Dynamics on the noncommutative plane

L. Martina
Dip. Fisica - Univ. Salento, Sez. INFN Lecce, Italy

Geometry of Differential Equations and Integrability
Hradec nad Moravicí , October 11-15 2010

Outline

P. A. Horváthy*, L. M. , P. C. Stichel†, :" Exotic galilean symmetry and non-commutative mechanics ", SIGMA 6 (2010), 060, arXiv:1003.0137, P. Aschieri *et al.* ed.s "Noncommutative Spaces and Fields"

M. Del Olmo, C. Duval, Z. Horvath, P. Horvathy, J. Lukierski, L.M., M. Plyushchay , P. Stichel, W.J. Zakrzewski

- The "Exotic" Galilean symmetry
- The method of coadjoint orbits for exotic mechanical models
- Quantization and Anyons
- Acceleration-dependent Lagrangian in configuration space
- Examples of generalized dynamical systems on noncommutative plane

*LMPT, Université de Tours, France
†An der Krebskuhle, Bielefeld, Germany

- Physical origin of the exotic structure
- Anomalous coupling of anyons
- Seiberg-Witten equivalence in E.M. interactions
- Galilean symmetry in Moyal field theory
- Noncommutativity in 3 dimensions: the semiclassical Bloch electron
- General Hamiltonian Structure
- Conclusions and Outlook

The "Exotic" Galilean symmetry

- V. Bargmann *Ann. Math.* **59**, 1 (1954).
- J.-M. Lévy-Leblond, (Loebl Ed.) (1972) $(2+1)\,D$ $[K_1, K_2] = i\kappa$

Physics carrying "exotic" structure ?

1. Kirillov - Konstant - Souriau method of the Group Coadjoint Orbits
2. Acceleration-dependent Lagrangian

1. D. R. Grigore, *Journ. Math. Phys.* **37**, 240 (1996); A. Ballesteros *et al. Journ. Math. Phys.* **33**, 3379 (1992); C. Duval, P. A. Horváthy, *Phys. Lett.* **B 479**, 284 (2000) .
2. J. Lukierski *et al.*, *Annals of Physics* (N. Y.) **260**, 224 (1997).

$$\{x_1, x_2\} = -\frac{\kappa}{m^2} \equiv \theta,$$

The $\widehat{G}\,(2+1)$ Galilei Group

$$
\begin{array}{rcccccc}
g = & (\phi, & \vec{a}, & \vec{v}, & \tau, & \zeta, & \eta) \\
 & \in & \in & \in & \in & \in & \in \\
 & \mathbb{R} & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R} & \mathbb{R} & \mathbb{R}
\end{array}
\qquad \phi \leftrightarrow R_\phi \in SO\,(2)
$$

$$
\begin{aligned}
g' \centerdot g = \quad & \Big(\phi + \phi',\ \vec{a}' + R_{\phi'}\vec{a} + \tau\,\vec{v}\,',\ \vec{v}\,' + R_{\phi'}\vec{v},\ \tau + \tau', \\
& \zeta + \zeta' + \tfrac{1}{2}\tau\vec{v}\,'^{\,2} + \vec{v}\,' \cdot R_{\phi'}\vec{a},\ \eta + \eta' + \tfrac{1}{2}\vec{v}\,'\widehat{\varepsilon}\ R_{\phi'}\vec{v}\ \Big)
\end{aligned}
$$

$$\widehat{\varepsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Action on 2+1 space-time

$$T_g : (\vec{r}, t) \to \big(R_\phi\vec{r} + \vec{a} + \vec{v}\,t,\ t + \tau\big)$$

The $\widehat{g}\,(2+1)$ Galilei Lie Algebra

$$\mathcal{B} = \{\mathcal{J}, \mathcal{P}_i, \mathcal{K}_i, \mathcal{H}_0, \mathcal{I}_i\} \qquad i = 1, 2$$

$$\begin{aligned}
&[\mathcal{P}_i, \mathcal{H}_0] = 0, && [\mathcal{K}_i, \mathcal{H}_0] = \mathcal{P}_i, \\
&[\mathcal{J}, \mathcal{P}_i] = \epsilon_{ij}\,\mathcal{P}_j, && \left[\mathcal{P}_i, \mathcal{P}_j\right] = 0, \\
&[\mathcal{J}, \mathcal{K}_i] = \epsilon_{ij}\,\mathcal{K}_j, && \left[\mathcal{K}_j, \mathcal{P}_i\right] = m\,\delta_{ij}\mathcal{I}_1, \\
&[\mathcal{J}, \mathcal{H}_0] = 0, && [\mathcal{K}_1, \mathcal{K}_2] = \kappa\mathcal{I}_2
\end{aligned}$$

The Group Coadjoint Orbit method

$$h \in \widehat{g}^*\,(2+1) \qquad h \leftrightarrow \left(j, \vec{p}, \vec{k}, E, m, \kappa\right)$$

$$Ad^*_{(\phi,\ \vec{a},\ \vec{v},\ \ \tau,\ \zeta,\ \eta)}\left(j, \vec{p}, \vec{k}, E, m, \kappa\right) =$$
$$\Big(j - \tfrac{1}{2}\kappa\vec{v}^{\,2} + m\vec{v}\,\widehat{\varepsilon}\,\vec{a} - \vec{k}\,\widehat{\varepsilon}\,R_{-\phi}\vec{v} - \vec{p}\,\widehat{\varepsilon}\,R_{-\phi}\vec{a},$$
$$R_\phi\,\vec{p} - m\vec{v}, \quad R_\phi\left(\vec{k} + \tau\vec{p}\right) + m\left(\vec{a} - \tau\vec{v}\right) + \kappa\widehat{\varepsilon}\,\vec{v},$$
$$E - \vec{v}\cdot R_\phi\,\vec{p} + \tfrac{1}{2}m\vec{v}^{\,2},\ m,\ \kappa\Big)$$

Inv. $\qquad \vec{p}^{\,2} - 2mE, \quad mj + \vec{p}\,\widehat{\varepsilon}\,\vec{k} - \kappa E, \quad m, \quad \kappa$

Orbit coordinates $\qquad \vec{x} = \frac{\vec{k}}{m}, \qquad \vec{p}$

Poisson Structure $\{x_1, x_2\} = \frac{\kappa}{m^2} = \theta, \qquad \{x_i, p_j\} = \delta_{ij}, \qquad \{p_1, p_2\} = 0$

Symplectic F. $$\Omega_0 = dp_i \wedge dx_i + \frac{1}{2}\theta\,\epsilon_{ij}\,dp_i \wedge dp_j$$

Free Hamiltonian $$H_0 = \frac{\vec{p}^{\,2}}{2m}$$

EOM $$m\dot{x}_i = p_i - m\theta\epsilon_{ij}\dot{p}_j, \qquad \dot{p}_i = 0.$$

Conserved Q. $$j = \vec{x}\hat{\varepsilon}\,\vec{p} + \frac{\theta}{2}\vec{p}^{\,2},\ \vec{K} = -m\vec{x} + \vec{p}\,t - m\theta\,\hat{\varepsilon}\,\vec{p}$$

Coupling to an external E.M. field

Lagrange - Souriau
2-form

$$\sigma = \Omega - dH \wedge dt$$

$$\Omega = \Omega_0 + eB\,dq_1 \wedge dq_2, \qquad H = H_0 + eV$$

$$B = B(\vec{x}), \qquad V = V(\vec{x})$$

DH-model
$$\left\{ \begin{array}{rcll} m^* \dot{x}_i &=& p_i - em\theta\,\epsilon_{ij}E_j, & \textit{anomalous velocity} \\ \dot{p}_i &=& eE_i + eB\,\epsilon_{ij}\dot{x}_j & \textit{Lorentz F.} \end{array} \right.$$

effective mass $\quad m^* = m(1 - e\theta B).$

Cartan 1-form

$$\lambda \;=\; (p_i - A_i)dx_i - \frac{\vec{p}^{\,2}}{2m}dt + \frac{\theta}{2}\epsilon_{ij}\,p_i\,dp_j,$$

$$A = \int_{\widetilde{\gamma}} \lambda \;=\; \int_{\widetilde{\gamma}} f_a(\xi)\,\dot{\xi}^a dt \neq \int_{\widetilde{\gamma}} \frac{\partial L}{\partial v_i}dx^i + \left(L - \frac{\partial L}{\partial v_i}v_i\right)dt = \int_{t_1}^{t_2} L dt$$

Poisson Str. $\quad \{x_1, x_2\} = \dfrac{m}{m^*}\theta, \quad \{x_i, p_j\} = \dfrac{m}{m^*}\delta_{ij},$

$$\{p_1, p_2\} = \frac{m}{m^*}eB$$

$m^* \to 0 \Rightarrow$ Symplectic reduction $\qquad$ Hall's motions

$$\{x_1, x_2\} = \frac{1}{eB_{cr}} = \theta, \quad H = V(\vec{x}) \quad \text{(Peierl's subst.)}$$

Quantization

$$\begin{array}{l} z = \frac{B(x_1+ix_2)-ip_1+p_2}{\sqrt{B}} \\ w = -\frac{g(p_2+ip_1)}{\sqrt{B}} \end{array}, \Omega_K = \frac{dz \wedge d\overline{z} + dw \wedge d\overline{w}}{2\imath}$$

Bargmann - Fock w.f. $\psi = f\,(z, w)\,\mathrm{exp}\left[-\frac{z\overline{z}+w\overline{w}}{4}\right]$

$$[\widehat{\overline{z}}, \widehat{z}] = [\widehat{\overline{w}}, \widehat{w}] = 2, \quad [\widehat{w}, \widehat{z}] = [\widehat{\overline{w}}, \widehat{\overline{z}}] = 0$$

$$\widehat{H} = \widehat{H}_0 + \widehat{V}, \quad \widehat{H}_0 = \frac{B}{2m^*}\,(\widehat{w}\widehat{\overline{w}} + 1)$$

$$m^* \rightsquigarrow 0 \quad \text{and} \quad \widehat{\overline{w}} f = 0 \Rightarrow \Psi = f\,(z)\,e^{-\frac{z\overline{z}}{4}}, \widehat{H}_0\Psi = \frac{B}{2m^*}\Psi$$

R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983)

ANYONS at the Lowest Landau Level

Acceleration-dependent Lagrangian in configuration space

LSZ model

$$\mathcal{L} = \frac{m}{2}\dot{x}_i^2 + \frac{m^2\theta}{2}\epsilon_{ij}\dot{x}_i\ddot{x}_j$$

2D Galilei covariant EOM

$$\mathcal{L}' = \mathcal{L} + p_i(\dot{x}_i - y_i) = p_i\dot{x}_i + \frac{m^2\theta}{2}\epsilon_{ij}y_i\dot{y}_j - \left(y_i p_i - \frac{m}{2}y_i^2\right)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{y}_i} = -\frac{m^2\theta}{2}\epsilon_{ij}y_j \quad \rightsquigarrow \textit{constrained sys.}$$

Symplectic reduction $\Rightarrow \{x_i, p_j\} = \delta_{ij}\ ,\ \{y_i, y_j\} = -\frac{1}{m^2\theta}\epsilon_{ij}$

$$K_i = -mx_i + p_i t - m^2\theta\epsilon_{ij}y_j : \qquad \{K_i, K_j\} = -m^2\theta\epsilon_{ij}$$

Galilei invariant decomposition $(\vec{x}, \vec{p}, \vec{y}) \to (\vec{X}, \vec{p}, \vec{Q})$ $\quad x_i = X_i - \epsilon_{ij} Q_j$, $\quad y_i = \frac{p_i}{m} + \frac{Q_i}{m\theta}$

$$\begin{aligned} \mathcal{L}' = & \left[p_i \dot{X}_i + \tfrac{\theta}{2} \epsilon_{ij} p_i \dot{p}_j - \tfrac{p_i^2}{2m} \right] + \left[\frac{Q_i^2}{2m\theta^2} + \frac{1}{2\theta} \epsilon_{ij} Q_i \dot{Q}_j \right] \\ = & \qquad \mathcal{L}_{DH} \qquad + \qquad \mathcal{L}_{int} \end{aligned}$$

$$\{X_i, X_j\} = \theta \epsilon_{ij}, \ \{X_i, p_j\} = \delta_{ij} \ , \quad \{Q_i, Q_j\} = -\theta \epsilon_{ij},$$

$$\Omega = \Omega_0 + \tfrac{1}{2\theta} \varepsilon_{ij} dQ_i \wedge dQ_j, \qquad H = \frac{\vec{p}^{\,2}}{2m} - \frac{1}{2m\theta^2} \vec{Q}^{\,2}$$

Commutative vs NonCommutative v. $X'_i = X_i + \frac{\theta}{2} \epsilon_{ij} p_j, \ \ p'_i = p_i$

Are N-C. v. usefull? $\to$ Yes, in presence of interactions

Minimal coupling to an E.M. field

$$\mathcal{L} = \mathcal{L}' + \mathcal{L}_{ext}^{gauge} + \mathcal{L}_{int}^{gauge}, \qquad \begin{array}{r} \mathcal{L}_{ext}^{gauge} = e(A_i\dot{X}_i + A_0) \\ \mathcal{L}_{int}^{gauge} = \frac{\vec{Q}^2}{2\theta}\Big(A_i\dot{X}_i + A_0\Big) \end{array}$$

$$\delta Q_i = \varphi(\vec{X},t)\,\epsilon_{ij}Q_j,\ \delta A_\mu = \partial_\mu\varphi$$

$$\begin{aligned} m^*\dot{X}_i &= P_i - e^* m\theta\epsilon_{ij}E_j, \\ \dot{P}_i &= e^* B\epsilon_{ij}\dot{X}_j + e^* E_i, \\ \dot{Q}_i &= \epsilon_{ij}Q_j\Big(A_k\dot{X}_k + A_0 + \frac{1}{m\theta}\Big). \end{aligned} \qquad \left(\begin{array}{r} e^* = e + \vec{Q}^2/2\theta \\ m^* = m(1 - e^*\theta B) \end{array}\right)$$

Radiation Damping Effect A. C. R. Mendes, *et al.* Eur.Phys.J. **C45** (2006) 257

General noncommutative mechanics

$$\mathcal{L} = p_i \dot{x}_i + \tilde{A}_i(\vec{x}, \vec{p})\dot{p}_i - H(\vec{p}, \vec{x})$$

$$\{x_i, x_j\} = \epsilon_{ij}\tilde{B} \ \left(\tilde{B} = \epsilon_{k\ell}\partial_{p_k}\tilde{A}_\ell(\vec{x}, \vec{p})\right), \qquad \{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0$$

$$x_i \to q_i = x_i - \tilde{A}_i(\vec{x}, \vec{p})$$

Commutative Coordinates

$$p_i \dot{x}_i + \tilde{A}_i \dot{p}_i = p_i \dot{q}_i + \frac{d}{dt}(\tilde{A}_i p_i)$$

Examples

- $\tilde{A}_i = \tilde{A}_i(\vec{p})$, $\{x_i, x_j\} = \epsilon_{ij}\tilde{B}(\vec{p}) \quad \{p_i, p_j\} = 0$ DH model
  $\{x_i, p_j\} = \delta_{ij}$ Berry phase in momentum s

- $\tilde{A}_i = f(p^2)(\vec{x}\cdot\vec{p})p_i,$ $\begin{array}{l} \{x_i, x_j\} = \frac{f(p^2)\epsilon_{ij}}{1-p^2 f(p^2)}\,\epsilon_{k\ell}x_k p_\ell, \\ \{x_i, p_j\} = \delta_{ij} + \frac{f(p^2)}{1-p^2 f(p^2)} p_i p_j \end{array}$

1) $f = \dfrac{\theta}{1+p^2\theta} \to$ H.S. Snyder, *Quantized space-time* *Phys. Rev.* **71**, 38 (1947)

2) $f \to \infty,\ H = \kappa \ln(p^2/2)$

cons. q. $G_i = p_i t + \dfrac{p^2}{2\kappa} x_i$ $\quad \{G_i, p_j\} = \frac{\delta_{ij}p^2 - 2p_i p_j}{2\kappa},$ $\{H, G_i\} = p_i\ ,\quad \{G_i, G_j\} = 0$

κ-deformed Galilei alg. $\{H, p_i, J, G_i\}$

de Azcarraga *et al.* *J. Math. Phys.* **36**, 6879 (1995)

Physical origin of the exotic structure

Group Coadjoint Orbit M. $SO(2,1)$ Skagerstam, Stern: *I J M P* **A 5**, 1575 (1990)

$$
\begin{aligned}
\Omega_r &= dp_\alpha \wedge dx^\alpha + \frac{s}{2}\epsilon^{\alpha\beta\gamma}\frac{p_\alpha dp_\beta \wedge dp_\gamma}{(p^2)^{3/2}}\,,\\
H_r &= \frac{1}{2m}\Big(p^2 - m^2c^2\Big).
\end{aligned}
$$

Lorentz gen. $J_\mu = \epsilon_{\mu\nu\rho}x^\nu p^\rho + s\frac{p_\mu}{\sqrt{p^2}}, \qquad \{J^\alpha, J^\beta\} = \epsilon^{\alpha\beta\gamma}J_\gamma,$

Jackiw-Nair limit $\quad s/c^2 \to m^2\theta \quad \Rightarrow \quad \Omega_r\Big|_{H_r=0} \to \Omega_0$

$c \to \infty \quad H_r \to 0 \quad \frac{\epsilon_{ij}J^j}{c} \to K_i = mx_i - p_i t + m\theta\epsilon_{ij}p_j$

non-relativistic shadow $J_0/c^2 = \{K_1, K_2\} \to m^2\theta \qquad J_0 - s \to \vec{x}\hat{\varepsilon}\,\vec{p} + \frac{\theta}{2}\vec{p}^{\,2} \equiv j$

Anomalous coupling of anyons

relativistic anyons in E.M. field: CNP - model C. Chou *et al*, *Phys. Lett.* **B304**, 105 (1993)

$$m\frac{dx^\alpha}{d\tau} = p^\alpha \qquad \frac{dp^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}p_\beta \quad \text{gyromagnetic ratio} \;\; g = 2$$

$$\Omega = \Omega_r + \tfrac{e}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta, \;\; H^{CNP} = \tfrac{1}{2m}\Big(p^2 - M_2^2c^2\Big)$$

$$M_g^2 = m^2 + \tfrac{ge}{2c^2}\,S \cdot F = m^2 - \tfrac{ge}{4mc^2}\,\frac{s\,\epsilon_{\alpha\beta\gamma}p^\alpha}{\sqrt{p^2}}F^{\beta\gamma}$$

Experiments : $0 \approx g < 2$ Duval, *Phys. Lett.* **B594** (2004), 402

$$D\frac{dx^\alpha}{d\tau} = G\frac{p^\alpha}{M_g} + (g-2)\,\frac{es}{4M_g^2}\,\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} \qquad \frac{dp^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}p_\beta$$

$$D = 1 + \frac{eF \cdot S}{2M_g^2c^2}, \qquad G = 1 + \frac{g}{2}\,\frac{eF \cdot S}{2M_g^2c^2}.$$

non-rel. JN limit

$$
\begin{aligned}
\widetilde{M}_g \widetilde{D} \dot{x}_i &= \tilde{G} p_i - \Big(1 - \frac{g}{2}\Big) e \widetilde{M}_g \theta \epsilon_{ij} E_j, \\
\dot{p}^\alpha &= \frac{e}{m} F^{\alpha\beta} p_\beta
\end{aligned}
$$

$$
\widetilde{M}_g = m\sqrt{1 - g\theta e B}, \qquad \widetilde{D} = [1 - (g+1)\,\theta e B]\,, \qquad \tilde{G} = [1 - (3g/2)]\,\theta e\, B
$$

Hall motions $\Leftrightarrow$ $B_{cr} = \frac{1}{1+g}\frac{1}{e\theta}$ or $\frac{2}{3g}\frac{1}{e\theta}$

non-rel. limit CNP -model ($g = 2$) $\not\Rightarrow$ DH -model ($g = 0$)

Seiberg-Witten equivalence in E.M. interactions

$$p_i\dot{x}_i - \frac{p_i^2}{2m} \longrightarrow \begin{cases} p_i\dot{x}_i - \frac{(p_i - eA_i)^2}{2m} + eA_0(\vec{x},t) & \text{minimal substitution} \\ (p_i + eA_i(\vec{x},t))\dot{x}_i - \frac{p_i^2}{2m} + eA_0(\vec{x},t) & \text{minimal addition} \end{cases}$$

Commutative Variables

$$\Leftrightarrow p_i \rightarrow p_i - eA_i \Leftrightarrow$$

Non Commutative Variables

DH - model

minimal addition $\mathcal{L}_{DH-em} = \mathcal{L}_{DH} + e(A_i\dot{X}_i + A_0)$

Souriau' method

loc. gauge tr. $A_\mu(\vec{x},t) \rightarrow A_\mu(\vec{x},t) + \partial_\mu\Lambda(\vec{x},t)$

$$\mathcal{L}_{DH-em} \rightarrow \mathcal{L}_{DH-em} + \frac{d}{dt}\Lambda$$

Invariance of the exotic-DH symplectic form ext. LSZ - model

minimal substitution $H = \frac{p_i^2}{2m} \rightarrow H_{e.m.} = \frac{(p_i - e\widehat{A}_i)^2}{2m} - e\widehat{A}_0$

Invariance of the exotic-LSZ symplectic form

$$\widetilde{\mathcal{L}}_{ext} = p_i \dot{x}_i + \frac{\theta}{2}\varepsilon_{ij} p_i \dot{p}_j - \frac{1}{2}(p_i - e\widehat{A}_i)^2 + e\widehat{A}_0$$

gen. gauge tr.

$$\begin{aligned} &\delta\widehat{A}_\mu(\vec{x},t) = \widehat{A}'_\mu(\vec{x}+\delta\vec{x},t) - \widehat{A}_\mu(\vec{x},t) \qquad = \partial_\mu \Lambda(\vec{x},t) \\ &\delta x_i = -e\theta\epsilon_{ij}\partial_j\Lambda, \qquad \delta p_i = e\partial_i\Lambda \end{aligned}$$

$$\delta\widetilde{\mathcal{L}}_{ext} = e\frac{d}{dt}(\Lambda + \frac{\theta}{2}\varepsilon_{ij}\partial_i\Lambda\, p_j)\,.$$

$$\begin{aligned} \delta_0\widehat{A}_\mu(\vec{x},t) &:= \widehat{A}'_\mu(\vec{x},t) - \widehat{A}_\mu(\vec{x},t) = \partial_\mu\Lambda(\vec{x},t) + e\{\widehat{A}_\mu(\vec{x},t), \Lambda(\vec{x},t)\} \\ \breve{F}_{\mu\nu} &= \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu + e\left\{\widehat{A}_\mu, \widehat{A}_\nu\right\} \end{aligned}$$

Generalized gauge transf. and invariant field strength

$$(X_i, P_i)_{(LSZ)} \leftrightarrow (x_i, p_i)_{(DH)}, \qquad \widehat{A}_\mu(\overrightarrow{X}, t) \to A_\mu(\overrightarrow{x}, t)$$

$$x_i = X_i + e\theta\varepsilon_{ij}\widehat{A}_j(\overrightarrow{X}, t), \qquad p_i = P_i - e\widehat{A}_j(\overrightarrow{X}, t)$$

gen. gauge tr. $\Rightarrow \delta x_i = 0 \ \Rightarrow \delta_0 x_i \ = \ e\{x_i, \Lambda\}$

DH - PB are satisfied $\qquad B(\vec{x}, t) = \frac{\breve{B}\left(\vec{X},t\right)}{1+e\theta\breve{B}\left(\vec{X},t\right)}$

$$F_{\mu\nu}(\vec{x}, t) = \frac{\breve{F}_{\mu\nu}(\vec{X}, t)}{1 + e\theta\breve{B}(\vec{X}, t)}$$

classical ($*_{Moyal} \to \cdot$) Seiberg-Witten transformation

$$\widetilde{\mathcal{L}}_{ext}\left(\widehat{A}_\mu(\overrightarrow{X}, t), \overrightarrow{X}, \dot{\overrightarrow{X}}, \overrightarrow{P}, \dot{\overrightarrow{P}})\right) = \mathcal{L}_{DH-em}(A_\mu(\overrightarrow{x}, t), \overrightarrow{x}, \dot{\overrightarrow{x}}, \overrightarrow{p}, \dot{\overrightarrow{p}}))$$

$$A_k\left(\overrightarrow{x}(\overrightarrow{X},t),t\right) = \frac{1}{2}\widehat{A}_l(\overrightarrow{X},t)\left(\delta_{kl} + \frac{e_{kl}(\overrightarrow{X},t)}{1+e\theta\widehat{B}}\right)$$

$$\begin{aligned} e_{00} &= 1\,,\ e_{i0} = 0 \\ e_{\mu k} &= \delta_{\mu k} + e\theta\varepsilon_{ik}\partial_i\widehat{A}_\mu \end{aligned}$$

$$A_0\left(\overrightarrow{x}(\overrightarrow{X},t),t\right) = \widehat{A}_0(\overrightarrow{X},t) - \frac{e\theta}{2(1+e\theta\widehat{B})}\widehat{A}_l(\overrightarrow{X},t)\varepsilon_{kj}\partial_t\widehat{A}_j(\overrightarrow{X},t)e_{kl}(\overrightarrow{X},t)$$

N. Seiberg, E. Witten, JHEP 9909, 032 (1999)

Galilean symmetry in Moyal field theory

Chern-Simons-Ginzburg-Landau th. FQHE Zhang, *Int. J. Mod. Phys.* **B6**, 25 (1992)

$$\mathcal{L}_{CSGL} = i\bar{\psi}D_t\psi - \frac{1}{2}\left|\vec{D}\psi\right|^2 + \kappa\left(\frac{1}{2}\epsilon_{ij}\partial_t A_i A_j + A_t B\right) \qquad \left(\kappa = \frac{n}{2\pi},\ n \in \mathbf{Z}\right)$$

Galilei invariant f.th. $\rightharpoonup$ "exotic" Galilean symmetry ? $\rightharpoonup$ Moyal f.th.

$$\Big(f \star g\Big)(\vec{x}) = e^{i\frac{\theta}{2}\left(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}\right)} f(\vec{x})g(\vec{y})\Big|_{\vec{x}=\vec{y}}$$

Szabo, Phys. Rept. **378** (2003) 207 $U(1)_*$-gauge group

$$\tilde{\psi} = e^{\imath\lambda(\vec{x})}\star\psi,\ \tilde{A_\mu} = e^{\imath\lambda(\vec{x})}\star(A_\mu + \imath\partial_\mu)\star e^{-\imath\lambda(\vec{x})},\ \tilde{F_{\mu\nu}} = e^{\imath\lambda(\vec{x})}\star F_{\mu\nu}\star e^{-\imath\lambda(\vec{x})}$$

$$\mathcal{L}^{*}_{CSGL} = i\bar{\psi}\star D_t\psi - \tfrac{1}{2}\overline{\vec{D}\psi}\star\vec{D}\psi + \kappa\left(\tfrac{1}{2}\epsilon_{ij}\partial_t A_i\star A_j + A_0\star B\right) - V\left(\psi,\bar{\psi}\right)$$

$$\begin{aligned} D_\mu\psi &= \partial_\mu\psi - ieA_\mu\star\psi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie\Big(A_\mu\star A_\nu - A_\nu\star A_\mu\Big) \end{aligned}$$

$V \equiv 0$

$$\begin{aligned} iD_t\psi + \frac{1}{2}\vec{D}^2\psi &= 0 \\ \kappa E_i - e\epsilon_{ik}{j^l}_k &= 0 \\ \kappa B + e\rho^l &= 0 \end{aligned}$$

$$B = \epsilon_{ij}F_{ij}, \qquad E_i = F_{i0}$$

$$\rho^l = \psi\star\bar{\psi}, \qquad \vec{j}^{\,l} = \frac{1}{2i}\left(\vec{D}\psi\star\bar{\psi} - \psi\star(\overline{\vec{D}\psi})\right)$$ chiral densities

Invariance under Galilean boost in *antifundamental representation*

$$\delta^r\psi = \psi \star (i\vec{b}\cdot\vec{x}) - t\vec{b}\cdot\vec{\nabla}\psi = (i\vec{b}\cdot\vec{x})\psi + \frac{\theta}{2}\vec{b}\times\vec{\nabla}\psi - t\vec{b}\cdot\vec{\nabla}\psi$$

$\delta A_i = -t\vec{b}\cdot\vec{\nabla}A_i,\ \delta A_0 = -\vec{b}\cdot\vec{A} - t\vec{b}\cdot\vec{\nabla}A_0$

$$\delta^r\rho^l = -t\vec{b}\cdot\vec{\nabla}\rho^l, \qquad \delta^r\rho^r = -t\vec{b}\cdot\vec{\nabla}\rho^r - \theta\vec{b}\times\vec{\nabla}\rho^r$$

$$\text{Noether th.} \Rightarrow \vec{K}^r = t\vec{P} - \int \vec{x}\,\rho^r\,d^2\vec{x}$$

$$P_i = \int \frac{1}{2i}\Big(\bar{\psi}\partial_i\psi - (\overline{\partial_i\psi})\psi\Big)d^2\vec{x} - \frac{\kappa}{2}\int \epsilon_{jk}A_k\partial_i A_j d^2\vec{x}$$

$$\Big\{K_i, K_j\Big\} = \epsilon_{ij}k, \qquad k \equiv -\theta\int|\psi|^2 d^2x \ = -\theta\, m$$

Noncommutativity in 3D: the semiclassical Bloch electron

$$\widehat{H}\left[\widehat{\vec{r}},\widehat{\vec{p}},f\left(\widehat{\vec{r}},t\right)\right] \qquad l_{latt} \ll l_{wp} \ll l_{mod}, \qquad \hbar/\Delta E_{gap} \ll T_f$$

$$\widehat{H}\left[\widehat{\vec{r}},\widehat{\vec{p}},f\left(\widehat{\vec{r}},t\right)\right] = \widehat{H}_{(\vec{r}_c,t)} + \widehat{W}_{(\vec{r}_c,t)}\,,$$

$$\widehat{W}_{(\vec{r}_c,t)} = \tfrac{1}{2}\left[\partial_f \widehat{H}\,\nabla_{\vec{r}_c} f\left(\vec{r}_c,t\right)\cdot\left(\widehat{\vec{r}}-\vec{r}_c\right) + h.c.\right]$$

quasi-static periodic $\widehat{H}_{(\vec{r}_c,t)}$, "slow" parameters $c = (\vec{r}_c, t)$

$$\widehat{H}_{(\vec{r}_c,t)}|\psi^{n,\vec{q}}_{(\vec{r}_c,t)}\rangle = E^{n,\vec{q}}_{(\vec{r}_c,t)}|\psi^{n,\vec{q}}_{(\vec{r}_c,t)}\rangle, \qquad \langle\psi^{n,\vec{q}}_{(\vec{r}_c,t)}|\psi^{n',\vec{q}'}_{(\vec{r}_c,t)}\rangle = \delta_{n,n'}\delta\left(\vec{q}-\vec{q}'\right),$$

$$\langle\vec{r}|\psi^{n,\vec{q}}_{(\vec{r}_c,t)}\rangle = e^{i\vec{q}\cdot\vec{r}}u^{n,\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right), \qquad u^{n,\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}+\vec{a}\right) = u^{n,\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right),$$

Karplus-Luttinger

$$\langle\psi^{\vec{q}}_{(\vec{r}_c,t)}|\,\widehat{\vec{r}}\,|\psi^{\vec{q}}_{(\vec{r}_c,t)}\rangle = \left[i\nabla_{\vec{q}} + \langle u^{\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right)|i\nabla_{\vec{q}}u^{\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right)\rangle_{cell}\right]\delta\left(\vec{q}'-\vec{q}\right).$$

$$\widehat{\vec{r}} = i\nabla_{\vec{q}} + \vec{\mathcal{Q}}\left(\vec{r}_c,\vec{q},t\right), \qquad \vec{\mathcal{Q}} = \langle u^{\vec{q}}_{(\vec{r}_c,t)}|i\nabla_{\vec{q}}u^{\vec{q}}_{(\vec{r}_c,t)}\rangle_{cell},$$ Berry connection

$$[\widehat{r}_j,\widehat{r}_l] = i\,\epsilon_{jl}\,\partial_{q_j}\mathcal{Q}_l(\vec{r}_c,\vec{q},t) = \Theta_{jl}\left(\vec{r}_c,\vec{q},t\right)$$

Hamiltonian Structure

closed Lagrange-Souriau 2-form

$$\sigma = [(1 - Q_i)\, dq_i - e\, E_i\, dt] \wedge (dr_i - g_i\, dt) +$$

$$\tfrac{1}{2}\, e\, \epsilon_{ijk} B_k\, dr_i \wedge dr_j + \tfrac{1}{2}\, \epsilon_{ijk}\, \Theta_k\, dq_i \wedge dq_j + Q_0\, \epsilon_{ij}\, dr_i \wedge dq_j,$$

only gauge invariant quantities

Cartan 1-form $\sigma = d\lambda$

$$\lambda = \left(\vec{q} + \overrightarrow{\mathcal{R}}\right) \cdot d\vec{r} + \vec{\mathcal{Q}} \cdot d\vec{q} + (\mathcal{T} - \mathcal{E} - \Delta\mathcal{E})\; dt.$$

$$\begin{cases} (1 + \Xi)\; \dot{\vec{r}} + \Theta\, \dot{\vec{q}} & = \nabla_{\vec{q}}\, [\mathcal{E} + \Delta\mathcal{E} - \mathcal{T}] + \partial_t \vec{\mathcal{Q}} \\ X\, \dot{\vec{r}} + (1 + \Xi)\; \dot{\vec{q}} & = -\nabla_{\vec{r}}\, [\mathcal{E} + \Delta\mathcal{E} - \mathcal{T}] - \partial_t \vec{\mathcal{R}} \end{cases}$$

$$\Xi_{ij} = \partial_{r_i} \mathcal{Q}_j - \partial_{q_j} \mathcal{R}_i, \qquad X_{ij} = \partial_{r_i} \mathcal{R}_j - \partial_{r_j} \mathcal{R}_i.$$

$\partial_t \vec{\mathcal{Q}} = \partial_t \vec{\mathcal{R}} \equiv 0 \Rightarrow \sigma = \omega - dH \wedge dt \qquad d\omega = 0 \Leftrightarrow$

$$\omega = (\delta_{i,j} + \Xi_{ij}\,)\, dr_{\,i} \wedge dq_{\,j} + \frac{1}{2}\, [X_{ij}\, dq_{\,i} \wedge dq_{\,j} - \Theta_{ij}\, dr_{\,i} \wedge dr_{\,j}]$$

$\mathcal{H} = \mathcal{E} + \Delta\mathcal{E} - \mathcal{T}$

$$
\begin{aligned}
\varepsilon_{ijk}\,\partial_{q_i}\Theta_{jk} &= 0, & \varepsilon_{ijk}\,\partial_{r_i}X_{jk} &= 0,\\
\partial_{q_j}\,\Xi_{ij} &= -\,\partial_{r_j}\,\Theta_{ij}, & \partial_{r_j}\,\Xi_{ij} &= \,\partial_{q_j}\,X_{ij},\\
(1-\delta_{hk})\,\varepsilon_{kij}\,\partial_{q_k}\Xi_{ij} &= \varepsilon_{hij}\,\partial_{r_h}\Theta_{ij}, & (1-\delta_{hk})\,\varepsilon_{kij}\,\partial_{r_k}\Xi_{ij} &= -\varepsilon_{hij}\,\partial_{q_h}X_{ij},
\end{aligned}
$$

$$
\omega = \omega_{\alpha\beta}\,d\xi_\alpha \wedge d\xi_\beta \Rightarrow \;\{f,g\} = \omega^{\alpha\beta}\partial_\alpha f \partial_\beta g
$$

$$
\sqrt{\mathrm{det}\left(\omega_{\alpha\beta}\right)} = 1 - \frac{1}{2}\mathrm{Tr}\left(\mathbf{\Xi}^2 + \mathbf{X}\,(\mathbf{1} + 2\,\mathbf{\Xi})\,\mathbf{\Theta}\right) \neq 0.
$$

monopole in momentum space

$$\mathbf{\Theta} = \theta \frac{\mathbf{k}}{k^3} \qquad (k \neq 0)$$

A. Bérard, H. Mohrbach *Phys. Rev.*(2004)

$B \equiv 0, \quad \vec{E} = E\widehat{x}, \quad \epsilon_n(\mathbf{k}) = \mathbf{k}^2/2$

$\mathbf{r}(t) = x(t)\widehat{\mathbf{k}}_0 + y(t)\widehat{\mathbf{E}} + z(t)\widehat{\mathbf{n}}, \qquad \widehat{\mathbf{k}}_0 \perp \widehat{\mathbf{E}},\ \widehat{\mathbf{n}} = \frac{\mathbf{k}_0 \wedge \mathbf{E}}{k_0 E}$

$$z(t) = \frac{\theta}{k_0} \frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}} \Rightarrow \quad \Delta z = \frac{2\theta}{k_0}$$

Fang et al. *Science* **302**, 92 (2003) $SrRuO_3$

perovskite structure in itinerant(metallic) ferromagnet

AHE as fingerprint of magnetic monopoles in crystal momentum space

Rashba-Dresselhaus spin-orbit Hamiltonian

$$H = \sum_i f_i\,(\mathbf{k})\,\sigma_i$$

Optical Hall Effect

$$\dot{\vec{r}} \approx \vec{p} - \frac{s}{\omega}\text{grad}(\frac{1}{n}) \times \vec{p}, \qquad \dot{\vec{p}} \approx -n^3\omega^2\text{grad}(\frac{1}{n}).$$

M. Onoda *et al. Hall effect of light. Phys. Rev. Lett.* **93**, 083901 (2004)

Conclusions and Outlook

- The method of coadjoint orbits allows to construct exotic (2-dim central extension for Galilei Gr.) mechanical models in (2+1)-dim. , the coupling with a gauge field (E.M.) is essential in the construction of a meaningful system.

- Quantization of the exotic models allows to identify the classical analogs of the Anyons

- It has been proved the equivalence of the DH exotic model with LSZ acceleration-dependent Lagrangian model

- Generalized models of noncommutative mechanics can be considered

- The second central extension can be considered as a "nonrelativistic shadow" of the particle spin in relativistic models.

- Exotic models may contain anomalous coupling of anyons, in particular DH model is not a non relativistic limit of the relativistic anyon model.

- In noncommutative models the "Minimal Coupling" and the "Minimal Addition" of a gauge field are not equivalent procedures (modulo total time derivatives). A local Seiberg-Witten transformations allows to map systems in different phase spaces (endowed with different symplectic structure) and fields acting on, in order to obtain the same physical results.

- In Field Theory based on the Moyal product, the exotic Galilei symmetry can be restored dynamically and preserved under chiral self-interactions.

- The semiclassical Bloch electron dynamics provides concrete examples of 2D and 3D systems in noncommutative variables. Monopoles in momentum space can be conveniently described in the presented formalism.

- The general Hamiltonian Structure of systems described by non commutative configuration variables is described.

- OUTLOOK: Applications to Physics: AHE, Grephene, Supersymmetry extensions, Spin Hall Effect, Optical Hall Effect.