q-difference Painlevé equations: symmetries and solutions

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Local and Nonlocal Geometry of PDE's and Integrability

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based on

Cluster integrable systems and q-Painlevé equations, JHEP 02 (2018) 077, arXiv:1711.02063

Cluster Toda chains and Nekrasov functions, to appear in L.D.Faddeev volume of Theor. Math. Phys. (January 2019), arXiv:1804.10145

with Misha Bershtein & Pasha Gavrylenko

some development ("Spin chain case" etc) yet to appear,
also with Kolya Semenyakin ...

Painlevé equations

- Non-autonomous systems (NO integrability);
- Related to isomonodromic deformations (Schlesinger system of PDE's);
- Cascade (classification), symmetry groups;
- Relation to conformal theories (Kiev formulas) and SUSY gauge theories.

DEAUTONOMIZATION of *integrable* systems!

Toda chains

- Integrable systems with exponential potentials

$$H = \sum_i \left(\frac{p_i^2}{2} + e^{\phi_{i+1}-\phi_i} \right)$$

- Lax representation (in terms of Chevalley generators e, f, h):

$$L = (p \cdot h) + \sum_{\alpha \in \Pi} (e_\alpha + f_\alpha) e^{(\alpha \cdot \phi)/2}$$

with the sum over the *simple* roots Π of $\mathfrak{g} = \mathrm{Lie}(G)$.

- Simple or affine groups: e.g. the Liouville ($G = SL(2)$)

$$H = \frac{p^2}{2} + e^{\phi}$$

or sine-Gordon ($G = \widehat{SL(2)}$)

$$H = \frac{p^2}{2} + z\left(e^{\phi} + e^{-\phi}\right)$$

Toda application

Seiberg-Witten integrable systems (from 4d SUSY gauge theories):

- Pure gauge theories $\equiv$ Toda chains with $\mathfrak{g} = \mathrm{Lie}(G)$ of the gauge group;
- Lax representation: $L \in \widehat{\mathfrak{g}} \otimes K(\Sigma)$, algebraic SW curve

$$\det(L(\mu) - \lambda) = 0 \tag{1}$$

  with differentials of two functions $E = \lambda$, $W = \log \mu$, i.e. $\Sigma \subset \mathbb{C} \times \mathbb{C}^{\times}$.
- "Relativization" (4d $\to$ 5d): symmetric situation $E = \log \lambda$, $W = \log \mu$ for $\Sigma \subset \mathbb{C}^{\times} \times \mathbb{C}^{\times}$. Lax operator $g\text{“} = \exp(L)''$ $\in \widehat{G}$: co-extended loop group.

Instead of sine-Gordon

$$H = e^{p} + e^{-p} + e^{\phi} + ze^{-\phi}$$

relativistic $\widehat{SL(2)}$-Toda.

Painlevé III

Painlevé III (D_8): the radial sine-Gordon equation

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \sinh\phi$$

or, for $r = e^t$

$$\frac{d^2\phi}{dt^2} = e^{2t}\sinh\phi$$

The Hamiltonian system

$$H = \frac{p^2}{2} + e^{2t}\cosh\phi$$

with time-dependent Hamiltonian: *deautonomization* of the sine-Gordon (or simplest affine Toda system).

q-Painlevé III

q-difference analog of Painlevé III (D_8): much simpler

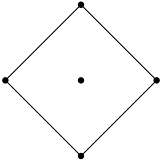
$$x(qz)x(q^{-1}z) = \left(\frac{x(z)+z}{x(z)+1}\right)^2$$

second order *difference* (non-local!) equation.

- In the $q \to 1$ limit (with $q = e^R$ and $R \to 0$) upon $x(z) \mapsto e^{i\phi(t)}$ and the scaling $z \mapsto (Re^t)^4$ turns into differential Painlevé III.
- Has rich discrete symmetry $Dih_4 \ltimes W(A_1^{(1)})$ where ... Dih_4 is dihedral group of square.

Where is this square?

First square



Newton Polygon (up to $SA(2,\mathbb{Z})$-tranform):

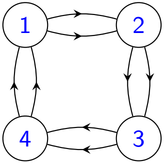
$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + u = 0 \tag{2}$$

spectral curve for relativistic $\widehat{SL(2)}$-Toda chain at $H(\vec{x}) = u$.

Remark: renormalizations of λ, μ and f_Δ fix 3 of coefficients $\{f_{a,b}\}$ in the equation.

Second square

X-cluster Poisson variety with (mutation class of) quiver $\mathcal{Q}$:



encoding logarithmically constant Poisson bracket

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad i, j = 1, \ldots, |\mathcal{Q}| \tag{3}$$

with the skew-symmetric matrix

$$\epsilon_{ij} = -\epsilon_{ji} = \#\text{arrows } (i \to j) = \pm 2 \tag{4}$$

Obviously $q = x_1 x_2 x_3 x_4$ and $z = x_1 x_3$ are in the center of Poisson algebra.

Second square

What about the second square?

- It defines the cluster variety with the MCG (symmetry group of the quiver) $\mathcal{G}_Q = Dih_4 \ltimes W(A_1^{(1)})$.
- The corresponding CLUSTER integrable system is $\widehat{SL(2)}$-Toda chain

$$H = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$$

  its spectral curve is given by the "first square" Newton polygon.
- The symmetry group $\mathcal{G}_Q \supset \mathcal{G}_\Delta$ contains (Abelian!) subgroup of algebraic discrete flows.
- DEAUTONOMIZATION of discrete flows gives rise to q-difference equations of the Painlevé type, for this square: q-Painlevé III.

Motivation and plan

What it has to do with JK-70 conference?

- Non-local (q-difference) equations are here much more *simple* (algebraic!), than continuous;
- Here is almost no difference between *equations* and *symmetries*;
- However, these simple equations do have important (highly-nontrivial!) solutions.

Plan for the rest

- Cluster integrable systems (e.g. on Poisson submanifolds in affine Lie groups);
- Mutations: symmetries and flows;
- Deautonomization and q-Painlevé-like equations;
- Solutions: SUSY gauge theories and topological strings.

Cluster integrable system

a la Goncharov-Kenyon and/or Fock-AM:

- Defined by *any* convex NP $\Delta \subset \mathbb{Z}^2 \subset \mathbb{R}^2$ for a curve $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$

$$f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \tag{5}$$

- Realized on a Poisson X-cluster variety $\mathcal{X}$, $\dim \mathcal{X} = 2\mathrm{Area}(\Delta)$. Poisson structure

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad \{x_i\} \in \left(\mathbb{C}^\times\right)^{2\mathrm{Area}(\Delta)}. \tag{6}$$

  is encoded in a quiver $\mathcal{Q}$, with $\epsilon_{ij} = \#\mathrm{arrows}(i \to j)$.

- Integrability: Pick's formula

$$2\mathrm{Area}(\Delta) - 1 = (B - 3) + 2g \tag{7}$$

Cluster Toda

For our example of $\widehat{SL(2)}$-Toda

- $\mathrm{Area}(\Delta) = 2$, $B = 4$, Pick's formula gives $g = 1$ of the curve

$$f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + u = 0 \qquad (8)$$

- Poisson bracket

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad \epsilon_{ij} = \pm 2 \qquad (9)$$

  in "almost Darboux" variables can be written as

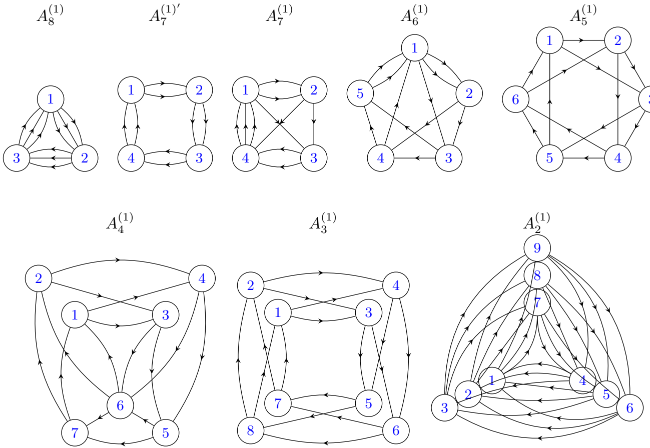
$$\{y_I, y_I\} = \{\mathrm{x}_I, \mathrm{x}_I\} = 0, \qquad \{y_I, \mathrm{x}_I\} = K_{IJ} y_I \mathrm{x}_J$$

  for $I, J = 1, 2$, and K being Cartan matrix of $\widehat{SL(2)}$.

- Easily generalized for $B = 4$ and arbitrary $g = N - 1$ for $\widehat{SL(N)}$ relativistic Toda.

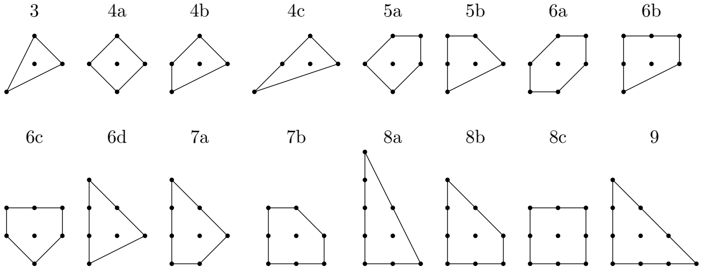
More examples: Quivers

$\mathcal{Q}$ of the "Painlevé cluster varieties" (with their q-Painlevé names), come from



More examples: Newton polygons

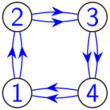
Δ with a single internal point and $3 \leq B \leq 9$ boundary points:



Here Σ: $f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is always a torus $g = 1$.

Mutations

Symmetries are generated by *mutations* on X-cluster variety:



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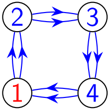

Mutation μ_1

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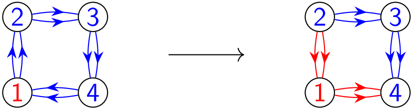

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Reverse all incoming
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$x_1' = 1/x_1$

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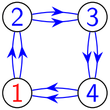

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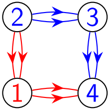

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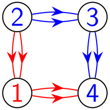

Complete cycles through mutation vertex
$x_4' = x_4(1 + x_1)^2$
$x_2' = x_2(1 + 1/x_1)^{-2}$

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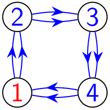

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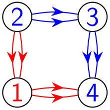

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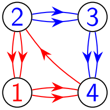

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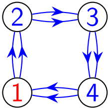

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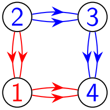

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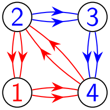

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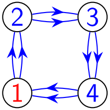

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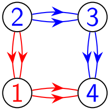

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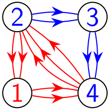

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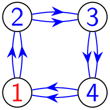

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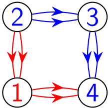

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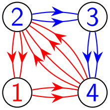

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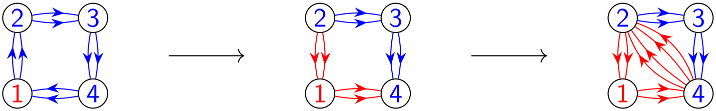

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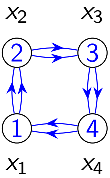
$$\mu_j : \epsilon_{ik} \mapsto -\epsilon_{ik}, \text{ if } i=j \text{ or } k=j, \quad \epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}|+\epsilon_{jk}|\epsilon_{ij}|}{2} \quad \text{otherwise.}$$

$$\mu_j : x_j \mapsto x_j^{-1}, \qquad x_i \mapsto x_i \left(1 + x_j^{\operatorname{sgn}\epsilon_{ij}}\right)^{\epsilon_{ij}}, \quad i \neq j. \qquad \{x_i', x_k'\} = \epsilon_{ik}' x_i' x_k'$$

Cluster automorphisms

All combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver $\mathcal{G}_{\mathcal{Q}} \supset \mathcal{G}_{\Delta}$ (discrete flows of IS).
Example – the flow T:



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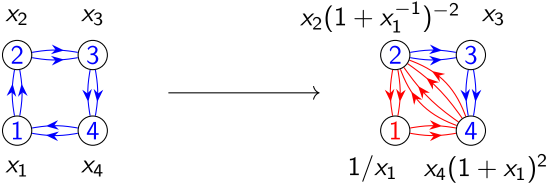

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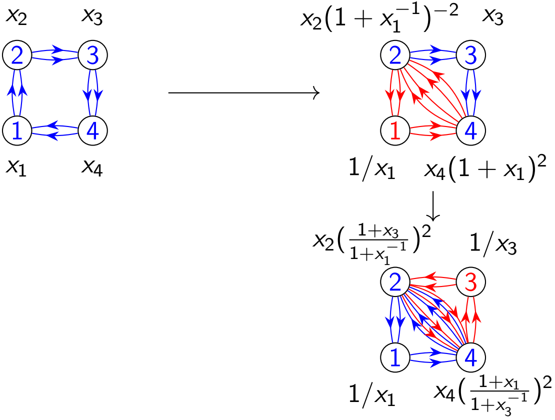

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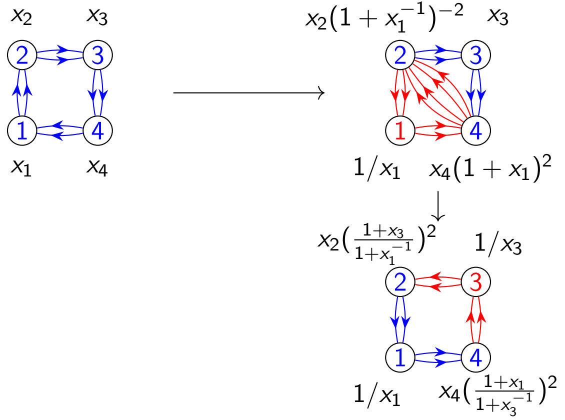

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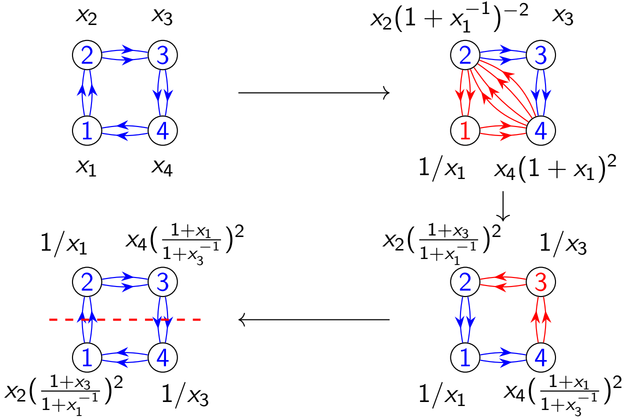


Deautonomization

For $q = 1$ the flow T

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 \left(\frac{1 + x_3}{1 + x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left(\frac{1 + x_1}{1 + x_3^{-1}} \right))^2, x_3^{-1} \right)$$

preserves Hamiltonian $H = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$.

Let $x_1 x_2 x_3 x_4 = q \neq 1$ (*no* integrable system!)

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right)$$

Casimir z as "time" $x_i = x_i(z)$, $T : x_i(z) \mapsto x_i(qz)$, satisfying

$$x_1(qz)x_1(q^{-1}z) = \left(\frac{x_1(z) + z}{x_1(z) + 1} \right)^2$$

or q-Painlevé III_3 equation $P(A_7^{(1)'})$.

Quantization

Remark:

- In addition to non-autonomous parameter q one may add quantum deformation p:

$$\hat{x}_i \hat{x}_j = p^{-2\epsilon_{ij}} \hat{x}_j \hat{x}_i$$

  just *quantizing* the X-cluster variety.
- Quantum mutations

$$\mu_j : \quad \hat{x}_j \mapsto \hat{x}_j^{-1}, \quad \hat{x}_i^{1/|\epsilon_{ij}|} \mapsto \hat{x}_i^{1/|\epsilon_{ij}|} \left(1 + p\hat{x}_j^{\operatorname{sgn} \epsilon_{ij}}\right)^{\operatorname{sgn} \epsilon_{ij}}, \; i \neq j$$

- Quantum q-Painlevé equations, e.g. quantum q-Painlevé III_3:

$$\begin{cases} \hat{x}_1(q^{-1}z)^{1/2}\, \hat{x}_1(qz)^{1/2} = \dfrac{\hat{x}_1(z) + pz}{\hat{x}_1(z) + p}, \\ \hat{x}_1(z)\hat{x}_1(q^{-1}z) = p^4 \hat{x}_1(q^{-1}z)\hat{x}_1(z). \end{cases}$$

- Important since SOLUTION still exists!

Tau-functions

For the tau-functions $x_1(z) = z^{1/2}\frac{\tau_3(z)^2}{\tau_1(z)^2}$ one gets bilinear (non-autonomous!) *Hirota equations*

$$\tau_1(qz)\tau_1(q^{-1}z) = \tau_1(z)^2 + z^{1/2}\tau_3(z)^2$$
$$\tau_3(qz)\tau_3(q^{-1}z) = \tau_3(z)^2 + z^{1/2}\tau_1(z)^2$$

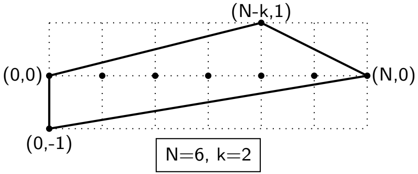
"Generic phenomenon": for the Toda family ($Y^{N,k}$-geometry)

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

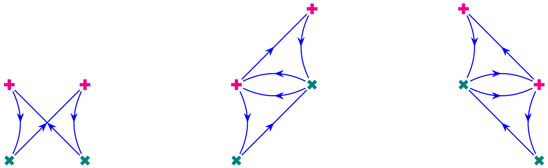
- generated by Toda discrete flows;
- are solved in terms of (dual) Nekrasov functions: "Kiev formulas".

Toda family

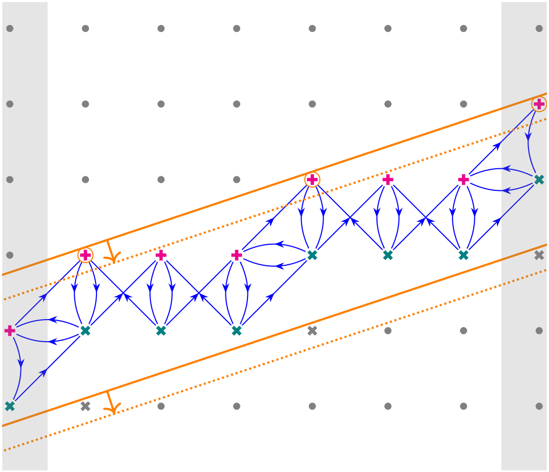
$Y^{N,k}$ polygons with $0 \le k \le N$: $B = 4$ boundary points, hyperelliptic curves $\Sigma_{N,k}$.



Quivers for $Y^{N,k}$ theories can be glued from blocks of three types 0, 1, -1, respectively. $N = N_1 + N_0 + N_{-1}$, $k = N_1 - N_{-1}$.

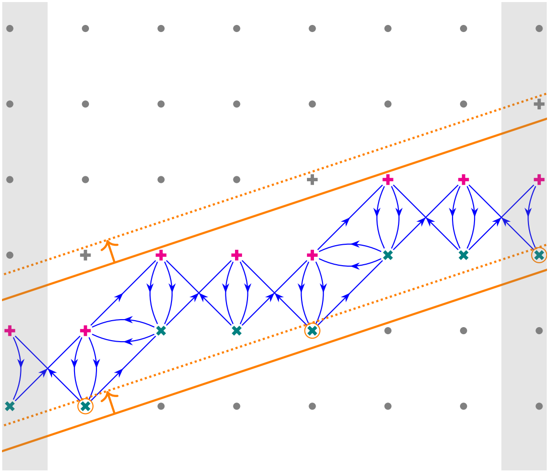


Toda discrete flow



N=6, k=2

Toda discrete flow



N=6, k=2

Solutions: integrable case $q = 1$

The autonomous version of Hirota equations

$$\tau_{n,m+1}\tau_{n,m-1} = \tau_{n,m}^2 + z^{1/N} \cdot \tau_{n+1,m}\tau_{n-1,m}$$

with periodicity (for (N,k)-Toda: $\tau_{n+N,m+k} = \tau_{n,m}$) is solved by

$$\tau_{n,m} = e^{\beta(nk-mN)^2}\Theta(Z + nV + mU)$$

where from the Fay identities

$$\exp(2N^2\beta) = \frac{E(x,y)E(u,v)}{E(x,v)E(u,y)}$$

and

$$2U = \mathcal{A}(x) - \mathcal{A}(y) - \mathcal{A}(u) + \mathcal{A}(v), \quad 2V = \mathcal{A}(x) - \mathcal{A}(y) + \mathcal{A}(u) - \mathcal{A}(v)$$

for the Abel map $\mathcal{A}(P) = \int^P \omega$ on $\Sigma_{N,k} \in Jac(\Sigma_{N,k})$.

Solutions: integrable case $q = 1$

E.g. for $(N, k) = (2, 0)$, $2U = 0$ and

$$\tau_{0,m} = \left(\frac{\theta_3(0)}{\theta_3(U)}\right)^{m^2} \theta_3(Z + mU), \quad \tau_{1,m} = e^{i\pi/4} \left(\frac{\theta_3(0)}{\theta_3(U)}\right)^{m^2} \theta_1(Z + mU)$$

are expressed in terms of Jacobi theta-functions

$$\theta_j(Z) = \sum_{n \in \mathbb{Z}+e_j} e^{2\pi i n Z} \mathrm{q}^{n^2} = \sum_{n \in \mathbb{Z}+e_j} s^n \mathcal{Z}(n)$$

or just *Fourier* transform of $\mathcal{Z}_{\mathrm{cl}}(a) = \mathrm{q}^{a^2}$.

Why this $\mathcal{Z}$ is called "classical"?

Solutions: non-autonomous case $q \neq 1$

Generic equations for the (N, k)-theory

$$\tau_j(qz)\,\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$

where $j \in \mathbb{Z}/N\mathbb{Z}$, are solved $\tau_j(z) = \tau_j^{N,k}(\vec{u}, \vec{s}; q|z)$ by the "Kiev-formula"

$$\tau_j^{N,k}(\vec{u}, \vec{s}; q|z) = \sum_{\vec{\Lambda} \in Q_{N-1}+\omega_j} s^{\Lambda}\mathcal{Z}_{N,k}(\vec{u}q^{\vec{\Lambda}}; q^{-1}, q|z) \tag{10}$$

where sum is over A_{N-1} root lattice, $\{\omega_j\}$ are fundamental weights, and $\mathcal{Z}_{N,k} = \mathcal{Z}_{\mathrm{cl}}^{N,k} \cdot \mathcal{Z}_{1-\mathrm{loop}}^{N} \cdot \mathcal{Z}_{\mathrm{inst}}^{N,k}$ are 5d Nekrasov functions.

This is again just a Fourier transform, e.g. in the simplest case:

$$\tau_j^{2,0}(u, s; q|z) = \sum_{n \in \mathbb{Z}+j/2} s^n \mathcal{Z}_{2,0}(uq^n; q^{-1}, q|z), \qquad j \in \mathbb{Z}/2\mathbb{Z}$$

Nekrasov functions

BUT:

$$\mathcal{Z}_{\rm cl}^{N,k} = \exp\left(\log z \frac{\sum(\log u_i)^2}{-2\log q_1 \log q_2} + k\frac{\sum(\log u_i)^3}{-6\log q_1\log q_2}\right)$$

$$\mathcal{Z}_{1-\rm loop}^{N} = \prod_{1\le i\ne j\le N}(u_i/u_j;q_1,q_2)_\infty$$

$$\mathcal{Z}_{\rm inst}^{N,k} = \sum_{\vec\lambda}\frac{z^{|\vec\lambda|}\prod_{i=1}^N \mathsf{T}_{\lambda^{(i)}}(u;q_1,q_2)^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j;q_1,q_2)}$$

with

$$\mathsf{N}_{\lambda,\mu}(u,q_1,q_2) = \prod_{s\in\lambda}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\prod_{s\in\mu}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1})$$

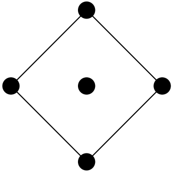
$$\mathsf{T}_\lambda(u;q_1,q_2) = u^{|\lambda|}q_1^{\frac12(\|\lambda^t\|-|\lambda^t|)}q_2^{\frac12(\|\lambda\|-|\lambda|)} = \prod_{(i,j)\in\lambda} uq_1^{i-1}q_2^{j-1},$$

and $\vec\lambda = (\lambda^{(1)},\ldots,\lambda^{(N)})$, $|\vec\lambda| = \sum|\lambda^{(i)}|$, $|\lambda| = \sum\lambda_j$, $\|\lambda\| = \sum\lambda_j^2$.

Summary

- q-difference equation (arising from cluster integrable systems) are more transparent, than differential ones;
- the corresponding tau-function (not of *integrable* systems!) satisfy simple Hirota equations, and ... do have solutions;
- there is q-isomonodromic system, following from Poisson structure on co-extended loop groups.

Directions of the generalization



4 boundary points, internal points on one line →

Non-autonomous discrete Hirota equations

↓ One internal point

? Generic NP, ?? symmetries $W(A_N^{(1)}) \subset \mathcal{G}_Q$

q-difference Painlevé equations

Thank you your attention!

Giuseppe, salute!