Symmetries of asymptotically flat spaces and gravitational memory effect

L. Martina

Dip. Matematica e Fisica "Ennio De Giorgi" - Università del Salento,
Sezione INFN, Lecce, Italy

Istituto Nazionale di Fisica Nucleare

Local and Nonlocal Geometry of PDEs and Integrability
Conference dedicated to the 70th birthday of Joseph Krasil'shchik

Y. Calo', F. Capone, L. M. arXiv:1805.07814

Outline

Motivations: known facts

1. In GR two metrics related by a a non trivial diffeomorphism $f : M \to M$, are equivalent if f is the identity outside a compact submanifold of M
2. There is a class of distinguished diffeomorphisms which do not tend to the identity at infinity: e.i. translations
3. They map asymptotically flat spacetimes into asymptotically flat spacetimes (AFS)
4. In AFS global conservation laws can be defined
5. Diffeomorphisms preserving flatness form the Asymptotic Symmetry Group, it have to include the Poincaré group
6. In fact, Einstein equations Asymptotic Symmetry Group is larger: $BMS_{glob} = ST \ltimes L_+^\uparrow$
7. ST is the abelian infinite dimensional group of the *Supertranslations*.
8. Conserved charge associated with BMS: Poincaré charges and *supermomenta*.

Motivations: new facts

1. The BMS group extension: $BMS_{ext} = ST \ltimes SR$, Conformal *BMS*
2. "BMS" Conserved charges and *Soft Theorems*
3. Gravitational Memory Effect
4. Construction of a perturbative quantum gravity S-matrix theory
5. Infinities due to Soft photons in QED correspond Soft Gravitons problem in Quantum Gravity (IR structure)
6. Carroll Group and Conformally Extended Carroll Group
7. Carroll Group for pp-Gravitational Waves
8. Carroll Group and BMS
9. Holographic description of gravity in asymptotically flat spacetimes

Basic References

1. H. Bondi. Gravitational waves in general relativity. Nature, **186** (1960).:535.
2. H. Bondi, M.G.J. van der Burg, and A.W.K Metzner. Gravitational waves in general relativity. VII. waves from axi-symmetric isolated systems. Proc. R. Soc. Lond. A, **269** (1962) 21-52
3. R. Sachs. Asymptotic symmetries in gravitational theory. Phys. Rev., **128** (1962) 2851-2864.
4. R.K. Sachs. Gravitational waves in general relativity. VIII. waves in asymptotically flat space-time. Proc. R. Soc. Lond. A, **270** (1962) 103-126.
5. G. Barnich and C. Troessaert. Aspects of the BMS/CFT correspondence. JHEP, **5,** (2010)
6. É.É. Flanagan and D.A. Nichols. Conserved charges of the extended Bondi-Metzner-Sachs algebra. Phys. Rev. **D, 95** (2015)
7. A. Strominger and A. Zhiboedov. Gravitational memory, BMS supertranslations and soft theorems. JHEP, **01** (2016) 86
8. A. Strominger, Lectures on the Infrared Structure of Gravity and Gauge Theory, arXiv:1703.05448v2 (2018) ,

Asymptotic flatness

Bondi coordinates (u, r, x_1, x_2)
Bondi gauge

$$g_{rr} = 0 = g_{rA}, \quad det\left(g_{AB}\right) = r^4 b\left(x_1, x_2\right)$$

$$ds^2 = -U\, e^{2\beta}\, du^2 - 2\, e^{2\beta}\, du\, dr + g_{AB}\left(dx^A - W^A\, du\right)\left(dx^B - W^B\, du\right)$$

$$g_{AB}\left(u, r, x^C\right) = r^2 \begin{pmatrix} e^{2G+2H} & 2\sin\theta\ \sinh\left(G-H\right) \\ 2\sin\theta\ \sinh\left(G-H\right) & e^{-2G-2H}\sin^2\theta \end{pmatrix}$$

Einstein Eq.s

$$\begin{array}{ll} \text{Main Eq.s}: & R_{rr} = 0, \quad R_{rA} = 0, \quad g^{AB} R_{AB} = 0, \quad g^{CA} R_{AB} = 0 \\ \text{Suppl. Eq.s}: & R_{uu} = 0, \quad R_{Au} = 0 \\ \text{Trivial Eq.}: & R_{ur} = 0 \end{array}$$

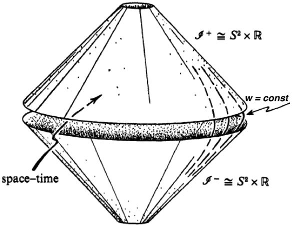
Paradigmatic AFS: The Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 \sin^2\theta \, d\phi^2\right)$$

Conformal Eddington - Finkelstein metrics

$u_\mp = t \mp r - 2m \log(r - 2m), \Omega = r^{-1} = w$

$d\tilde{s}^2_\mp = \Omega^2 \, d\, s^2_\mp = -\left(w^2 - 2m\, w^3\right) du^2_\mp \mp 2du_\mp \, dw + d\theta^2 \sin^2\theta + d\phi^2$



: The null infinity for the conformal Schwarzschild metric (Penrose)

Asymptotics

1) Select a suitable conformal rescaled metric d $^2 = \Omega(x)^2 ds^2$ in such a way that infinity becomes a *place* in an unphysical spacetime $\tilde{M} = M \bigcup \mathscr{I}^+$ with conformal boundary $\mathscr{I}^+$: i.e. the the metric $\tilde{g}$ should be smooth at $\mathscr{I}^+$, and $\Omega = 0$ e $d\Omega \neq 0$ at $\mathscr{I}^+$.
2) Asymptotics $\frac{1}{r} \to 0$ at null infinity

The fall-off conditions must be sufficiently weak /strong

- to allow non trivial phenomena in the interior of the spacetime gravitational waves
- to provide well defined total mass and radiating energy flux

$$ds^2 = -U\, e^{2\beta} du^2 - 2\; e^{2\beta} dudr + g_{AB}\, \left(dx^A - W^A\, du\right) \left(dx^B - W^B\, du\right)$$

$$g_{AB} = r^2 \left(\gamma_{AB} + \frac{C_{AB}}{r} + O\left(r^{-2}\right)\right), \quad \gamma_{AB} = \mathrm{diag}\left(1, \sin^2\theta\right) \; (*g.c.)$$

$$U = 1 - \frac{2m}{r} - \frac{2M}{r^2} + O(r^{-3}), \quad \beta = -\frac{C_{AB}\, C^{AB}}{32\, r^2} + O(r^{-3})$$

$m\left(u, x^C\right)$: Bondi mass aspect, $C_{AB} = C_{AB}\left(u, x^C\right)$ initial data symmetric tensor

$$W^A = -\frac{D_B\, C^{AB}}{2r^2} - \frac{1}{r^3}\left(\frac{2}{3} N^A - \frac{1}{16} D^A \left(C^{BC} C_{BC}\right) + \tfrac{1}{2}\, C^{AB} D^C C_{BC}\right) + O(r^{-4}),$$

D_A covariant derivative on the sphere.

The News tensor

$$R_{uu} = 0 \quad \longrightarrow \quad \partial_u\, m = \frac{1}{4} D_A\, D_B\, N^{AB} - \frac{1}{8} N_{AB}\, N^{AB} \quad N_{AB} = \partial_u\, C_{AB}$$

News tensor

Bondi angular momentum aspect $N_A = N_A(u, x^C)$

$$\begin{aligned} \partial_u N_A \quad &= \quad D_A\, m + \frac{1}{4}\left(D_A D_B D_C\, C^{BC} - D^2 D^c\, C_{AC}\right) \\ &\quad\; \frac{1}{4} D_B\left(N^{BC}\, C_{CA} + 2 D_B N^{BC}\, C_{CA}\right) \end{aligned}$$

The BMS algebra

BMS symmetries: diffeomorphisms $\mathscr{I}^+ \to \mathscr{I}^+$ generated by the asymptotic Killing vectors ξ

$$\mathcal{L}_\xi\, g_{uu} = O\left(\tfrac{1}{r}\right), \quad \mathcal{L}_\xi\, g_{ur} = O\left(\frac{1}{r^2}\right),$$

$$\mathcal{L}_\xi\, g_{uA} = O\left(1\right), \quad \mathcal{L}_\xi\, g_{AB} = O\left(r\right)$$

$$\mathcal{L}_\xi\, g_{rr} = 0, \quad \mathcal{L}_\xi\, g_{rA} = 0, \quad g^{AB}\mathcal{L}_\xi\, g_{AB} = 0 \quad \text{gauge preserving}$$

$$\xi = f\,\partial_u + \xi^A\,\partial_A + \xi^r\,\partial_r$$

$$\begin{cases} f = & \alpha\left(x^B\right) + \tfrac{1}{2} u\; D_B Y^B\left(x^B\right) \\ \xi^A = & Y^A - \frac{D^A f}{r} + \frac{C^{AB} D_B f}{2\,r^2} + O\left(r^{-3}\right) \\ \xi^r = & -\tfrac{1}{2} r D_A Y^A + \tfrac{1}{2} D^2 f + O(r^{-1}) \end{cases}$$

$$2D_{(A}Y_{B)} - h_{AB} D_C Y^C = 0 \to \quad Y^A = D^A \chi_+ + \epsilon^{AB} D_B \chi_-, \;\left(D^2 + 2\right)\chi_\pm = 0$$

The BMS Group

$$BMS^{glob} = ST \ltimes L^{\uparrow}$$

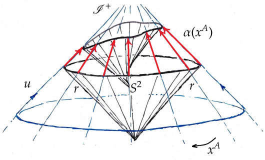
SuperTranslations

$\xi_\alpha = \alpha\left(x^B\right)\,\partial_u \quad \longrightarrow \quad \tilde{u} = u + \epsilon\alpha\left(x^B\right),\ \tilde{x}^A = x^A$

Rotations $\xi_Y = \frac{u}{2} D_A Y^A \partial_u + Y^A\,\partial_A \quad \longrightarrow$

$\tilde{u} = u\,\Omega\left(x^B\right), \qquad \tilde{x}^A = X^A\left(x^B\right):\ \tilde{\gamma}_{AB} = \Omega\left(x^B\right)^2\,\gamma_{AB}$

$$[\xi_\alpha, \xi_\beta] = 0, \quad [\xi_\alpha, \xi_Y] = \xi_\gamma, \quad [\xi_Y, \xi_Z] = \xi_{Y\nabla Z - Z\nabla Y}$$



$$\frac{1}{16\pi}\int d^2x \left\{4\alpha m - 2u_0 Y^A D_A m + 2Y^A N_A - \frac{1}{8} Y^a D_A\left(C_{BC}C^{BC}\right)\right\}$$

Conformal Transf. of S^2

Stereographic coordinates $z = e^{\imath\phi} \cot\left(\frac{\theta}{2}\right)$, $\quad \bar{z} = e^{-\imath\phi} \cot\left(\frac{\theta}{2}\right)$

$$
\begin{aligned}
ds^2 \;&=\; -\left(1 - \frac{2m}{r}\right) du^2 - 2du\,dr + 2r^2\gamma_{z\bar{z}}\, dzd\bar{z} + \\
&\quad r\, C_{z\,z} dz^2 + c.c. + D^z C_{z\,z} du\, dz + c.c. + O\left(r^{-2}\right)
\end{aligned}
$$

$$
\xi_\alpha = \alpha\partial_u - \tfrac{1}{r}\left(D^z\alpha\partial_z + D^{\bar{z}}\alpha\partial_{\bar{z}}\right) + D^2\,\alpha\,\partial_r
$$

$$
\xi_Y = \frac{u\,D_z Y^z}{2}\,\partial_u + \left(Y^z + \frac{u\,D_z^2\,Y^z}{2\,r}\right)\partial_z - \frac{u\,D^{\bar{z}} D_z\,Y^z}{2\,r}\,\partial_{\bar{z}}
$$

Killing eq.s

$\longrightarrow \quad \partial_z\, Y^z = 0, \quad \partial_{\bar{z}}\, Y^{\bar{z}} = 0 \Rightarrow SL\left(2, \mathbb{C}\right)/\mathbb{Z} \cong SO\left(1,3\right)$

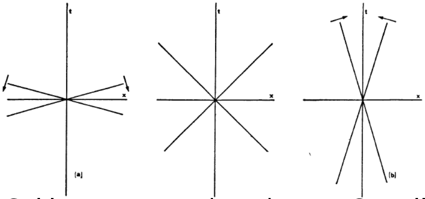
G. Barnich and C. Troessaert JHEP, 12:105, 2011 *meromorphic conformal transformations*

$$
BMS^{ext} = ST \ltimes SupeRotations
$$

The Carroll group

J.M. Lévy-Leblond, Ann. Inst. H. Poincaré, 3(1965)1.

$$\begin{cases} \Delta x' & = \frac{\Delta x + u \Delta t}{\sqrt{1-u^2}} \\ \Delta t' & = \frac{\Delta t + u \Delta x}{\sqrt{1-u^2}} \end{cases} \quad u \ll 1, \ \frac{\Delta x}{\Delta t} \gg 1 \longrightarrow \begin{cases} \Delta x' & = \Delta x \\ \Delta t' & = \Delta t + u \Delta x \end{cases}$$



Galilei Minkowski Carroll

$$s = Cx^0, \ \mathbf{b} = C\beta, \ f = C\, l^0$$
$$C \to \infty$$

$$\begin{cases} s' & = s + \mathbf{b} \cdot R\,\mathbf{x} + f \\ \mathbf{x}' & = R\,\mathbf{x} + \mathbf{l} \end{cases} \qquad \begin{array}{c} (f', \mathbf{l}', \mathbf{b}', R') \cdot (f, \mathbf{l}, \mathbf{b}, R) = \\ (f + f' + \mathbf{b}' \cdot R'\,\mathbf{l}, \mathbf{l}' + R'\mathbf{l}, \mathbf{b}' + R'\mathbf{b}, R'R) \end{array}$$

$$Carr\,(3+1) = E\,(3) \rtimes \mathbb{R}^4$$

Carroll manifolds

C Duval et al., Class.Quant.Grav. 31 (2014) 085016
(C, g, ζ, ∇) :

1. C $d+1$-dim manifold
2. g 2-contravariant symmetric positive
3. $Ker\,(g) = \mathrm{Span}\,\{\zeta\}$
4. $\nabla g = 0$, $\nabla \zeta = 0$

Carroll Group: g-preserving diffeomorph. $C \to C$

Ex.: $C = \mathbb{R}^3 \times \mathbb{R} \ni (\boldsymbol{x}, s)\,,\ g = \delta_{ab} dx^a \otimes dx^b,\ \zeta = \partial_s,\ \Gamma^k_{ij} = 0$

$X \in$ Carroll algebra: $\mathcal{L}_X g = 0,\ \mathcal{L}_X \zeta = 0,\ \mathcal{L}_X \nabla = 0$

$$X = \left(\omega^a_b x^b + \alpha^a\right) \partial_{x^a} + \left(\phi - v_a x^a\right) \partial_s \in carr\,(d+1)$$

$$Carr\,(d+1) \ni a = \begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix}, \quad R \in O\,(d)$$

Conformal Carroll Group

$\left(\mathbb{R}^{d+1}, g, \zeta, \nabla \equiv 0\right)$ and $N \in \mathbb{N}$: a is a Conformal Carroll transf. of level N if

$$a^*\left(g \otimes \zeta^{\otimes N}\right) = g \otimes \zeta^{\otimes N} \iff a^* g = \Omega^2 g,\ a_* \zeta = \Omega^{-\frac{2}{N}} \zeta$$

$$X \in Ccarr_N\left(d+1\right) \iff \mathcal{L}_X\, g \otimes \zeta^{\otimes N} = 0 \longrightarrow \begin{cases} \mathcal{L}_X\, g & = & \lambda\, g \\ \mathcal{L}_X\, \zeta & = & \mu\zeta \\ \lambda + N\mu & = & 0 \end{cases}$$

$$X = X^a \partial_a + T\partial_s : \qquad \partial_a X_b + \partial_b X_a = \lambda \delta_{ab},\ \partial_s X^a = 0,\ \partial_s T = -\mu$$

$$\begin{aligned} X &= \left(\omega^a_b x^b + \alpha^a + \left(\chi - 2\kappa^b x_b\right) x^a + \kappa^a x_b x^b\right) \partial_{x^a} + \\ & \quad \left(\frac{2}{N}\left(\chi - 2\kappa^b x_b\right) s - F\left(x^a\right)\right) \partial_s \end{aligned}$$

$$\omega^a_b \in so\left(d\right),\ \alpha^a, \kappa^a \in \mathbb{R}^d, \chi \in \mathbb{R},\ F \in C^\infty\left(\mathbb{R}^d, \mathbb{R}\right),\ \lambda = 2\left(\chi - 2\kappa^b x_b\right)$$

$$X|_{\chi=\kappa^a=0, F=\phi - v_a x^a} \in carr\left(d+1\right) \subset Ccarr_N\left(d+1\right)$$

Conformal Carroll Group and BMS

$$\left(C = \Sigma \times \mathbb{R}, g = g^{\Sigma}_{ab} dx^a \otimes dx^b,\ \zeta = \partial_s,\ \Gamma^c_{ab} = \left\{\Gamma^{\Sigma}{}^{c}_{ab}, 0\right\}\right)$$

$$X \in Ccarr_N\left(C, g, \zeta, \nabla\right) \iff \begin{cases} \mathcal{L}_Y\, g^{\Sigma} = \lambda\, g^{\Sigma} \quad \leftrightarrow \lambda = \frac{2}{d}\nabla^{\Sigma}_a Y^a \\ X = Y^a \partial_a + \left(\frac{\lambda}{N} s + F\left(x^a\right)\right)\partial_s \\ g^{\Sigma} \to \Omega^2\, g^{\Sigma},\ F \to \Omega^{-2N} F \end{cases}$$

$$x'^a = \varphi\left(x^b\right) \in Conf\left(\Sigma, g^{\Sigma}\right),\ s' = \Omega^{\frac{2}{N}}\left(s + \alpha\left(x^b\right)\right), \alpha \in C^{\infty}\left(\Sigma, \mathbb{R}\right)$$

Ex.1 : $\Sigma = S^2\ \ Ccarr\left(S^2 \times \mathbb{R}, g, \zeta\right) \cong sl\left(2, \mathbb{C}\right) \ltimes C^{\infty}\left(S^2, \mathbb{R}\right)$

$$N = 2: \quad X = Y^a \partial_a + \left(\tfrac{1}{2}\nabla^{\Sigma}_a Y^a s + F\left(x^a\right)\right)\partial_s \in BMS$$

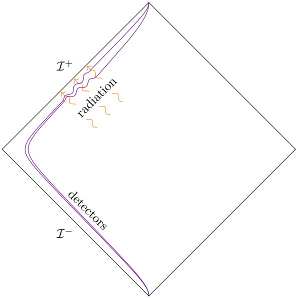
$$\mathscr{I}^+ \simeq S^2 \times \mathbb{R} = \left\{\mathbb{R}^{d,1} \ni \left(x_0 > 0, \mathbf{x}\right) : x^0 = |\mathbf{x}|, g_M \to g = x_0^2\, g^{S^{d-1}}\right\}$$

Carroll manifold

Gravitational Memory Effect

Ya. B. Zel'dovich, A.G. Polnarev, Astron. Zh. 51 (1974) 30
V. Braginsky, K. S. Thorne, Nature (1987) 123
H. Bondi, F.A.E. Pirani, Nature 332 (1988) 212
D. Christodolou, Phys. Rev Lett. 67 (1991) 1486
D. Garfinkle, S. Hollands, A. Ishibashi, A. Tolish and R. M. Wald, Class. Quantum Grav. 34, 145015 (2017)

P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, Phys. Lett. B 772 (2017) 743.



Plane Gravitational waves

Brinkmann coordinates

$$g = \delta_{ij} dX^i dX^j + 2dUdV + K_{ij}(U)\, X^i X^j\, dU^2, \quad tr(K) = 0$$

$$K_{ij}(U)\, X^i X^j = \tfrac{1}{2}\mathcal{A}_+(U)\left[\left(X^1\right)^2 - \left(X^2\right)^2\right] + \mathcal{A}_x(U)\, X^1 X^2$$

$R^i_{UjU} = -R^V_{ijU} = -K_{ij} \Rightarrow R_{\mu\nu} = 0$ Ricci - flat Brinkmann metrix

Isometries

Baldwin-Jeffrey-Rosen coord.

$$(\mathbf{X}, U, V) \to \left(P(u)\,\mathbf{x}, u, v - \frac{1}{4}\mathbf{x} \cdot \dot{a}(u)\,\mathbf{x} \right)$$

$$g = a_{ij}(u)\, dx^i\, dx^j + 2du\, dv$$

$$a = P^\dagger\, P$$

$$\ddot{P} = K\, P, \quad P^\dagger\, \dot{P} = \dot{P}^\dagger\, P,\ \det(P) \neq 0$$

$$K = P\left(\tfrac{1}{2}\dot{L} + L^2\right) P^{-1},\ L = a^{-1}\,\dot{a}$$

$$R_{uu} = tr\left(\tfrac{1}{2}\dot{L} + L^2\right) = 0$$

Ex.: Flat metrics $\quad R_{uiuj} = 0 \Rightarrow \ddot{a} - \tfrac{1}{2}\dot{a}\, a^{-1}\, a = 0$

$$a(u) = a_0^{\frac{1}{2}} \left[\mathbf{1} + (u - u_0)\, c_0\right]^2 a_0^{\frac{1}{2}}$$

Manifestly flat BJR coordinate chart

$$\hat{\mathbf{x}} = \left[\mathbf{1} + (u - u_0)\, c_0\right] a_0\, \mathbf{x},\ \hat{u} = u,\ \hat{v} = v - \tfrac{1}{2}\mathbf{x} \cdot c_0 \left[\mathbf{1} + (u - u_0)\, c_0\right]^{-1} \mathbf{x}$$

Simmetries of pp-Waves
J-M. Souriau, Coll. Int. CNRS 220 (1973)

$$\mathbf{x} \to \mathbf{x} + H(u)\,\mathbf{b} + \mathbf{c}, \quad u \to u, \quad v \to v - \mathbf{b}\cdot\mathbf{x} - \tfrac{1}{2}\mathbf{b}\cdot H(u)\,\mathbf{b} + f$$

$$H(u) = \int_{u_0}^{u} a(t)^{-1}\, dt, \qquad \mathbf{b}, \mathbf{c} \in \mathbb{R}^2, f \in \mathbb{R}$$

no rotations: C_{BR} normal sub-group of $Carr\,(2+1)$
$Carr\,(2+1)\,/\,C_{BR} \cong O(2)$

$$(\mathbf{b}, \mathbf{c}, f)\cdot(\mathbf{b}', \mathbf{c}', f') = (\mathbf{b}+\mathbf{b}', \mathbf{c}+\mathbf{c}', f+f'-\mathbf{b}\cdot\mathbf{c}')$$

C_{BR} group is implemented on an u = const. null hypersurface it does depend on the (fixed) value of u

C_{BR} is a subgroup of the Poincaré group, represented as

$$A_c = \begin{pmatrix} \mathbf{1} & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & 0 \\ -\mathbf{b}^T & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The coordinate transformation followed backwards yields the Carroll action in terms of the Brinkmann coordinates

Geodesic motion

C_{BR}-Lie Algebra $\quad K_i = H_{ij}\partial_j - x_i\,\partial_v,\ P_i = \partial_i,\ P_0 = \partial_v$

$$[K_i, P_j] = \delta_{ij} P_0$$

Affine parameter s: $\frac{d\,u}{d\,s} = \mu$

$$e = \tfrac{1}{2}\frac{d\,\mathbf{x}}{d\,s}\cdot a\,(u)\,\frac{d\,\mathbf{x}}{d\,s} + \mu\,\frac{d\,v}{d\,s}$$

Other conserved q.s $\quad \mathbf{p} = a\,(u)\,\frac{d\,\mathbf{x}}{d\,s}, \qquad \mathbf{k} = \mathbf{x} - H\,(u)\,\mathbf{p}$

Geodesics

$\mathbf{x}\,(u) = H\,(u)\,\mathbf{p} + \mathbf{k}, \quad v\,(u) = -\tfrac{1}{2}\mathbf{p}\cdot H\,(u)\,\mathbf{p} + eu + d$

Action of Isometries $(\mathbf{p}, \mathbf{k}, e, d) \to (\mathbf{p} + \mathbf{b}, \mathbf{k} + \mathbf{c}, e, d + f - \mathbf{b}\cdot\mathbf{k})$

To "srtraighten out" a geodesic : $\mathbf{p} = 0,\ \mathbf{k} = \mathbf{x}_0,\ d = 0$

Sandwich waves

$$R_{uu} = 0 \quad u_i < u < u_f$$

$$tr\left(\tfrac{1}{2}\dot{L} + L^2\right) = 0 \quad L = a^{-1}\,\dot{a}$$

Introducing $\chi = (\det a)^{\frac{1}{4}}\,, \quad q = \chi^{-2}\,a$

$$\ddot{\chi} + \omega^2(u)\,\chi = 0, \qquad \omega^2(u) = tr\left[\left(q^{-1}\,\dot{q}\right)^2\right]$$

$$\chi(u) > 0 \Rightarrow \ddot{\chi}(u) < 0 \;\Rightarrow\; \exists u_s > u_i : \chi(u_s) = 0 \Rightarrow \text{metric singularity}$$

Geodesics in flat before/after-zone

Before zone : $u < u_i, \quad a(u) = \mathbf{1}_2$

Geodesic of the detector at rest in the before-zone $\hat{\mathbf{p}}_0 = 0$

$$\mathbf{x}_b = \hat{\mathbf{x}}_0, \quad v_b = e\,(u - \hat{u}_0) + \hat{v}_0$$

In generic flat metrics

$$H(u) = -a_0^{-\frac{1}{2}} c_0^{-1} \left[(\mathbf{1} + (u - u_0)\; c_0)^{-1} - \mathbf{1} \right] a_0^{-\frac{1}{2}}$$

Transformed constants of motion

$\mathbf{p} = -a_0^{\frac{1}{2}} c_0\, \hat{\mathbf{x}}_0, \quad \mathbf{k} = a_0^{-\frac{1}{2}} c_0\, \hat{\mathbf{x}}_0, \quad d = \hat{v}_0 - e\,\hat{u}_0 + \frac{1}{2}\hat{\mathbf{x}}_0 \cdot c_0 \hat{\mathbf{x}}_0$

Reference metrics

Transformed geodesics

$$\mathbf{x}(u) = \left[-H(u)\, a_0^{\frac{1}{2}} c_0 + a_0^{-\frac{1}{2}}\right] \hat{\mathbf{x}}, \;\; u = \hat{u},$$

$$v(u) = \hat{v} + \tfrac{1}{2}\hat{\mathbf{x}} \cdot c_0 \left[\mathbf{1} - a_0^{\frac{1}{2}} H(u)\, a_0^{\frac{1}{2}} c_0\right] \hat{\mathbf{x}}$$

Hold everywhere including the inside-zone.

$$g = d\mathbf{x} \cdot a(u)\, d\mathbf{x} + 2du\, dv = d\hat{\mathbf{x}} \cdot \hat{a}(u)\, d\hat{\mathbf{x}} + 2du\, d\hat{v}$$

$$\hat{a}(u) = \left[a_0^{-\frac{1}{2}} - c_0\, a_0^{\frac{1}{2}} H(u)\right] a(u) \left[a_0^{-\frac{1}{2}} - H(u)\, a_0^{\frac{1}{2}}\, c_0\right]$$

Particles at rest in the before-zone are **not** at rest in the after-zone.
→ Gravity Memory effect.

Integrable Pulses

Andrzejewski - Prencel, Phys. Lett B 782 (2018) 421-426

$$A = \tfrac{1}{2}K_{11} = \frac{A0}{\left(e^2+u^2\right)^2}$$

$$P = \sqrt{e^2+u^2}\begin{pmatrix} \mu_2\Phi_{1,2}(u) & \mu_3\Phi_{1,2}() \\ \mu_6\Phi_{3,4}(u) & \mu_7\Phi_{3,4}(u) \end{pmatrix}$$

$$\Phi_{1,2}(u) = \mu_1\cos\left(\omega_1(u)\right)+\sin\left(\omega_2(u)\right), \Phi_{3,4}(u) = \mu_5\cos\left(\omega_3(u)\right)+\sin\left(\omega_4(u)\right)$$

$$a = \left(e^2+u^2\right)\begin{pmatrix} \mu_2^2{\Phi_{1,2}}^2+\mu_6^2{\Phi_{3,4}}^2 & \mu_2\mu_3{\Phi_{1,2}}^2+\mu_6\mu_7{\Phi_{3,4}}^2 \\ \mu_2\mu_3{\Phi_{1,2}}^2+\mu_6\mu_7{\Phi_{3,4}}^2 & \mu_3^2{\Phi_{1,2}}^2+\mu_7^2{\Phi_{3,4}}^2 \end{pmatrix}$$

$$\omega_{\mathbf{1}} = \sqrt{1-\frac{A0}{2e^2}}\tan^{-1}\left(\frac{u}{e}\right), \omega_{\mathbf{2}} = \sqrt{\frac{2e^{\mathbf{2}}-A0}{e^{\mathbf{2}}}}\tan^{-1}\left(\frac{u}{e}\right),$$

$$\omega_{\mathbf{3}} = \sqrt{\frac{\mathbf{A0}}{\mathbf{2}e^{\mathbf{2}}}+1}\tan^{-1}\left(\frac{u}{e}\right), \omega_{\mathbf{4}} \to \sqrt{\frac{\mathbf{A0}+\mathbf{2}e^{\mathbf{2}}}{e^{\mathbf{2}}}}\tan^{-1}\left(\frac{u}{e}\right)$$

Integrable Pulses II

Geodesic equations in Brinkmann coord.

$$\ddot{X}^{\pm} = \frac{\pm 1}{2} A X^{\pm}, \quad \ddot{V} + \sum_{\pm} \pm X^{\pm} \left(A \dot{X}^{\pm} + \frac{1}{4} \dot{A} X^{\pm} \right) = 0$$

$$\sqrt{\sum_{\pm} \left(\dot{X}^{\pm}_{fin1} - \dot{X}^{\pm}_{fin2} \right)^2} \approx \sqrt{\sum_{\pm} \left(\dot{X}^{\pm}_{in1} - \dot{X}^{\pm}_{in2} \right)^2 A_0 / \sqrt{1 \pm \frac{A0}{2e^2}}}$$

1. The future null conformal boundary $\mathscr{I}^+$ of an asymptotically flat spacetime emitting gravitational radiation is a Carroll manifold
2. The associated conformal Carroll group is isomorphic to the Bondi-Metzner-Sachs group.
3. The Carroll symmetry allows to integrate the geodesic eq.s in BJR coordinates.
4. The Carroll symmetry is implemented on the geodesic conservation laws
5. In BJR the past inertial coordinates is connected to the future non-inertial coordinate system.
6. This diffeomorphism does not tend to identity at infinity.
7. AFS before/after a Sandwich wave are non-equivalent.
8. The corrispondence Brinkman $\Leftrightarrow$ BJR coordinates requires to solve a parametric oscillator equation, solved in various cases

Open problems

1. How to compute the Bondi *News* tensor with the pp-waves
2. How to include the finite Carroll isometries of the pp-waves into the BMS scheme
3. How to relate pp-waves with the soft graviton theorem Study of the Einstein - Maxwell equation: soft photon theorem and S-scattering for gravity.
4. The relation with the extended BMS group (superrotations).
5. The future null conformal boundary $\mathscr{I}^+$ of an asymptotically flat spacetime in $d+1>4$
6. The asymptotic conditions null spaces in $d+1>4$
7. Relation with the Conformal Field Theory