Structure preserving discretizations of the Liouville equation

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Workshop on Integrable Nonlinear Equations
Mikulov, 18-24/10/2015

Aims

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1. How symmetry structure in PDEs can be preserved in PΔEs;
2. Compare three different structure preserving procedures of discretization for the Liouville equation (as prototype):
   - maximal finite subgroup of the point symmetry group,
   - infinite point-symmetry group as discrete higher symmetries ,
   - linearizability .
3. Geometric integration: focus on preserved geometrical properties under discretization addressed to numerical solutions.

References

- D Levi, L Martina and P Winternitz: *Lie-point symmetries of the discrete Liouville equation*, J. Phys A: Math. Theor. **48**, 2 (2015) 025204.
- D Levi, L Martina and P Winternitz: *Structure Preserving Discretizations of the Liouville Equation and their Numerical Tests*, SIGMA **11** (2015), 080

Outline

Motivations

Point-Symmetry preserving discretization

- Dorodnitsyn V A 1991 *J. Soviet Math.* **55** 1490–1517; Dorodnitsyn V A 2011 , CRC Press; Dorodnitsyn V A, Kozlov R and Winternitz P 2000, *J. Math. Phys.* **41** 480–504.
- Winternitz P 2004, **644** of Lecture Notes in Physics pp 185–243, B. Grammaticos, et al., nlin.SI/0309058; Winternitz P 2011, LMS Lecture Series, Ed.s Levi, Olver, Thomova, Winternitz;
- Levi D and Winternitz P 1991 *Phys. Lett. A* **152** 335–338; Levi D and Winternitz P 1996 *J.Math. Phys.* **37** 5551–5576; Levi D and Winternitz P 2006 *J. Phys. A* **39**, no. 2, R1-R63

Motivations

Implementation on ODEs

- Bourlioux A, Rebelo R and Winternitz P 2008 *J. Nonlinear Math. Phys.* **15** 362–372; Bourlioux A, Cyr-Gagnon C and Winternitz P 2006 *J.Phys. A* **39** 6877–6896;

Implementation on PDEs

- F Valiquette and P Winternitz, 2005 *J.Phys. A* **39** 9765;
- Levi D and Rodriguez M A 2014 *arXiv:1407.0838*;
- Bihlo A and Popovych R O 2012 , *SIAM J. Sci. Comput.* **34** B810–B839.; Bihlo A and Nave J C 2013 , *SIGMA* **9**; Bihlo A 2013 *J. Phys. A: Math. Theor.* **46** 062001; Bihlo A, Coiteux-Roy X and Winternitz P 2014, arXiv:1409.4340.

A toy model: the Liouville Equation

The **Liouville equation**

$$z_{xy} = e^{z}.$$

Constant Curvature surfaces
sl(2) reduction of the **2-dim Toda lattice** $\partial\bar{\partial} z_j = \sum_i^{\ell} \beta\alpha_{i\,j} \exp\left[\beta\alpha_i \cdot z\right]$
A.V. MIkhailov JEPT Lett. (1979)

$$U_{xy} = \Phi\left(U, U_x, U_y\right) \qquad U_t = f\left(U, U_x, U_{xx}, \ldots\right), U_{t'} = g\left(U, U_y, U_{yy}, \ldots\right)$$

S.I. Svinolupov, V.V. Sokolov, Fun.AAppl. (1982)

$$\text{Algebraic form} \quad u\, u_{xy} - u_x\, u_y = u^3, \qquad u = e^{z}.$$

Linearizability

$$u = 2\frac{\phi_x\, \phi_y}{\phi^2}, \qquad \phi_{xy} = 0 \qquad \text{(B. T.)}$$

$$\phi\left(x, y\right) = \psi\left(x\right) + \chi\left(y\right) \quad \Rightarrow \quad z = \ln\left(2\frac{\psi_x\, \chi_y}{\left(\psi + \chi\right)^2}\right)$$

The Point Symmetry Group

∞-dim Lie point symmetry algebra (Medolaghi, 1898)

$$\begin{aligned} X(f(x)) &= f(x)\,\partial_x - f_x(x)\; u\,\partial_u, \\ Y(g(y)) &= g(y)\,\partial_y - g_y(y)\; u\,\partial_u \quad \forall f, g \text{ smooth f.s} \end{aligned}$$

$$\left[X(f), X(\tilde{f})\right] = X\left(f_x\tilde{f} - f\tilde{f}_x\right), \quad \left[Y(g), Y(\tilde{g})\right] = Y\left(g\,\tilde{g}_y - \tilde{g}\,g_y\right), \quad \left[X(f), Y(g)\right] = 0.$$

$$L = vir_x \oplus vir_y$$

$$\tilde{u}(\tilde{x},\tilde{y}) = u(x(\tilde{x}), y(\tilde{y}))\, x_{\tilde{x}}\, y_{\tilde{y}}, \quad f(\tilde{x})\; x_{\tilde{x}} = f(x), \quad g(\tilde{y})\; y_{\tilde{y}} = g(y)$$

Maximal finite subalgebra

$$\begin{aligned} sl_x(2,\mathbb{R}) \quad \bigoplus \quad sl_y(2,\mathbb{R}) \subset L \\ f, g \quad \in \quad \mathcal{P}_2 \end{aligned}$$

Invariants of $SL_x(2,\mathbb{R}) \otimes SL_y(2,\mathbb{R})$

A basis of 2^o order "strong" invariants $I(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$

$$I_1 = \frac{uu_{xy} - u_x\, u_y}{u^3}, \qquad I_2 = \frac{\left(2uu_{xx} - 3u_x^2\right)\, \left(2uu_{yy} - 3u_y^2\right)}{u^6}$$

$$\begin{aligned} & \mathrm{pr}^{(2)} X(f)\ I_1 = \mathrm{pr}^{(2)}\ Y(g)\ I_1 = 0 \qquad \forall\, f, g \\ \Rightarrow \quad & VIR(x) \otimes VIR(y) - \mathrm{Invariant} \\ & \mathrm{pr}^{(2)} X(f)\ I_2 = \frac{2f_{xxx}\left(3u_y^2 - 2uu_{yy}\right)}{u^4}, \quad \mathrm{pr}^{(2)}\ Y(g)\ I_2 = \frac{2g_{yyy}\left(3u_x^2 - 2uu_{xx}\right)}{u^4} \\ \Rightarrow \quad & f_{xxx} = g_{yyy} = 0 \end{aligned}$$

The symmetry preserving discretization

Idea:

- Replace a given **PDE** by a **system of difference equations** formed out of invariants of the action of **its** point symmetry group on a **stencil**.
- The difference system describes **both** a **lattice** and the **PΔE**
- In the continuous limit

$$\text{the lattice eq.} \rightarrow \text{identity}, \qquad \text{P}\Delta\text{E} \rightarrow \text{PDE}$$

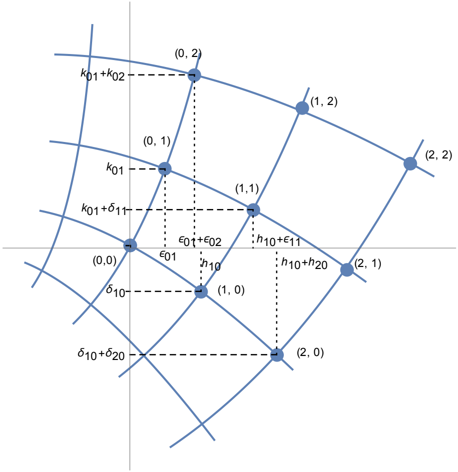
Given a PDE

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots\right) = 0 \quad \Rightarrow \mathcal{L} \ \text{(Lie symmetry alg.)}$$

PΔE

$$\mathcal{L} \quad \Rightarrow \qquad E_\alpha\left(x_{m+i,n+j}, y_{m+i,n+j}, u_{m+i,n+j}\right) = 0,$$
$$\alpha = 1, \ldots, l, \quad i_{min} \leq i \leq i_{max}, \quad j_{min} \leq j \leq j_{max}.$$

A *Stencil* $= \left\{N \ \text{adjacent points} \in \mathbb{R}^2\right\}$ in the limit sufficient to reproduce all derivatives in PDE $\Rightarrow |i_{max} - i_{min}|$ and $|j_{max} - j_{min}|$.



2nd order PDE $\Rightarrow$ 6-point stencil : $\{(0,0),(1,0),(0,1),(1,1),\ (2,0),(0,2)\}$

$$u_{xy} = F(x, y, u, u_x, u_y)$$

not involving u_{xx}, u_{yy} $\rightarrow$ **4-points**: $\mathbf{s}_4^0 \equiv \{(0,0),(1,0),\ (0,1),(1,1)\}$.

Invariant discretization on 4-points

$$Z \in \mathcal{L}: \qquad Z = \xi(x,y,u)\partial_x + \eta(x,y,u)\partial_y + \phi(x,y,u)\partial_u$$

Prolongation on the lattice: consider the action of the vector field $\hat{Z}$ at some reference point $\{x_{0,0},\ y_{0,0},\ u_{0,0}\}$ and apply it to all points figuring on the stencil:

$$\mathrm{pr}Z = \sum_{i,j}(\xi_{i,j}\partial_{x_{i,j}} + \eta_{i,j}\partial_{y_{i,j}} + \phi_{i,j}\partial_{u_{i,j}}).$$

$$X^D(f) = \mathrm{pr}\, X(f) \;=\; \sum_{(m,n)\in \mathbf{s_4^0}} \left[f(x_{m,n})\,\partial_{x_{mn}} - f'(x_{m,n})\; u_{mn}\,\partial_{u_{mn}}\right],$$

$$Y^D(g) = \mathrm{pr}\, Y(g) \;=\; \sum_{(m,n)\in \mathbf{s_4^0}} \left[g(y_{m,n})\,\partial_{y_{mn}} - \dot{g}(y_{m,n})\; u_{mn}\,\partial_{u_{mn}}\right].$$

Strong invariants $\quad X^D(f)\; I(x_{ij}, y_{ij}, u_{ij}) = 0,\;\; Y^D(g)\, I(x_{ij}, y_{ij}, u_{ij}) = 0.$
$f = 1, x, x^2 \;\; g = 1, y, y^2$

6 functionally independent invariants on $SL_x(2) \otimes SL_y(2)$:

$$\xi_1 = \frac{(x_{0,1}-x_{0,0})(x_{1,1}-x_{1,0})}{(x_{0,0}-x_{1,0})(x_{0,1}-x_{1,1})} = \frac{\epsilon_{0,1}\epsilon_{1,1}}{h_{1,0}(h_{1,0}+\epsilon_{1,1}-\epsilon_{0,1})},$$
$$\eta_1 = \frac{(y_{0,0}-y_{1,0})(y_{0,1}-y_{1,1})}{(y_{0,1}-y_{0,0})(y_{1,1}-y_{1,0})} = \frac{\delta_{1,0}\delta_{1,1}}{k_{0,1}(k_{0,1}+\delta_{1,1}-\delta_{1,0})}$$

$$H_1 = u_{0,0}\ u_{0,1}\ \epsilon_{0,1}^2\ k_{0,1}^2$$
$$H_2 = u_{1,0}\ u_{1,1}\ \epsilon_{1,1}^2\ (k_{0,1}+\delta_{1,1}-\delta_{1,0})^2$$

$$H_3 = \frac{u_{1,0}\ (h_{1,0}-\epsilon_{0,1})^2\ (k_{0,1}-\delta_{1,0})^2}{u_{0,0}\ \epsilon_{0,1}^2\ k_{0,1}^2}$$

$$H_4 = \frac{u_{1,1}\ \epsilon_{1,1}^2\ (k_{0,1}+\delta_{1,1}-\delta_{1,0})^2}{u_{0,0}\ h_{1,0}^2\ \delta_{1,0}^2}$$

Invariant lattice: $\xi_1 = A, \qquad \eta_1 = B$

Invariant orthogonal lattice

$$\xi_1 = 0,\ \eta_1 = 0 \iff x_{m,n} = x_m,\ y_{m,n} = y_n.$$

Checking the invariance on $VIR(x) \otimes VIR(y)$

$$\begin{aligned} \hat{X}^D(x^3)\xi_1 &= (x_{1,1} - x_{0,0})(x_{1,0} - x_{0,1})\xi_1 \,|_{\xi_1=0} = 0 \\ &\Downarrow \\ &\forall\ n \quad \hat{X}^D(x^n)\xi_1|_{\xi_1=0} = 0 \end{aligned} \tag{1}$$

The orthogonal lattice $\xi_1 = 0,\ \eta_1 = 0$ **is the only** $VIR(x) \otimes VIR(y)$ invariant lattice.

The invariants $H_1, \cdots, H_4$ all vanish or become infinite on $\xi_1 = 0,\ \eta_1 = 0$.
The only **finite** $SL_x(2) \otimes SL_y(2)$ -invariants

$$J_1 = H_1 H_3 = u_{0,1} u_{1,0} h_{1,0}^2 k_{0,1}^2, \tag{2}$$

$$J_2 = \frac{1}{\xi_1^2} \frac{H_2}{H_3} = u_{0,0} u_{1,1} h_{1,0}^2 k_{0,1}^2. \tag{3}$$

No Virasoro - invariants

$$\hat{X}^D(x^3) J_1 = -h_{1,0}^2 J_1, \qquad \hat{X}^D(x^3) J_2 = -h_{1,0}^2 J_2.$$

$J_2 - J_1$ is a weak Virasoro invariant $\Rightarrow$ Wave equation

Expanding u in Taylor series on the lattice

$$J_2 - J_1 = h_{1,0}^3 \ k_{0,1}^3 \ (uu_{xy} - u_x u_y).$$

The **Liouville equation** is approximated by the $SL_x(2) \otimes SL_y(2)$ invariant difference scheme

$$\mathbf{J_2 - J_1 = a} \operatorname{sign}(\mathbf{J_1}) \ |\mathbf{J_1}|^{3/2} + \mathbf{b} \operatorname{sign}(\mathbf{J_2}) \ \mathbf{J_1}|\mathbf{J_2}|^{1/2}$$
$$+\mathbf{c} \operatorname{sign}(\mathbf{J_1}) \ |\mathbf{J_1}|^{1/2}\mathbf{J_2} + \mathbf{d} \operatorname{sign}(\mathbf{J_1}) \ |\mathbf{J_2}|^{3/2}, \quad (\mathbf{a+b+c+d=1})$$
$$\xi_1 = 0, \quad \eta_1 = 0,$$

$$\lim_{h,k \to 0} \frac{1}{h^3 k^3} \{J_2 - J_1 - [a|J_1|^{3/2} + bJ_1|J_2|^{1/2} + c|J_1|^{1/2} J_2 + d|J_2|^{3/2}]\} =$$
$$= [uu_{xy} - u_x u_y - u^3][1 + \mathcal{O}(h,k)]$$

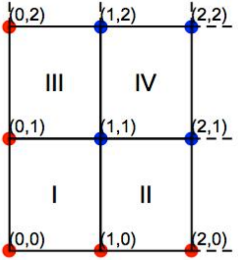
Recurrence formula

$b = d = 0,\ c = 1 - a,\ a \in \mathbb{R}$

$$\begin{aligned} u_{m+1,n+1} &= \frac{u_{m,n+1} u_{m+1,n}}{u_{m,n}} A_{m,n+1;m+1,n}, \\ A_{m,n+1;m+1,n} &= \frac{1 + a\, h\, k\, \operatorname{sign}(u_{m+1,n} u_{m,n+1}) \sqrt{|\, u_{m,n+1} u_{m+1,n}|}}{1 + (a-1)\, h\, k\, \operatorname{sign}(u_{m+1,n} u_{m,n+1}) \sqrt{|u_{m,n+1} u_{m+1,n}|}}. \end{aligned}$$

$$\begin{aligned} u_{m_0,0} &= 0 & u_{m_0+1,n} &= u_{m_0+1,n-1} \frac{u_{m_0-1,n}}{u_{m_0-1,n-1}} \\ u_{0,n_0} &= 0 & u_{m,n_0+1} &= u_{m-1,n_0+1} \frac{u_{m,n_0-1}}{u_{m-1,n_0-1}} \\ u_{0,n_0} = u_{m_0,0} &= 0 & u_{m_0+1,n_0+1} &= u_{m_0-1,n_0+1} \frac{u_{m_0+1,n_0-1}}{u_{m_0-1,n_0-1}} \end{aligned}$$

The 9-point - stencil



$$
\begin{aligned}
J_1 &= u_{01}u_{10}h^2k^2, & J_2 &= u_{00}u_{11}h^2k^2,\\
J_3 &= u_{11}u_{20}h^2k^2, & J_4 &= u_{10}u_{21}h^2k^2,\\
J_5 &= u_{11}u_{02}h^2k^2, & J_6 &= u_{01}u_{12}h^2k^2,\\
J_7 &= u_{12}u_{21}h^2k^2, & J_8 &= u_{11}u_{22}h^2k^2.
\end{aligned}
$$

$$J_4J_6 = J_1J_7$$

$$J_2 - J_1 = J_4 - J_3 = J_6 - J_5 = J_8 - J_7 = h^3 k^3 (u u_{xy} - u_x u_y)[1 + \mathcal{O}(h, k)]$$

$$\alpha[4(J_2 - J_1) - (J_6 - J_5 + J_4 - J_3)] + \beta[4(J_8 - J_7) - 3(J_6 - J_5 + J_4 - J_3)] \\ = 2h^3 k^3 (u u_{xy} - u_x u_y)(\alpha - \beta)[1 + \mathcal{O}(h^2,\ hk,\ k^2)]$$

$$2\alpha[4A_3 - 2A_1 - (A_4 + A_5)] + 2\beta[4A_6 + 2A_1 - 3(A_4 + A_5)] = \\ = \sum_{a,b=1}^{6} \gamma_{a,b} A_a \sqrt{|A_b|}, \qquad \sum_{a,b=1}^{6} \gamma_{a,b} = 2(\alpha - \beta).$$

$$A_1 = \tfrac{1}{2}(3J_1 - J_8)\,, A_2 = \tfrac{1}{2}(J_1 + J_4 + J_6 - 2J_8)\,, A_3 = \cdot$$

$u_{22} \leftrightarrow J_8$

$I)$

$$J_8 = J_7 + 3(J_2 - J_1) - \frac{1}{\sqrt{2}}(3J_1 - J_7)\sqrt{|3J_1 - J_7|}$$

$$u_{22} = \frac{1}{u_{11}}[u_{12}u_{21} + 3(u_{11}u_{00} - u_{10}u_{01}) \\ -\frac{1}{\sqrt{2}}hk(3u_{01}u_{10} - u_{12}u_{21})\sqrt{|3u_{01}u_{10} - u_{12}u_{21}|}]$$

II)

$$J_8 = \frac{J_7 + 3(J_2 - J_1) - \frac{3}{\sqrt{2}} J_1 \operatorname{sign}(3J_1 - J_7)\sqrt{|3J_1 - J_7|}}{1 - \frac{1}{\sqrt{2}} \operatorname{sign}(3J_1 - J_7)\sqrt{|3J_1 - J_7|}}$$

$$u_{22} = \{u_{12}u_{21} + 3(u_{11}u_{00} - u_{10}u_{01}) -$$

$$\frac{3}{\sqrt{2}} hk u_{01}u_{10} \operatorname{sign}(3u_{01}u_{10} - u_{12}u_{21})\sqrt{|3u_{01}u_{10} - u_{12}u_{21}|}\}$$

$$\{u_{11}[1 - \frac{hk}{\sqrt{2}} \operatorname{sign}(3u_{01}u_{10} - u_{12}u_{21})\sqrt{|3u_{01}u_{10} - u_{12}u_{21}|}]\}^{-1}$$

- 7 points involved.
- Boundary conditions;
- Numerical instabilities close to zeroes.

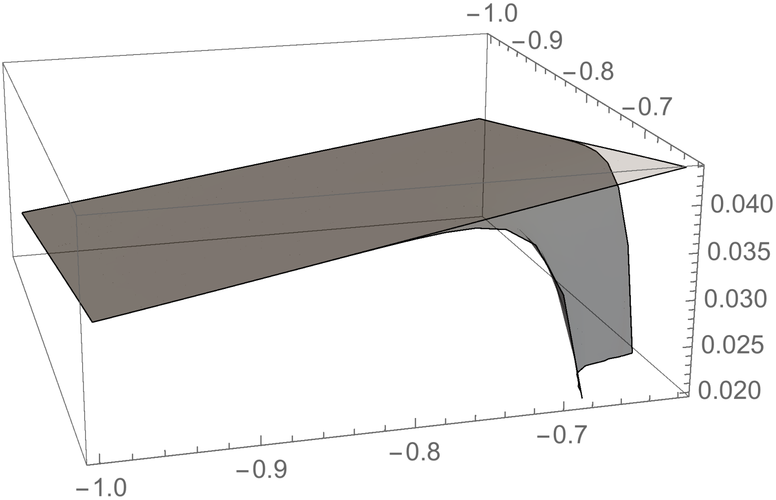


Figure : The 4- point (light) and 7-point (dark) recursion for the solution $f_1 = 2\{\left(x^2+1\right)\left(y^2+1\right)\left(\tan^{-1}(x)+\tan^{-1}(y)+6\right)^2\}_{-1}$. Steps h = k =0.02 on a grid 20 x 20.

Discrete Integrability

Adler, Startsev *Theor. Math. Phys.* **121** (1999)
discretization of the algebraic Liouville equation on a 4-point lattice

$$u_{i+1,j+1}(1+\frac{1}{u_{i+1,j}})(1+\frac{1}{u_{i,j+1}})u_{i,j}=1.$$

Linearizable via

$$u_{i,j}=-\frac{(v_{i+1,j}-v_{i,j})(v_{i,j+1}-v_{i,j})}{v_{i+1,j}v_{i,j+1}}, \qquad v_{i+1,j+1}-v_{i+1,j}-v_{i,j+1}+v_{i,j}=0.$$

General solution

$$u_{i,j}=-\frac{(c_{i+1}-c_i)(k_{j+1}-k_j)}{(c_{i+1}+k_j)(c_i+k_{j+1})} \qquad \forall c_i,\ k_j$$

It has NO continuous point symmetries. Generalized symmetries.

$$c_{m+1}=\phi_1(x)+h\tfrac{d\phi_1}{dx}+\mathcal{O}(h^2), \quad k_{n+1}=\phi_2(y)+k\tfrac{d\phi_2}{dy}+\mathcal{O}(k^2)$$

$$u_{m,n}=-hk\tfrac{\phi_{1,x}\phi_{2,y}}{(\phi_1+\phi_2)^2}+\mathcal{O}(h^3,h^2k,hk^2,k^3)$$

The invariant discretization by the pseudo-group approach

Alternative symmetry preserving discretization R. Rebelo and F. Valiquette (2014)
The **entire** $VIR(x) \otimes VIR(y)$ is preserved as **generalized symmetries**

$$\begin{aligned}\hat{X}(f,g) &= \sum_{i,j}\Big\{f(x_i)\partial_{x_i} + g(y_j)\partial_{y_j} \\ &- \left[\frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j} + \frac{g(y_{i+1})-g(y_i)}{y_{i+1}-y_i}\right]\partial_{u_{ij}}\Big\}\end{aligned}$$

$$u_{11}u_{00} - u_{10}v_{01} - u_{00}u_{01}u_{10}(x_{10}-x_{00})(y_{01}-y_{00}) = 0,$$
$$x_{01} = x_{00}, \qquad y_{10} = y_{00}.$$

Numerical experiments

1. preserving the $SL_x(2,\mathbb{R}) \otimes SL_y(2,\mathbb{R})$

$$u_{11}u_{00} - u_{10}u_{01} = h\,k\,[au_{01}u_{10} + (1-a)u_{00}u_{11}]\,\mathrm{sign}\,(u_{01}u_{10})\sqrt{|u_{01}u_{10}|},$$

2. preserving linearizability

$$u_{11}u_{00} - u_{10}u_{01} = h\,k\;u_{00}u_{11}[\frac{u_{01}+u_{10}}{2} - h\,k\;u_{01}u_{10}],$$

3. preserving generalized symmetries

$$u_{11}u_{00} - u_{10}u_{01} = h\,k\;u_{00}u_{01}u_{10},$$

4. The *standard* discretization

$$u_{11}u_{00} - u_{10}u_{01} = h\,k\;u_{00}^3,$$

Higher order corrections in h, k

5 test functions

$$
\begin{aligned}
f_1 &= \frac{2}{\left(x^2+1\right)\left(y^2+1\right)\left(\tan^{-1}(x)+\tan^{-1}(y)+6\right)^2}, \\
f_2 &= \frac{8\left(1-4\left(x+\frac{1}{2}\right)\right)(1-4y)\exp\left(-2x(1+2x)-2y(y+2)\right)}{\left(e^{-2x(1+2x)}+e^{2y(1-2y)}+1\right)^2}, \\
f_3 &= -\frac{3.38\sin(1.3(x+0.01))\cos(1.3(y+0.01))}{(\cos(1.3(x+0.01))+\sin(1.3(y+0.01))+3)^2} \\
f_4 &= \frac{8xy}{\left(x^2+y^2+2\right)^2} \\
f_5 &= \frac{383.1\, e^{3.862(2.5(x-0.5)+0.4y+2.5)}}{\left(e^{9.655(x+0.5)}+12.83e^{1.545y}\right)^2}
\end{aligned}
$$

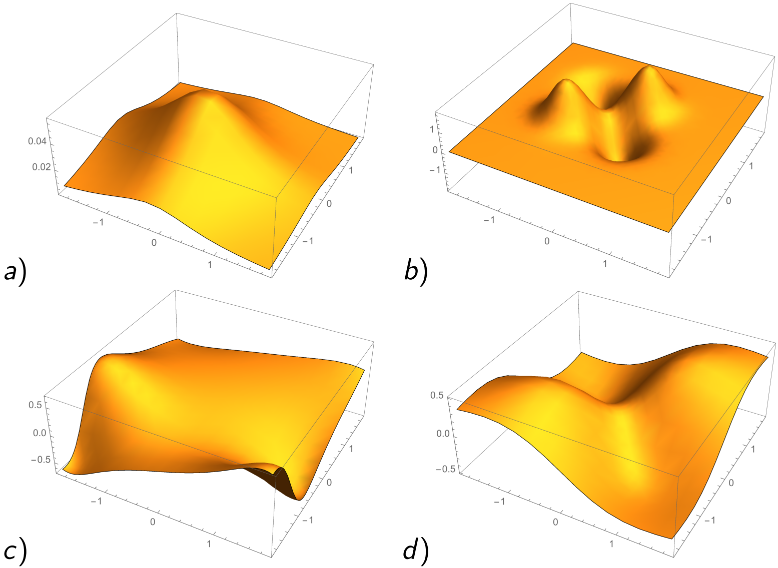


Figure : Plots of the exact solutions a) f_1, b) f_2, c) f_3 and d) f_4 in the domain $\mathcal{D}_1$ used in the numerical integrations.

Global numerical estimator

$$\chi_\alpha(F) = \sqrt{\frac{\sum_{ij}\left(F_{ij}^{\alpha} - F_{ij}\right)^2}{\sum_{ij} F_{ij}^2}}$$

Table : Relative mean square distance between the numerical solutions and the analytic one in the domain $\mathcal{D}_0$.

	χ_{Inv}	χ_{AS}	χ_{RV}	χ_{stand}
f_1	5.2×10^{-6}	2.7×10^{-6}	3.1×10^{-4}	9.2×10^{-4}
f_2	3.4×10^{-4}	1.5×10^{-4}	7.6×10^{-3}	2.2×10^{-2}
f_3	4.7×10^{-5}	1.5×10^{-5}	3.0×10^{-3}	9.2×10^{-3}
f_4	4.3×10^{-5}	7.9×10^{-5}	5.2×10^{-3}	2.0×10^{-2}
f_5	3.8×10^{-2}	3.0×10^{-2}	2.8×10^{-1}	4.3×10^{-1}

$\mathcal{D}_0 = [-1.5, 1.08] \times [-1.0, 1.58]$, $h = k = 0.02$, latice points 130×130

Point-like numerical estimator

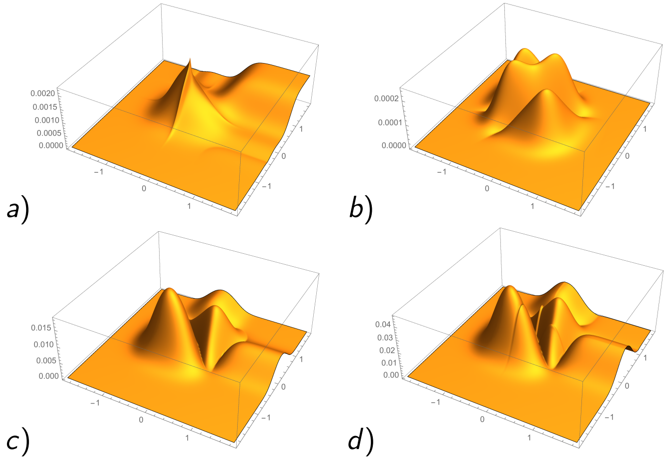


Figure : $R_{ij} = \left| \frac{F^{\alpha}_{ij} - F_{ij}}{F_{ij}} \right|$ for f_2 : a) Invariant, b) Adler-Startsev, c) R-V., d) Standard. Maximal error 2×10^{-3}, 2×10^{-4}, 1.5×10^{-2}, 4×10^{-2}.

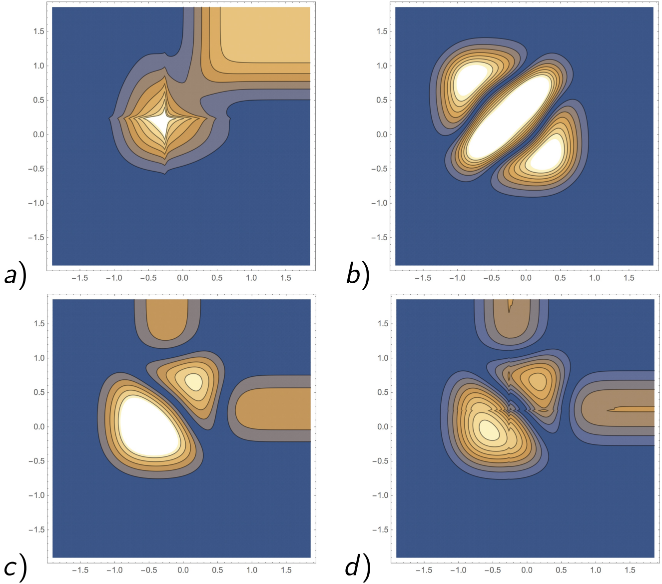


Figure :

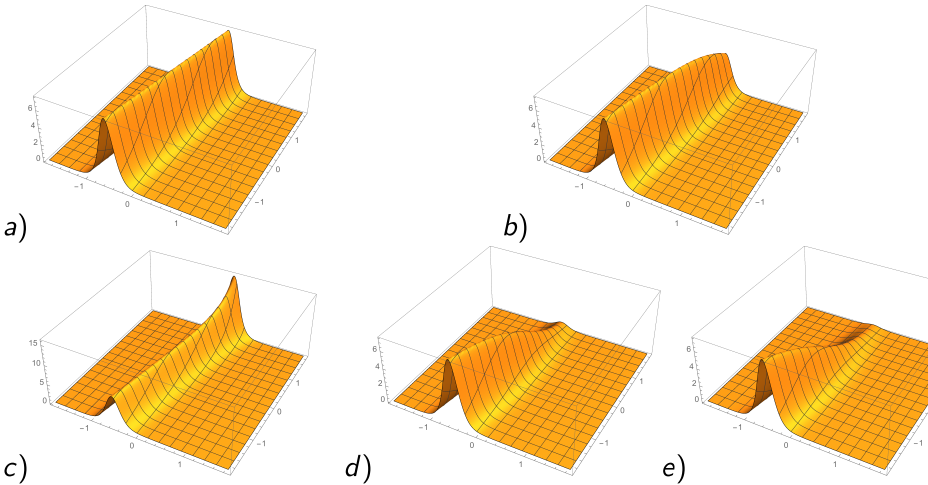


Figure : The function f_5: a) analytic, b) symmetry preserving, c) Adler-Startsev, d) Rebelo-Valiquette, e) Standard.

Stability on the step size

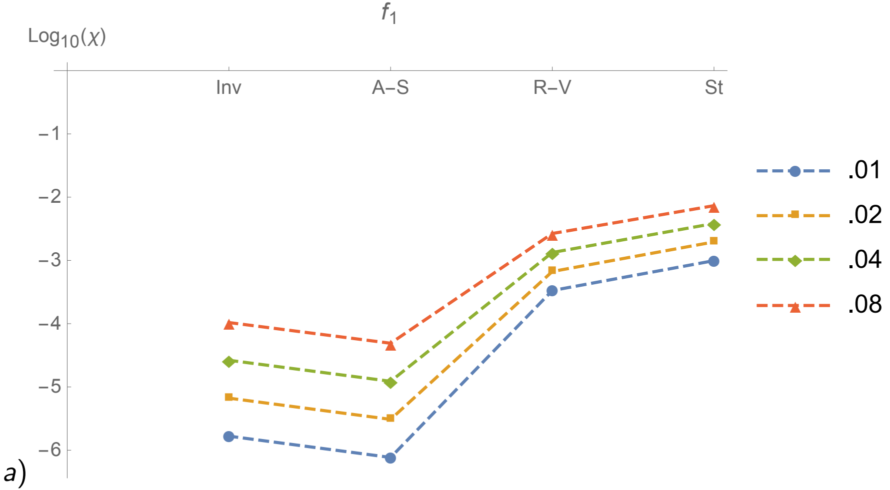


Figure :

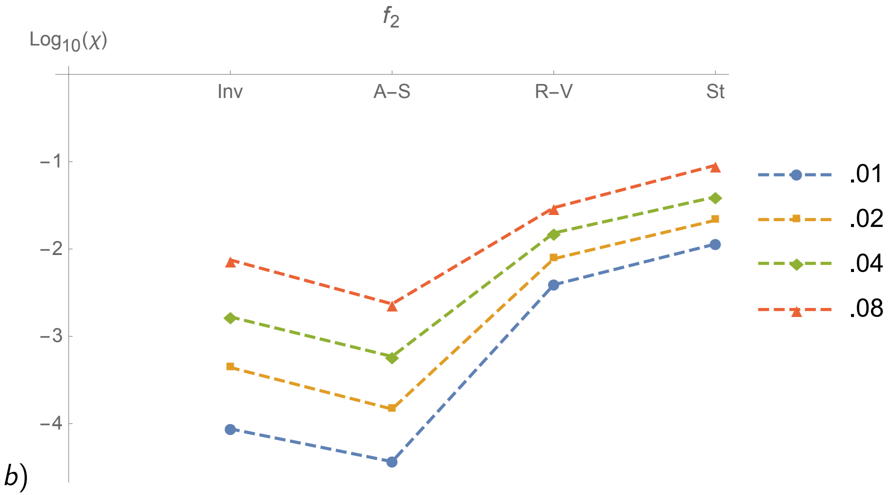


Figure :

Stability on the parameter a

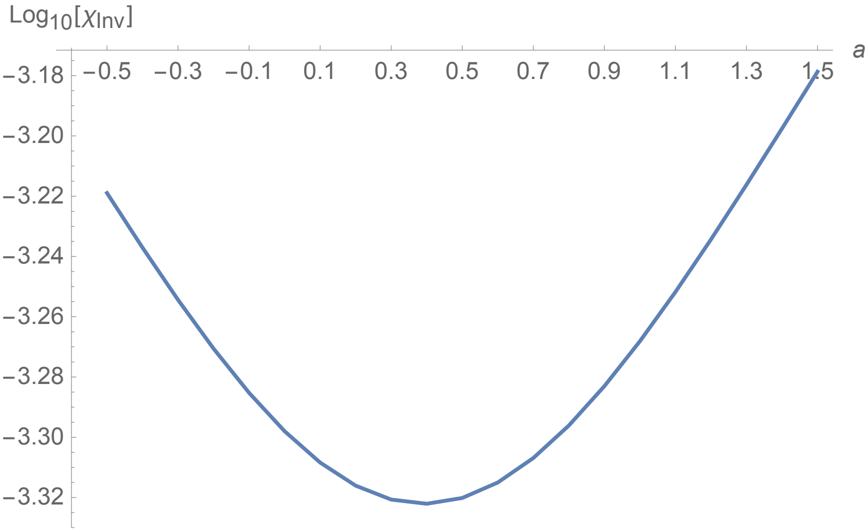


Figure : Plot of χ_{Inv} for the function f_2 in the parameter range $a \in [-0.5, 1.5]$.

Conclusions

- Explicit construction of symmetry preserving discretization is obtained for PDE;
- Maximal finite subgroup of Infinite-dimensional symmetry group is preserved;
- Lattice equations may be weak invariant of the full group;
- Discretization preserving generalized symmetriy group can be obtained by the pseudo-group approach.
- Linearizability preserving discretizations are more stable than the other approches, but ...
- Numerical experiments based on the preserving symmetry discretization works well like linearizable, except around zeros

- Discretizations of other equations Liouville type
- Use of symmetries for controlling
- The 3WI equation, DS eq. ....
- Multidimensional non integrable eq.s