On symmetries of the Gibbons–Tsarev equation

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The Gibbons–Tsarev equation

\[ \text{GT: } z_{yy} + z_x z_{xy} - z_y z_{xx} + 1 = 0. \]


The Benney chain:

\[ a^{(n)}_\tau + a^{(n+1)}_\xi + na^{(n-1)}_\xi a^{(0)}_\xi, \quad n = 0, 1, 2, \ldots \]

with unknowns \(a^{(n)}(\tau, \xi)\).

Gibbons–Tsarev reductions:

\(a^{(n)}, n \geq N, \) are functions of \(a^{(0)}, a^{(1)}, \ldots a^{(N-1)}\).

We obtain the above Gibbons–Tsarev equation \(\mathcal{E}\), if we set \(N = 2\) and

\[ x = a^{(1)}, \quad y = a^{(0)}, \quad z(x, y) = -a^{(2)} + \frac{1}{2} a^{(0)^2}. \]
What is known?
- A nonlinear and a linear Lax pair.
- Infinitely many nonlocal conservation laws.
- Infinite dimensional algebra of nonlocal infinitesimal symmetries.


What is not known?
- No recursion operator known.
- No matrix zero-curvature representation known.

What are our new results?
- Explicit formulas for infinitely many nonlocal conservation laws.
- Remarkable and explicit formulas for infinitely many nonlocal infinitesimal symmetries.
- A uniqueness theorem for nonlocal symmetries.
Janet monomial notation for jet coordinates

\[ z_X \leftrightarrow \frac{\partial^{i+j} z}{\partial x^i \partial y^j} \quad \text{if} \quad X = x^i y^j, \]

where \( x^i y^j \) is an arbitrary commutative monomial.

**Difiety \( E \)**

Internal coordinates: \( x, y, \) and \( z_X \) such that \( X = x^k \) or \( X = x^k y. \)

If not stated otherwise, sums are over all internal coordinates \( z_X. \)

The total derivatives

\[ D_x = \frac{\partial}{\partial x} + \sum z_{xX} \frac{\partial}{\partial z_X}, \quad D_y = \frac{\partial}{\partial y} + \sum z_{yX} \frac{\partial}{\partial z_X}, \]

where

\[ z_{yyX} = D_X(z_{yy}z_{xx} - z_xz_{xy} - 1) \]

for every \( X, \) i.e., the non-internal jet coordinates become functions of the internal ones.
Weights

The Gibbons–Tsarev equation is homogeneous under the weights

\[ |x| = 3, \quad |y| = 2, \quad |z| = 4. \]

Then

\[ |z_x| = 4 - 3k, \quad |z_x y| = 2 - 3k. \]

The weights are due to existence of the scaling symmetry, see below.

The operators of total derivatives have the weights

\[ |D_x| = -3, \quad |D_y| = -2. \]
Local symmetries

are

\[ S^{(i)} = \sum D_X Z^{(i)} \frac{\partial}{\partial z^X}, \quad i = -4, \ldots, 0, \]

(sum over all internal coordinates), where

\[ Z^{(-4)} = 1, \quad z\text{-translation}, \]
\[ Z^{(-3)} = z_x, \quad x\text{-translation}, \]
\[ Z^{(-2)} = z_y, \quad y\text{-translation}, \]
\[ Z^{(-1)} = yz_x - 2x, \quad \text{generalized Galilean boost}, \]
\[ Z^{(0)} = 2z - \frac{3}{2} xz_x - yz_y, \quad \text{scaling}. \]

Provably, this is the complete set of local symmetries.

The vector field \( S^{(i)} \) is of weight \( i \) (as an operator).
Commutators.

The commutators of symmetries satisfy

\[ [S^{(i)}, S^{(j)}] = (j - i)S^{(i+j)}, \]

if we set \( S^{(\alpha)} = 0 \) for \( \alpha < -4 \).

It will be shown below that this set of five symmetries can be extended to a hierarchy of nonlocal symmetries infinite in both positive and negative directions, satisfying the same commutation relations.
Local conservation laws

A computation using cosymmetries reveals conservation laws $\rho^{(i)}$ of the weight $i + 5$, namely $\rho^{(i)} = P^{(i)} \, dx + Q^{(i)} \, dy$, where

\[
P^{(0)} = z_x^2 + z_y + y,
Q^{(0)} = z_x z_y,
P^{(1)} = z_x^3 + 2z_x z_y - x,
Q^{(1)} = z_x^2 z_y + z_y^2 - 2z,
\]

\[
P^{(2)} = z_x^4 + 3z_x^2 z_y + 3y z_x^2 + z_y^2 - xz + 3yz - 2z,
Q^{(2)} = z_x^3 z_y + 2z_x^2 z_y + 3yz z_y - xyz - 3xy,
\]

\[
P^{(3)} = z_x^5 + 4z_x^3 z_y + 4yz z_y^2 + 3zz_y + 2xz^2 + 8yz z_y
- 2z_x + 2z_y - 4xy,
Q^{(3)} = z_x^4 z_y + 3z_x^2 z_y^2 + 4yz z_y^2 + z_y^3 + 2xz z_y + 4yz^2
- 2z_x - 8yz - 3x^2,
\]
More local conservation laws

\[
P^{(4)} = z^6_x + 5z^4_xz_y + 5yz^4_x + 6z^2_xz^2_y + 3xz^3_x + 15yz^2_xz_y + z^3_y \\
+ (z + \frac{15}{2}y^2)z^2_x + 6xz_xz_y + 5yz^2_x + (z + \frac{15}{2}y^2)z_y \\
- 5yz - 4x^2,
\]

\[
Q^{(4)} = z^5_x + 4z^3_xz_y + 5yz^3_x + 3z_xz^3_y + 3xz^2_xz_y + 10yz_xz^2_y \\
+ (z + \frac{15}{2}y^2)z_xz_y + 3xz^2_y - \frac{3}{2}x(4z + 5y^2),
\]

\[
P^{(5)} = z^7_x + 6z^5_xz_y + 6yz^5_x + 10z^3_xz^2_y + 4xz^4_x + 24yz^3_xz_y + 4z_xz^3_y \\
+ 4\left(\frac{1}{2}z + 3y^2\right)z^3_x + 12xz^2_xz_y + 18yz_xz^2_y + 12xz^2_x \\
+ 4\left(z + 6y^2\right)z_xz_y + 4xz^2_y + 12yz_xz_y - 4xz,
\]

\[
Q^{(5)} = z^6_x + 5z^4_xz_y + 6yz^4_x + 6z^2_xz^3_y + 4xz^3_xz_y + 18yz^2_xz^2_y + z^4_y \\
+ 4\left(\frac{1}{2}z + 3y^2\right)z^2_xz_y + 8xz_xz^2_y + 6yz^3_x + 12xyz_xz_y \\
+ 4\left(\frac{1}{2}z + 3y^2\right)z^2_y - 2z^2 - 24yz^2 - 6x^2y,
\]

\[
P^{(-5)} = -\frac{1}{2}z_xz^2_xz - z_xz_yz_x, \\
Q^{(-5)} = -\frac{1}{2}z_yz^2_xz - \frac{1}{2}z^2_xz_y.
\]
Infinite hierarchies of conservation laws

Two ways (equivalent results).

Two Lax pairs

I. The nonlinear Lax pair given by
\[ \phi_x = \frac{1}{z_y + z_x \phi - \phi^2}, \quad \phi_y = \frac{\phi - z_x}{z_y + z_x \phi - \phi^2}. \]

II. The linear Lax pair
\[ \psi_x = \frac{1}{\lambda^2 - z_x \lambda - z_y} \cdot \psi\lambda, \quad \psi_y = \frac{\lambda - z_x}{\lambda^2 - z_x \lambda - z_y} \cdot \psi\lambda, \]

obtained from the former by the reversion procedure.


Both systems have the Gibbons–Tsarev equation as the compatibility condition.
**Linear potentials** $\psi^{(i)}$

Inserting

$$\psi = \lambda + \frac{\psi^{(1)}}{\lambda} + \cdots + \frac{\psi^{(k)}}{\lambda^k} + \cdots,$$

into the linear Lax pair, we obtain a tower of linear systems

$$\psi^{(k)}_x = \sigma_{k-2} - \sum_{i=1}^{k-3} i \sigma_{k-i-3} \psi^{(i)}_z,$$

$$\psi^{(k)}_y = z_y \psi^{(k-1)}_x - (k - 2) \psi^{(k-2)},$$

where

$$\sigma_k = \sum_{0 \leq j \leq k-j} \binom{k-j}{j} z_x^{k-2j} z_y^j, \quad k > 0.$$

These potentials are not quite convenient for the construction of nonlocal symmetries.
Nonlinear potentials $\phi^{(i)}$

Inserting

$$\phi = \lambda^{-1} + \phi^{(1)}\lambda + \cdots + \phi^{(k)}\lambda^k + \cdots$$

into the nonlinear Lax pair, we obtain a tower of nonlinear systems

$$\phi_x^{(m)} = -2 \sum_{i=1}^{m-2} \phi^{(i)}\phi_x^{(m-i-1)} - \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-2} \phi^{(i)}\phi^{(j)}\phi_x^{(m-i-j-2)}$$

$$+ z_x \left( \phi_x^{(m-1)} + \sum_{i=1}^{m-2} \phi^{(i)}\phi_x^{(m-i-2)} \right) + z_y \phi_x^{(m-2)} + \delta_2^m,$$

$$\phi_y^{(m)} = -2 \sum_{i=1}^{m-2} \phi^{(i)}\phi_y^{(m-i-1)} - \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-2} \phi^{(i)}\phi^{(j)}\phi_y^{(m-i-j-2)}$$

$$+ z_x \left( \phi_y^{(m-1)} + \sum_{i=1}^{m-2} \phi^{(i)}\phi_y^{(m-i-2)} - \delta_2^m \right) + z_y \phi_y^{(m-2)} - \phi^{(m-2)}.$$
Comparison of the potentials $\psi^{(i)}$ and $\phi^{(i)}$

The weights are the same: $|\phi^{(k)}| = |\psi^{(k)}| = k + 1$.

The first three potentials are local:

$\psi^{(1)} = y, \quad \phi^{(1)} = -y,$
$\psi^{(2)} = x, \quad \phi^{(2)} = -x,$
$\psi^{(3)} = z - \frac{1}{2} y^2, \quad \phi^{(3)} = -z - \frac{1}{2} y^2.$

Mutual relations (will be proved below):

$\psi^{(k)} = -\sum_{m \geq 1} (-1)^m \sum_{i_1 + \cdots + i_m = k+1} \frac{1}{k} \binom{k}{m} \phi^{(i_1-1)} \cdots \phi^{(i_m-1)},$
$\phi^{(k)} = -\sum_{m \geq 1} \sum_{i_1 + \cdots + i_m = k+1} \frac{1}{k} \binom{k}{m} \psi^{(i_1-1)} \cdots \psi^{(i_m-1)}$. 
Relation among the potentials – part I

Recall that
\[ \psi(\lambda) = \lambda + \frac{\psi^{(1)}}{\lambda} + \frac{\psi^{(2)}}{\lambda^2} + \cdots, \quad \phi(\lambda) = \frac{1}{\lambda} + \phi^{(1)} \lambda + \phi^{(2)} \lambda^2 + \cdots \]

The formal series
\[ \frac{1}{\phi(\lambda)} = -\lambda + \phi^{(1)} \lambda^3 + \phi^{(2)} \lambda^4 + (\phi^{(3)} - \phi^{(1)^2}) \lambda^5 + \cdots \]

is a power series of lowest degree 1.

Therefore, the composition series
\[ \psi(\phi) = \phi(\lambda) + \frac{\psi^{(1)}}{\phi(\lambda)} + \frac{\psi^{(2)}}{\phi(\lambda)^2} + \cdots \]

is well defined (no infinite sums occur).
Relation among the potentials – part II

Substituting $\phi$ for $\lambda$ in

$$\psi_x = \frac{1}{\lambda^2 - z_x \lambda - z_y} \cdot \psi', \quad \psi_y = \frac{\lambda - z_x}{\lambda^2 - z_x \lambda - z_y} \cdot \psi',$$

we obtain

$$(\psi(\phi))_x = \psi_x(\phi) + \psi'(\phi)\phi_x = 0,$$

$$(\psi(\phi))_y = \psi_y(\phi) + \psi'(\phi)\phi_y = 0$$

since

$$\phi_x = -\frac{1}{\phi' - z_x \phi - z_y}, \quad \phi_y = -\frac{\phi - z_x}{\phi' - z_x \phi - z_y}.$$  

Therefore, $\psi(\phi) = c(\lambda)$.

The converse is also true, if $\psi(\phi) = c(\lambda)$, then the two systems above are equivalent.
Relation among the potentials – part III

Let us choose $c(\lambda) = 1/\lambda$, i.e.,

$$\psi(\phi(\lambda)) = 1/\lambda.$$ 

This is compatible with the choices we made for $\psi^{(i)}$, $\phi^{(i)}$, $i = 1, 2, 3$.

**Proposition.** The power series $1/\psi(1/\lambda)$ and $1/\phi(\lambda)$ are compositionally inverse one to another.

Indeed, substituting $1/\phi(\lambda)$ for $\lambda$ in $1/\psi(1/\lambda)$, we obtain

$$1/\psi(\phi(\lambda)) = 1/c(\lambda) = \lambda.$$ 

**Corollary.** The coefficients $\psi^{(i)}$ determine the coefficients $\phi^{(i)}$ and vice versa.
A reminder about diffieties

Recall the diffiety $\mathcal{E}$, an infinite dimensional manifold, equipped with the total derivatives, written in the “internal” coordinates $z_X$, $y^2 \| X$, as

$$D_x = \frac{\partial}{\partial x} + \sum z_{xX} \frac{\partial}{\partial z_X}, \quad D_y = \frac{\partial}{\partial y} + \sum z_{yX} \frac{\partial}{\partial z_X},$$

$$[D_x, D_y] = 0.$$ 

As a consequence of the Gibbons–Tsarev equation, the “external” variables $z_{yyX}$ become functions

$$z_{yyX} = D_x (z_y z_{xx} - z_x z_{xy} - 1).$$

Symmetries

$$S = \sum D_X Z(x, y, z_X) \frac{\partial}{\partial z_X}, \quad \sum D_X Z \frac{\partial \text{GT}}{\partial z_X} = 0,$$

are vector fields $S$ on $\mathcal{E}$, satisfying $[S, D_x] = [S, D_y] = 0 = S(\text{GT}).$
Coverings

An $n$-dimensional covering over $\mathcal{E}$ is the projection $\tilde{\mathcal{E}} = \mathcal{E} \times \mathbb{R}^n \rightarrow \mathcal{E}$. Nonlocal variables $q^i$ are defined to be any coordinates in the fibres $\mathbb{R}^n$. The total derivatives assume the form

$$\tilde{D}_x = D_x + \sum_{i=1}^n X^i \frac{\partial}{\partial q^i}, \quad \tilde{D}_y = D_y + \sum_{i=1}^n Y^i \frac{\partial}{\partial q^i},$$

where $X^i, Y^i$ are functions of the internal coordinates $x, y, z_X$ of $\mathcal{E}$ and of the nonlocal variables $q^i$. They are required to satisfy

$$[\tilde{D}_x, \tilde{D}_y] = 0.$$

The two coverings

We have two infinite-dimensional coverings, one with the nonlocal variables $\psi^{(k)}$, the other with the nonlocal variables $\phi^{(k)}$. They are equivalent.
Nonlocal symmetries

Nonlocal symmetries are symmetries in a covering $\tilde{\mathcal{E}}$.

Of the two coverings we choose that with the potentials $\phi^{(i)}$:

$$\mathbf{GT} : \quad \mathbf{GT}, \quad \phi^{(i)}_x = X^{(i)}, \quad \phi^{(i)}_y = Y^{(i)}.$$  

Symmetries will be vector fields $\mathcal{S}$ on $\tilde{\mathcal{E}}$, satisfying $[\mathcal{S}, \tilde{D}_x] = [\mathcal{S}, \tilde{D}_y] = 0 = \mathcal{S}(\mathbf{GT})$. In particular,

$$\mathcal{S} = \sum D_X Z(z_X, \phi^{(*)}) \frac{\partial}{\partial z_X} + \sum \Phi^{(i)}(z_X, \phi^{(*)}) \frac{\partial}{\partial \phi^{(i)}},$$

where

$$\sum D_X Z \frac{\partial \mathbf{GT}}{\partial z_X} = 0, \quad D_x \Phi^{(i)} = \sum D_X Z \frac{\partial X^{(i)}}{\partial z_X} + \sum \Phi^{(j)} \frac{\partial X^{(i)}}{\partial \phi^{(j)}},$$

and analogously for $D_y \Phi^{(i)}$.

The first summand in $\mathcal{S}$ is called a shadow.
The shadows I

To determine nonlocal symmetries, we are looking for shadows in the first place, i.e., for functions $Z(z_X, \phi^{(\ell)})$ satisfying

$$\sum D_XZ \frac{\partial GT}{\partial z_X} = 0.$$ 

Using the expansion

$$\phi = \frac{1}{\lambda} + \phi^{(1)} \lambda + \phi^{(2)} \lambda^2 + \cdots$$

we introduce different nonlocal variables

$$\phi_\Lambda \xrightarrow{\ast} \frac{d^i \phi}{d\lambda^i}, \quad \Lambda = \lambda^i$$

Consider the covering $\tilde{\tau}_\lambda: \tilde{\mathcal{E}}_\lambda \rightarrow \mathcal{E}$, where $\tilde{\mathcal{E}}_\lambda = \mathcal{E} \times J(\lambda; \phi)$. $J(\lambda; \phi)$ is the jet space with the coordinates $\lambda$ and $\phi_\Lambda$. 

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The shadows II

Total derivatives in $\tilde{\mathcal{E}}_\Lambda$:

$$
\tilde{D}_x = D_x + \sum_{\Lambda} \phi_{\lambda} \frac{\partial}{\partial \phi_{\lambda}}, \quad \tilde{D}_y = D_y + \sum_{\Lambda} \phi_{y\lambda} \frac{\partial}{\partial \phi_{\lambda}},
$$

$$
\tilde{D}_\lambda = \frac{d}{d\lambda} + \sum_{\Lambda} \phi_{\lambda} \frac{\partial}{\partial \phi_{\lambda}},
$$

The diffiety $\tilde{\mathcal{E}}_\Lambda$ corresponds to the covering $\tilde{\mathcal{G}}T$, considered over an extended set $x, y, \lambda$ of independent variables, along with

$$
z_\lambda = 0.
$$
Proposition. Expand

\[ Z = (\phi^2 - z_x \phi - z_y) \phi_\lambda^2 \]

as a formal Laurent series of the form

\[ \sum_{n=-4}^{\infty} Z^{(n)} \lambda^{n-2}. \]

Then \( Z^{(n)} \) are shadows of symmetries of the Gibbons–Tsarev equation in the covering \( \tilde{E} \).

Proof. By coordinate computation, using the fact that GT does not involve \( \lambda \).
The shadows IV

The formula

\[ Z = (\phi^2 - z_x \phi - z_y) \phi^2 \]

leads to an explicit formula for the shadows.

Let \( \sum^{(*)} \) denote summation where indices run through all integers from \(-1\) to infinity.

**Proposition.** Let

\[ A^{(k,n)}_2 = \sum^{(*)}_{i_1 + \ldots + i_{k+2} = n} i_1 i_2 \phi^{(i_1)} \ldots \phi^{(i_{k+2})}, \quad k \geq 0. \]

Then

\[ Z^{(n)} = A^{(1,n)}_2 z_x + A^{(0,n)}_2 z_y - A^{(2,n)}_2. \]
We look for nonlocal symmetries in the form
\[ a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} + \sum_{i>3} f^{(i)} \frac{\partial}{\partial \phi^{(i)}} + \cdots \]

where \( a, b, c, f^{(i)} \) are functions on \( \tilde{E} \).

The corresponding vertical field, obtained by subtracting \( aD_x + bD_y \), is
\[ S = \sum_{\Xi} \tilde{D}_{\Xi}(c - az_x - bz_y) \frac{\partial}{\partial z_{\Xi}} + \sum_{i>3} (f^{(i)} - aX^{(i)} - bY^{(i)}) \frac{\partial}{\partial \phi^{(i)}}. \]

Variables \( x, y, z \) can be expressed in terms of \( \phi^{(i)}, = 1, 2, 3 \).
Consequently, we can rewrite the symmetry in terms of \( \phi^{(i)} \) and \( z_{\Xi} \), \( |\Xi| > 0 \), alone.
From shadows to symmetries II

The symmetry in terms of $\phi^{(i)}$ and $z_{\Xi}$:

$$ S = \sum_{i>0} \Phi^{(i)} \frac{\partial}{\partial \phi^{(i)}} + \sum_{|\Xi|>0} \bar{D}_{z_{\Xi}} Z \frac{\partial}{\partial z_{\Xi}}, $$

$$ Z = (z_y - \phi^{(1)}) f^{(1)} + z_x f^{(2)} - f^{(3)}, $$

$$ \Phi^{(i)} = f^{(i)} + f^{(2)} X^{(i)} + f^{(1)} Y^{(i)}. $$

The same symmetry in terms of $\phi_{\Lambda}$ and $z_{\Xi}$:

$$ S = \sum_{\Lambda} \partial_{\Lambda} \Phi \frac{\partial}{\partial \phi_{\Lambda}} + \sum_{|\Xi|>0} \bar{D}_{z_{\Xi}} Z \frac{\partial}{\partial z_{\Xi}}, $$

$$ Z = (z_y - \phi^{(1)}) f^{(1)} + z_x f^{(2)} - f^{(3)}, $$

$$ \Phi = f - \frac{f^{(2)} + f^{(1)} (\phi - z_x)}{\phi^2 - z_x \phi - z_y}, \quad f = \sum_{i>0} f^{(i)} \lambda^i. $$
From shadows to symmetries III

The field $\mathcal{S}$ is a symmetry if it preserves the GT equation and the covering system, i.e.,

$$
\bar{D}_x \Phi + \frac{\phi \bar{D}_x Z + \bar{D}_y Z - (2\phi - z_x) \Phi}{(\phi^2 - z_x \phi - z_y)^2} = 0,
$$

$$
\bar{D}_y \Phi + \frac{z_y \bar{D}_x Z + (\phi - z_x) \bar{D}_y Z - ((\phi - z_x)^2 + z_y) \Phi}{(\phi^2 - z_x \phi - z_y)^2} = 0.
$$

Notation

Assuming that the coefficients $f^{(i)}$ are independent of $\lambda$, we denote

$$
\mathcal{S}_f = \sum_{i>0} f^{(i)} \frac{\partial}{\partial \phi^{(i)}} = \sum_{\Lambda} \partial_{\Lambda} f \frac{\partial}{\partial \phi_{\Lambda}} \quad \text{if} \quad f = \sum_{i>0} f^{(i)} \lambda^i.
$$

Thus, $\mathcal{S}_f$ is the usual prolongation of a vertical generator $f \partial / \partial \phi$. 

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From shadows to symmetries IV

The symmetries we are looking for can be written as

$$ S = \mathcal{S}_\phi + \sum_{|\Xi| > 0} D_{\Xi} Z \frac{\partial}{\partial z_{\Xi}}, $$

where

$$ Z = (z_y - \phi^{(1)}) f^{(1)} + z_x f^{(2)} - f^{(3)}, $$

$$ \Phi = f - \frac{f^{(2)} + f^{(1)} (\phi - z_x)}{\phi^- - z_x \Phi - z_y}, $$

$$ f = \sum_{i > 0} f^{(i)} \lambda^i. $$
From shadows to symmetries V

Let us consider the function

\[ f = \lambda^n \phi_\lambda = -\lambda^{n-2} + \sum_{i \geq 1} i\phi^{(i)} \lambda^{n+i-1}. \]

The case \( n \geq 3 \). The series \( \lambda^n \phi_\lambda \) is a polynomial without the constant term, which is the condition required by the definition of \( \Theta_f \).

If \( n = 3 \), then \( f = \lambda^3 \phi_\lambda = -\lambda + \sum_{i \geq 1} i\phi^{(i)} \lambda^{2+i} \), i.e., \( f^{(1)} = -1 \), \( f^{(2)} = 0 \), \( f^{(3)} = \phi^{(1)} \). In this case, \( Z = -z_y = -Z^{(-2)} \) and we obtain the lift

\[ S^{(-2)} = \Theta \lambda^3 \phi_\lambda = -\frac{\partial}{\partial \phi^{(1)}} + \sum_{i \geq 1} i\phi^{(i)} \frac{\partial}{\partial \phi^{(2+i)}} \]

of the \( y \)-translation.
From shadows to symmetries VI

If $n = 4$, then $f = \lambda^4 \phi_{\lambda} = -\lambda^2 + \sum_{i \geq 1} i \phi^{(i)} \lambda^{3+i}$, i.e.,

$f^{(1)} = f^{(3)} = 0$, $f^{(2)} = -1$. In this case, $Z = -z_x = -Z^{(-3)}$ and we obtain the lift

$$S^{(-3)} = \mathcal{E}_{\lambda^4 \phi_{\lambda}} = -\frac{\partial}{\partial \phi^{(2)}} + \sum_{i \geq 1} i \phi^{(i)} \frac{\partial}{\partial \phi^{(3+i)}}$$

of the $x$-translation.

If $n = 5$, then $f = \lambda^5 \phi_{\lambda} = -\lambda^3 + \sum_{i \geq 1} i \phi^{(i)} \lambda^{4+i}$, i.e.,

$f^{(1)} = f^{(2)} = 0$, $f^{(3)} = -1$. In this case, $Z = 1 = Z^{(-4)}$, and obtained the lift

$$S^{(-4)} = \mathcal{E}_{\lambda^5 \phi_{\lambda}} = -\frac{\partial}{\partial \phi^{(3)}} + \sum_{i \geq 1} i \phi^{(i)} \frac{\partial}{\partial \phi^{(4+i)}}$$.
If $n \geq 6$, then the coefficients $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ are zero. Obviously, $Z = 0$ and we obtain “invisible” symmetries

$$S^{(1-n)} = \mathcal{S}_{\lambda^n \phi_\lambda} = -\frac{\partial}{\partial \phi^{(n-2)}} + \sum_{i \geq 1} i \phi^{(i)} \frac{\partial}{\partial \phi^{(n+i-1)}}, \quad n \geq 6.$$  

The case $n < 3$. In this case, the series

$$f = \lambda^n \phi_\lambda = -\lambda^{n-2} + \sum_{i \geq 1} i \phi^{(i)} \lambda^{n+i-1}$$

contains non-positive terms and so we cannot construct the corresponding field $\mathcal{S}_f$. Surprisingly enough, it suffices to remove the extra terms by adding a suitable polynomial in $\phi$. 
From shadows to symmetries VIII

For any formal series $T = \sum_i c_i t^i$ we use the notation $[t^n]T = c_n$.

We introduce the operators

$$P_{\varphi} f = \sum_{k=0}^{m} [\varphi^k] f \cdot \varphi^k,$$

$$\mathbf{P}_{\varphi} f = f - P_{\varphi} f = f - \sum_{k=0}^{m} [\varphi^k] f \cdot \varphi^k.$$

Then $\mathbf{P}_{\varphi} f$ is a positive series in $\lambda$ and a polynomial in $\phi$.

Consider the family $f_n = \mathbf{P}_{\varphi}(\lambda^n \varphi^\lambda)$, $n \leq 2$. The first three members are

$$f_2 = \mathbf{P}_{\varphi}(\lambda^2 \varphi^\lambda) = \lambda^2 \varphi^\lambda + 1,$$

$$f_1 = \mathbf{P}_{\varphi}(\lambda \varphi^\lambda) = \lambda \varphi^\lambda + \varphi,$$

$$f_0 = \mathbf{P}_{\varphi}(\varphi^\lambda) = \varphi^\lambda + \varphi^2 - 3\phi^{(1)}.$$

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From shadows to symmetries IX

Proposition. For any \( n \leq 2 \), all functions \( f = f_n = \mathcal{P}_n(\varphi_\lambda^{\lambda^n}) \) provide nonlocal symmetries \( S^{1-n} \) in \( \tilde{E} \), which are extensions of the shadows \( Z^{1-n} \).

To obtain explicit formulas for the symmetries, we denote
\[
A^{(k,n)}_r = [\lambda^{n-r}] \phi^k \phi^{(r)}.
\]

The case of \( r = 2 \) has been already introduced earlier.

For \( r = 0 \) we have simply
\[
A^{(k,n)}_0 = \sum_{j_1 + \cdots + j_k = n}^{(*)} \phi^{(j_1)} \phi^{(j_2)} \cdots \phi^{(j_k)}, \quad k > 0.
\]
From shadows to symmetries \( X \)

Vector fields \( S^n = \mathfrak{S}_{f_{1-n}} \) admit the explicit formula

\[
S^n = \sum_{m \geq 1} \left( (n + m) \phi^{(n+m)} + \sum_{k=0}^{n+1} A_0^{(k,m)} A_2^{(-k-1,n)} \right) \frac{\partial}{\partial \phi^{(m)}}.
\]

Alternatively, we can also write

\[
S^n = - \sum_{m \geq 1} \sum_{k=0}^{m} A_0^{(-k-1,m)} A_2^{(k,n)} \frac{\partial}{\partial \phi^{(m)}}.
\]

The Lie algebra structure

The symmetries \( S^n = \mathfrak{S}_{f_{1-n}} \) satisfy

\[
[S^n, S^m] = (m - n) S^{n+m},
\]

i.e., constitute a basis of the Witt algebra.

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