mathcal{S}^{(i+j)},$$

if we set $\mathcal{S}^{(\alpha)} = 0$ for $\alpha < -4$.

It will be shown below that this set of five symmetries can be extended to a hierarchy of nonlocal symmetries infinite in both positive and negative directions, satisfying the same commutation relations.

Local conservation laws

A computation using cosymmetries reveals conservation laws $\rho^{(i)}$ of the weight $i+5$, namely $\rho^{(i)} = P^{(i)}\,\mathrm{d}x + Q^{(i)}\,\mathrm{d}y,$ where

$$
\begin{aligned}
P^{(0)} &= z_x^2 + z_y + y,\\
Q^{(0)} &= z_x z_y,\\
P^{(1)} &= z_x^3 + 2z_x z_y - x,\\
Q^{(1)} &= z_x^2 z_y + z_y^2 - 2z,\\
P^{(2)} &= z_x^4 + 3z_x^2 z_y + 3y z_x^2 + z_y^2 - x z_x + 3y z_y - 2z,\\
Q^{(2)} &= z_x^3 z_y + 2z_x z_y^2 + 3y z_x z_y - x z_y - 3xy,\\
P^{(3)} &= z_x^5 + 4z_x^3 z_y + 4y z_x^3 + 3z_x z_y^2 + 2x z_x^2 + 8y z_x z_y\\
&\quad - 2z z_x + 2x z_y - 4xy,\\
Q^{(3)} &= z_x^4 z_y + 3z_x^2 z_y^2 + 4y z_x^2 z_y + z_y^3 + 2x z_x z_y + 4y z_y^2\\
&\quad - 2z z_y - 8yz - 3x^2,
\end{aligned}
$$

More local conservation laws

$$P^{(4)} = z_x^6 + 5z_x^4 z_y + 5yz_x^4 + 6z_x^2 z_y^2 + 3xz_x^3 + 15yz_x^2 z_y + z_y^3 + (z + \tfrac{15}{2}\, y^2) z_x^2 + 6xz_x z_y + 5yz_y^2 + (z + \tfrac{15}{2}\, y^2) z_y - 5yz - 4x^2,$$

$$Q^{(4)} = z_x^5 z_y + 4z_x^3 z_y^2 + 5yz_x^3 z_y + 3z_x z_y^3 + 3xz_x^2 z_y + 10yz_x z_y^2 + (z + \tfrac{15}{2}\, y^2) z_x z_y + 3xz_y^2 - \tfrac{3}{2}\, x(4z + 5y^2),$$

$$P^{(5)} = z_x^7 + 6z_x^5 z_y + 6yz_x^5 + 10z_x^3 z_y^2 + 4xz_x^4 + 24yz_x^3 z_y + 4z_x z_y^3 + 4(\tfrac{1}{2}\, z + 3y^2) z_x^3 + 12xz_x^2 z_y + 18yz_x z_y^2 + 12xyz_x^2 + 4(z + 6y^2) z_x z_y + 4xz_y^2 + 12xyz_y - 4xz,$$

$$Q^{(5)} = z_x^6 z_y + 5z_x^4 z_y^2 + 6yz_x^4 z_y + 6z_x^2 z_y^3 + 4xz_x^3 z_y + 18yz_x^2 z_y^2 + z_y^4 + 4(\tfrac{1}{2}\, z + 3y^2) z_x^2 z_y + 8xz_x z_y^2 + 6yz_y^3 + 12xyz_x z_y + 4(\tfrac{1}{2}\, z + 3y^2) z_y^2 - 2z^2 - 24y^2 z - 6x^2 y,$$

$$P^{(-5)} = -\tfrac{1}{2}\, z_x z_{xx}^2 - z_{xy} z_{xx},$$

$$Q^{(-5)} = -\tfrac{1}{2}\, z_y z_{xx}^2 - \tfrac{1}{2}\, z_{xy}^2.$$

Infinite hierarchies of conservation laws

Two ways (equivalent results).

Two Lax pairs

I. The nonlinear Lax pair given by

$$\phi_x = \frac{1}{z_y + z_x\phi - \phi^2}, \qquad \phi_y = \frac{\phi - z_x}{z_y + z_x\phi - \phi^2}.$$

II. The linear Lax pair

$$\psi_x = \frac{1}{\lambda^2 - z_x\lambda - z_y} \cdot \psi_\lambda, \qquad \psi_y = \frac{\lambda - z_x}{\lambda^2 - z_x\lambda - z_y} \cdot \psi_\lambda,$$

obtained from the former by the *reversion* procedure.

M.V. Pavlov, Jen Hsu Chang, Yu Tung Chen, Integrability of the Manakov-Santini hierarchy, `arXiv:0910.2400`, 2009.

Both systems have the Gibbons–Tsarev equation as the compatibility condition.

Linear potentials $\psi^{(i)}$

Inserting

$$\psi = \lambda + \frac{\psi^{(1)}}{\lambda} + \cdots + \frac{\psi^{(k)}}{\lambda^k} + \cdots,$$

into the linear Lax pair, we obtain a tower of linear systems

$$\psi_x^{(k)} = \sigma_{k-2} - \sum_{i=1}^{k-3} i\sigma_{k-i-3}\psi^{(i)},$$

$$\psi_y^{(k)} = z_y\psi_x^{(k-1)} - (k-2)\psi^{(k-2)},$$

where

$$\sigma_k = \sum_{0\le j\le k-j} \binom{k-j}{j} z_x^{k-2j} z_y^j, \qquad k > 0.$$

These potentials are not quite convenient for the construction of nonlocal symmetries.

Nonlinear potentials $\phi^{(i)}$

Inserting

$$\phi = \lambda^{-1} + \phi^{(1)}\lambda + \cdots + \phi^{(k)}\lambda^k + \cdots$$

into the nonlinear Lax pair, we obtain a tower of nonlinear systems

$$\phi_x^{(m)} = -2\sum_{i=1}^{m-2}\phi^{(i)}\phi_x^{(m-i-1)} - \sum_{i=1}^{m-2}\sum_{j=1}^{m-i-2}\phi^{(i)}\phi^{(j)}\phi_x^{(m-i-j-2)}$$

$$+ z_x\left(\phi_x^{(m-1)} + \sum_{i=1}^{m-2}\phi^{(i)}\phi_x^{(m-i-2)}\right) + z_y\phi_x^{(m-2)} + \delta_2^m,$$

$$\phi_y^{(m)} = -2\sum_{i=1}^{m-2}\phi^{(i)}\phi_y^{(m-i-1)} - \sum_{i=1}^{m-2}\sum_{j=1}^{m-i-2}\phi^{(i)}\phi^{(j)}\phi_y^{(m-i-j-2)}$$

$$+ z_x\left(\phi_y^{(m-1)} + \sum_{i=1}^{m-2}\phi^{(i)}\phi_y^{(m-i-2)} - \delta_2^m\right) + z_y\phi_y^{(m-2)} - \phi^{(m-2)}.$$

Comparison of the potentials $\psi^{(i)}$ and $\phi^{(i)}$

The weights are the same: $|\phi^{(k)}| = |\psi^{(k)}| = k + 1$.

The first three potentials are local:

$$\psi^{(1)} = y, \qquad \phi^{(1)} = -y,$$
$$\psi^{(2)} = x, \qquad \phi^{(2)} = -x,$$
$$\psi^{(3)} = z - \tfrac{1}{2}\, y^2, \qquad \phi^{(3)} = -z - \tfrac{1}{2}\, y^2.$$

Mutual relations (will be proved below):

$$\psi^{(k)} = -\sum_{m\geq 1} (-1)^m \sum_{i_1+\cdots+i_m=k+1} \frac{1}{k} \binom{k}{m} \phi^{(i_1-1)} \ldots \phi^{(i_m-1)},$$

$$\phi^{(k)} = -\sum_{m\geq 1} \sum_{i_1+\cdots+i_m=k+1} \frac{1}{k} \binom{k}{m} \psi^{(i_1-1)} \ldots \psi^{(i_m-1)}$$

Relation among the potentials – part I

Recall that

$$\psi(\lambda) = \lambda + \frac{\psi^{(1)}}{\lambda} + \frac{\psi^{(2)}}{\lambda^k} + \cdots, \qquad \phi(\lambda) = \frac{1}{\lambda} + \phi^{(1)}\lambda + \phi^{(2)}\lambda^k + \cdots$$

The formal series

$$1/\phi(\lambda) = -\lambda + \phi^{(1)}\lambda^3 + \phi^{(2)}\lambda^4 + (\phi^{(3)} - \phi^{(1)\,2})\lambda^5 + \cdots$$

is a power series of lowest degree 1.

Therefore, the composition series

$$\psi(\phi) = \phi(\lambda) + \frac{\psi^{(1)}}{\phi(\lambda)} + \frac{\psi^{(2)}}{\phi(\lambda)^2} + \cdots$$

is well defined (no infinite sums occur).

Relation among the potentials – part II

Substituting ϕ for λ in

$$\psi_x = \frac{1}{\lambda^2 - z_x\lambda - z_y} \cdot \psi', \qquad \psi_y = \frac{\lambda - z_x}{\lambda^2 - z_x\lambda - z_y} \cdot \psi',$$

we obtain

$$(\psi(\phi))_x = \psi_x(\phi) + \psi'(\phi)\phi_x = 0,$$
$$(\psi(\phi))_y = \psi_y(\phi) + \psi'(\phi)\phi_y = 0$$

since

$$\phi_x = -\frac{1}{\phi^2 - z_x\phi - z_y}, \qquad \phi_y = -\frac{\phi - z_x}{\phi^2 - z_x\phi - z_y}.$$

Therefore, $\psi(\phi) = c(\lambda)$.

The converse is also true, if $\psi(\phi) = c(\lambda)$, then the two systems above are equivalent.

Relation among the potentials – part III

Let us choose $c(\lambda) = 1/\lambda$, i.e.,

$$\psi(\phi(\lambda)) = 1/\lambda.$$

This is compatible with the choices we made for $\psi^{(i)}$, $\phi^{(i)}$, $i = 1, 2, 3$.

Proposition. *The power series* $1/\psi(1/\lambda)$ *and* $1/\phi(\lambda)$ *are compositionally inverse one to another.*

Indeed, substituting $1/\phi(\lambda)$ for λ in $1/\psi(1/\lambda)$, we obtain $1/\psi(\phi(\lambda)) = 1/c(\lambda) = \lambda$.

Corollary. *The coefficients* $\psi^{(i)}$ *determine the coefficients* $\phi^{(i)}$ *and vice versa.*

A reminder about diffieties

Recall the diffiety $\mathcal{E}$, an-infinite dimensional manifold, equipped with the total derivatives, written in the "internal" coordinates z_X, $y^2 \nmid X$, as

$$D_x = \frac{\partial}{\partial x} + \sum z_{xX} \frac{\partial}{\partial z_X}, \qquad D_y = \frac{\partial}{\partial y} + \sum z_{yX} \frac{\partial}{\partial z_X},$$

$$[D_x, D_y] = 0.$$

As a consequence of the Gibbons–Tsarev equation, the "external" variables z_{yyX} become functions

$$z_{yyX} = D_X(z_y z_{xx} - z_x z_{xy} - 1).$$

Symmetries

$$\mathcal{S} = \sum D_X Z(x, y, z_X) \frac{\partial}{\partial z_X}, \qquad \sum D_X Z \frac{\partial\,\mathrm{GT}}{\partial z_X} = 0,$$

are vector fields $\mathcal{S}$ on $\mathcal{E}$, satisfying $[\mathcal{S}, D_x] = [\mathcal{S}, D_y] = 0 = \mathcal{S}(\mathrm{GT})$.

Coverings

An n-dimensional covering over $\mathcal{E}$ is the projection $\tilde{\mathcal{E}} = \mathcal{E} \times \mathbb{R}^n \longrightarrow \mathcal{E}$. *Nonlocal variables* q^i are defined to be any coordinates in the fibres $\mathbb{R}^n$. The total derivatives assume the form

$$\widetilde{D}_x = D_x + \sum_{i=1}^{n} X^i \frac{\partial}{\partial q^i}, \qquad \widetilde{D}_y = D_y + \sum_{i=1}^{n} Y^i \frac{\partial}{\partial q^i},$$

where X^i, Y^i are functions of the internal coordinates x, y, z_X of $\mathcal{E}$ and of the nonlocal variables q^i. They are required to satisfy

$$[\widetilde{D}_x, \widetilde{D}_y] = 0.$$

The two coverings

We have two infinite-dimensional coverings, one with the nonlocal variables $\psi^{(k)}$, the other with the nonlocal variables $\phi^{(k)}$.

They are equivalent.

Nonlocal symmetries

Nonlocal symmetries are symmetries in a covering $\tilde{\mathcal{E}}$.

Of the two coverings we choose that with the potentials $\phi^{(i)}$:

$$\widetilde{\mathrm{GT}}: \qquad \mathrm{GT}, \quad \phi^{(i)}_x = X^{(i)}, \quad \phi^{(i)}_y = Y^{(i)}.$$

Symmetries will be vector fields $\mathcal{S}$ on $\tilde{\mathcal{E}}$, satisfying $[\mathcal{S}, \widetilde{D}_x] = [\mathcal{S}, \widetilde{D}_y] = 0 = \mathcal{S}(\widetilde{\mathrm{GT}})$. In particular,

$$\mathcal{S} = \sum D_X Z(z_X, \phi^{(*)}) \frac{\partial}{\partial z_X} + \sum \Phi^{(i)}(z_X, \phi^{(*)}) \frac{\partial}{\partial \phi^{(i)}},$$

where

$$\sum D_X Z \frac{\partial\, \mathrm{GT}}{\partial z_X} = 0, \quad D_x \Phi^{(i)} = \sum D_X Z \frac{\partial X^{(i)}}{\partial z_X} + \sum \Phi^{(j)} \frac{\partial X^{(i)}}{\partial \phi^{(j)}},$$

and analogously for $D_y \Phi^{(i)}$.

The first summand in $\mathcal{S}$ is called a *shadow*.

The shadows I

To determine nonlocal symmetries, we are looking for shadows in the first place, i.e., for functions $Z(z_X, \phi^{(*)})$ satisfying

$$\sum D_X Z \frac{\partial\, \mathrm{GT}}{\partial z_X} = 0.$$

Using the expansion

$$\phi = \frac{1}{\lambda} + \phi^{(1)}\lambda + \phi^{(2)}\lambda^k + \cdots$$

we introduce different nonlocal variables

$$\phi_\Lambda \leftrightsquigarrow \frac{\mathrm{d}^i\phi}{\mathrm{d}\lambda^i}, \qquad \Lambda = \lambda^i$$

Consider the covering $\tilde{\tau}_\lambda \colon \tilde{\mathcal{E}}_\lambda \longrightarrow \mathcal{E}$, where $\tilde{\mathcal{E}}_\lambda = \mathcal{E} \times J(\lambda;\phi)$. $J(\lambda;\phi)$ is the jet space with the coordinates λ and ϕ_Λ.

The shadows II

Total derivatives in $\tilde{\mathcal{E}}_\lambda$:

$$\widetilde{D}_x = D_x + \sum_\Lambda \phi_{x\Lambda} \frac{\partial}{\partial \phi_\Lambda}, \qquad \widetilde{D}_y = D_y + \sum_\Lambda \phi_{y\Lambda} \frac{\partial}{\partial \phi_\Lambda},$$

$$\widetilde{D}_\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda} + \sum_\Lambda \phi_{\lambda\Lambda} \frac{\partial}{\partial \phi_\Lambda},$$

The diffiety $\tilde{\mathcal{E}}_\lambda$ corresponds to the covering $\widetilde{\mathrm{GT}}$, considered over an extended set x, y, λ of independent variables, along with

$$z_\lambda = 0.$$

The shadows III

Proposition. *Expand*

$$Z = (\phi^2 - z_x\phi - z_y)\phi_\lambda^2$$

as a formal Laurent series of the form

$$\sum_{n=-4}^{\infty} Z^{(n)}\lambda^{n-2}.$$

Then $Z^{(n)}$ *are shadows of symmetries of the Gibbons–Tsarev equation in the covering* $\tilde{\mathcal{E}}$*.*

Proof. By coordinate computation, using the fact that GT does not involve λ.

The shadows IV

The formula

$$Z = (\phi^2 - z_x\phi - z_y)\phi_\lambda^2$$

leads to an explicit formula for the shadows.

Let $\sum^{(\bullet)}$ denote summation where indices run through all integers from -1 to infinity.

Proposition. Let

$$A_2^{(k,n)} = \sum_{i_1+\cdots+i_{k+2}=n}^{(\bullet)} i_1 i_2 \phi^{(i_1)} \cdots \phi^{(i_{k+2})}, \qquad k \geq 0.$$

Then

$$Z^{(n)} = A_2^{(1,n)} z_x + A_2^{(0,n)} z_y - A_2^{(2,n)}.$$

From shadows to symmetries I

We look for nonlocal symmetries in the form

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + \sum_{i>3} f^{(i)}\frac{\partial}{\partial \phi^{(i)}} + \cdots$$

where a, b, c, $f^{(i)}$ are functions on $\tilde{\mathcal{E}}$.

The corresponding vertical field, obtained by subtracting $a\widetilde{D}_x + b\widetilde{D}_y$, is

$$\mathcal{S} = \sum_{\Xi} \widetilde{D}_{\Xi}(c - az_x - bz_y)\frac{\partial}{\partial z_{\Xi}} + \sum_{i>3}(f^{(i)} - aX^{(i)} - bY^{(i)})\frac{\partial}{\partial \phi^{(i)}}.$$

Variables x, y, z can be expressed in terms of $\phi^{(i)}$, $= 1, 2, 3$. Consequently, we can rewrite the symmetry in terms of $\phi^{(i)}$ and z_{Ξ}, $|\Xi| > 0$, alone.

From shadows to symmetries II

The symmetry in terms of $\phi^{(i)}$ and z_Ξ:

$$\mathcal{S} = \sum_{i>0} \Phi^{(i)} \frac{\partial}{\partial \phi^{(i)}} + \sum_{|\Xi|>0} \widetilde{D}_\Xi Z \frac{\partial}{\partial z_\Xi},$$

$$Z = \left(z_y - \phi^{(1)}\right) f^{(1)} + z_x f^{(2)} - f^{(3)},$$

$$\Phi^{(i)} = f^{(i)} + f^{(2)} X^{(i)} + f^{(1)} Y^{(i)}.$$

The same symmetry in terms of ϕ_Λ and z_Ξ:

$$\mathcal{S} = \sum_{\Lambda} \partial_\Lambda \Phi \frac{\partial}{\partial \phi_\Lambda} + \sum_{|\Xi|>0} \widetilde{D}_\Xi Z \frac{\partial}{\partial z_\Xi},$$

$$Z = \left(z_y - \phi^{(1)}\right) f^{(1)} + z_x f^{(2)} - f^{(3)},$$

$$\Phi = f - \frac{f^{(2)} + f^{(1)} \left(\phi - z_x\right)}{\phi^2 - z_x \phi - z_y}, \qquad f = \sum_{i>0} f^{(i)} \lambda^i.$$

From shadows to symmetries III

The field $\mathcal{S}$ is a symmetry iff it preserves the GT equation and the covering system, i.e.,

$$\widetilde{D}_x\Phi + \frac{\phi\widetilde{D}_x Z + \widetilde{D}_y Z - (2\phi - z_x)\Phi}{(\phi^2 - z_x\phi - z_y)^2} = 0,$$

$$\widetilde{D}_y\Phi + \frac{z_y\widetilde{D}_x Z + (\phi - z_x)\widetilde{D}_y Z - ((\phi - z_x)^2 + z_y)\Phi}{(\phi^2 - z_x\phi - z_y)^2} = 0.$$

Notation

Assuming that the coefficients $f^{(i)}$ are independent of λ, we denote

$$\mathfrak{S}_f = \sum_{i>0} f^{(i)} \frac{\partial}{\partial \phi^{(i)}} = \sum_{\Lambda} \partial_\Lambda f \frac{\partial}{\partial \phi_\Lambda} \quad \text{if} \quad f = \sum_{i>0} f^{(i)} \lambda^i.$$

Thus, $\mathfrak{S}_f$ is the usual prolongation of a vertical generator $f\,\partial/\partial\phi$.

From shadows to symmetries IV

The symmetries we are looking for can be written as

$$\mathcal{S} = \mathfrak{S}_\Phi + \sum_{|\Xi|>0} \widetilde{D}_\Xi Z \frac{\partial}{\partial z_\Xi},$$

where

$$Z = \left(z_y - \phi^{(1)}\right) f^{(1)} + z_x f^{(2)} - f^{(3)},$$

$$\Phi = f - \frac{f^{(2)} + f^{(1)}\,(\phi - z_x)}{\phi^2 - z_x\phi - z_y},$$

$$f = \sum_{i>0} f^{(i)} \lambda^i.$$

From shadows to symmetries V

Let us consider the function

$$f = \lambda^n \phi_\lambda = -\lambda^{n-2} + \sum_{i \geq 1} i\phi^{(i)} \lambda^{n+i-1}.$$

The case $n \geq 3$. The series $\lambda^n \phi_\lambda$ is a polynomial without the constant term, which is the condition required by the definition of $\mathfrak{S}_f$.

If $n = 3$, then $f = \lambda^3 \phi_\lambda = -\lambda + \sum_{i \geq 1} i\phi^{(i)} \lambda^{2+i}$, i.e., $f^{(1)} = -1$, $f^{(2)} = 0$, $f^{(3)} = \phi^{(1)}$. In this case, $Z = -z_y = -Z^{(-2)}$ and we obtain the lift

$$\mathcal{S}^{(-2)} = \mathfrak{S}_{\lambda^3 \phi_\lambda} = -\frac{\partial}{\partial \phi^{(1)}} + \sum_{i \geq 1} i\phi^{(i)} \frac{\partial}{\partial \phi^{(2+i)}}$$

of the y-translation.

From shadows to symmetries VI

If $n = 4$, then $f = \lambda^4\phi_\lambda = -\lambda^2 + \sum_{i\geq 1} i\phi^{(i)}\lambda^{3+i}$, i.e., $f^{(1)} = f^{(3)} = 0$, $f^{(2)} = -1$. In this case, $Z = -z_x = -Z^{(-3)}$ and we obtain the lift

$$\mathcal{S}^{(-3)} = \mathfrak{S}_{\lambda^4\phi_\lambda} = -\frac{\partial}{\partial\phi^{(2)}} + \sum_{i\geq 1} i\phi^{(i)}\frac{\partial}{\partial\phi^{(3+i)}}$$

of the x-translation.

If $n = 5$, then $f = \lambda^5\phi_\lambda = -\lambda^3 + \sum_{i\geq 1} i\phi^{(i)}\lambda^{4+i}$, i.e., $f^{(1)} = f^{(2)} = 0$, $f^{(3)} = -1$. In this case, $Z = 1 = Z^{(-4)}$, and obtained the lift

$$\mathcal{S}^{(-4)} = \mathfrak{S}_{\lambda^5\phi_\lambda} = -\frac{\partial}{\partial\phi^{(3)}} + \sum_{i\geq 1} i\phi^{(i)}\frac{\partial}{\partial\phi^{(4+i)}}.$$

From shadows to symmetries VII

If $n \geq 6$, then the coefficients $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ are zero. Obviously, $Z = 0$ and we obtain "invisible" symmetries

$$\mathcal{S}^{(1-n)} = \mathfrak{S}_{\lambda^n \phi_\lambda} = -\frac{\partial}{\partial \phi^{(n-2)}} + \sum_{i \geq 1} i\phi^{(i)} \frac{\partial}{\partial \phi^{(n+i-1)}}, \qquad n \geq 6.$$

The case $n < 3$. In this case, the series

$$f = \lambda^n \phi_\lambda = -\lambda^{n-2} + \sum_{i \geq 1} i\phi^{(i)} \lambda^{n+i-1}$$

contains non-positive terms and so we cannot construct the corresponding field $\mathfrak{S}_f$. Surprisingly enough, it suffices to remove the extra terms by adding a suitable polynomial in ϕ.

From shadows to symmetries VIII

For any formal series $T = \sum_i c_i t^i$ we use the notation $[t^n]T = c_n$.

We introduce the operators

$$P_\varphi f = \sum_{k=0}^{m} [\varphi^k] f \cdot \varphi^k, \qquad \mathfrak{P}_\varphi f = f - P_\varphi f = f - \sum_{k=0}^{m} [\varphi^k] f \cdot \varphi^k.$$

Then $\mathfrak{P}_\varphi f$ is a positive series in λ and a polynomial in ϕ.

Consider the family $f_n = \mathfrak{P}_\varphi(\lambda^n \varphi_\lambda)$, $n \leq 2$. The first three members are

$$f_2 = \mathfrak{P}_\varphi(\lambda^2 \varphi_\lambda) = \lambda^2 \varphi_\lambda + 1,$$

$$f_1 = \mathfrak{P}_\varphi(\lambda \varphi_\lambda) = \lambda \varphi_\lambda + \varphi,$$

$$f_0 = \mathfrak{P}_\varphi(\varphi_\lambda) = \varphi_\lambda + \varphi^2 - 3\phi^{(1)}.$$

From shadows to symmetries IX

Proposition. *For any $n \leq 2$, all functions*

$$f = f_n = \mathfrak{P}_\varphi(\varphi_\lambda \lambda^n)$$

provide nonlocal symmetries $\mathcal{S}^{1-n}$ in $\tilde{\mathcal{E}}$, which are extensions of the shadows Z^{1-n}.

To obtain explicit formulas for the symmetries, we denote

$$A_r^{(k,n)} = [\lambda^{n-r}]\phi^k \phi'^r.$$

The case of $r = 2$ has been already introduced earlier.

For $r = 0$ we have simply

$$A_0^{(k,n)} = \sum_{j_1+\cdots+j_k=n}^{(\bullet)} \phi^{(j_1)}\phi^{(j_2)}\cdots\phi^{(j_k)}, \qquad k > 0.$$

From shadows to symmetries X

Vector fields $\mathcal{S}^n = \mathfrak{S}_{f_{1-n}}$ admit the explicit formula

$$\mathcal{S}^n = \sum_{m \geq 1} \left((n+m)\phi^{(n+m)} + \sum_{k=0}^{n+1} A_0^{(k,m)} A_2^{(-k-1,n)} \right) \frac{\partial}{\partial \phi^{(m)}}.$$

Alternatively, we can also write

$$\mathcal{S}^n = -\sum_{m \geq 1} \sum_{k=0}^{m} A_0^{(-k-1,m)} A_2^{(k,n)} \frac{\partial}{\partial \phi^{(m)}}.$$

The Lie algebra structure

The symmetries $\mathcal{S}^n = \mathfrak{S}_{f_{1-n}}$ satisfy

$$[\mathcal{S}^n, \mathcal{S}^m] = (m-n)\mathcal{S}^{n+m},$$

i.e., constitute a basis of the Witt algebra.