A Manifold view point of Lie Algebras

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Lie algebra

The definition of Lie algebra from the book of Claude Chevalley is the following: (Theory of Lie Groups, page 103)

Let K be a field, and let g be a vector space of finite dimension over K. Suppose moreover that there is given a law of composition $(X, Y) \longrightarrow [X, Y]$ in g with the following properties:

1. It is bilinear, i.e.,

$$[a_1 X_1 + a_2 X_2, Y] = a_1 [X_1, Y] + a_2 [X_2, Y]$$
$$[X, a_1 Y_1 + a_2 Y_2] = a_1 [X, Y_1] + a_2 [X, Y_2]$$
$$a_1, a_2 \in K; \ X, Y, X_1, Y_1, X_2, Y_2 \in g$$

2. It satisfies the following conditions:

$$[X, X] = 0 \quad , \quad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for any $X, Y, Z \in g$. Then g, equipped with this law of composition, is called a Lie algebra over K

In what I am going to say I shall assume $K \equiv \mathbb{R}$
(I want to work on real manifolds)

Vector space

$V \qquad V^* = \mathrm{Lin}(V, \mathbb{R}) \qquad (V^*)^* = \mathrm{Lin}(V^*, \mathbb{R}) \rightleftarrows V$ (reflexive)

Tensors

$$T^r_s(V) = L^{r+s}(\underbrace{V^*, \dots, V^*}_{1 \quad\; r}; \underbrace{V, \dots, V}_{1 \quad\; s}; \mathbb{R})$$ contravariant order r, covariant order s

$\tau_1 \otimes \tau_2 \in T^{r_1+r_2}_{s_1+s_2}(V)$; $\tau_1 \in T^{r_1}_{s_1}(V) \quad \tau_2 \in T^{r_2}_{s_2}(V)$

coordinates

We consider an ordered basis of V $\hat{e} \equiv (e_1, \dots, e_n)$ and a dual basis for V^*, $\hat{e}^* = (\alpha^1, \dots, \alpha^n)$ $\quad \alpha^j(e_k) = \delta^j_k$

A coordinate system on V

$$x^j = \alpha^j(u) = x^j(u)$$

coordinate system on V^*

$$x_j = e_j(\alpha) = x_j(\alpha)$$

Remark - $T^1_1(V) \simeq L(V, V)$

Law of composition

- binary, bilinear

$$\beta \in T^2_1(V)$$

$$\beta = C^m_{jk}\, \alpha^j \otimes \alpha^k \otimes e_m$$

$\beta(u,u) = 0 \Rightarrow$ skew-symmetry

$\forall\ u \in V$ $\quad \beta(u) \in T^1_1(V)$

$\beta(u)$ is a derivation

$$\beta(u)\big(\beta(v,w)\big) = \beta\big(\beta(u)\cdot v, w\big) + \beta\big(v, \beta(u)\cdot w\big)$$

Jacobi identity

Manifold view point

$$V \qquad TV \rightleftarrows V \times V \qquad T^*V \rightleftarrows V \times V^*$$

$$V^* \qquad TV^* \rightleftarrows V^* \times V^* \qquad T^*V^* \rightleftarrows V^* \times V$$

Distinguished section

$$\Delta : V \to TV \qquad u \mapsto (u, u)$$

coordinates

$$\Delta \equiv x^j \frac{\partial}{\partial x^j}$$

$$\beta \equiv c^m_{jk}\, dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^m}$$

on V^*

$$\Delta \equiv x_j \frac{\partial}{\partial x_j}$$

$$\beta \equiv c^m_{jk}\, dx_m \otimes \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_k}$$

contracting with Δ and using skew-symmetry

$$\beta \equiv c^m_{jk}\, x_m \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

Linear structure on a contractible manifold

M contractible finite dimensional manifold, $\dim M = n$

Δ a vector field

i) complete

ii) $\exists!\ m_0 \vdash \Delta(m_0) = 0$

iii) $L_\Delta f = 0 \Rightarrow f$ constant

iv) $L_\Delta h = h$ admits n functionally independed solutions

M may be given a vector space structure

use $h^1, h^2, \ldots, h^n$ as coordinate functions
linearizing coordinates

notice that $\Delta(m_0) = 0 \Rightarrow h^j(m_0) = 0 \quad \forall j \in \{1, 2, \ldots, n\}$

Manifold point of view :

From vector spaces $\rightarrow$ modules

tensors $\rightarrow$ tensor fields

The manifold view point allows to perform any "change of coordinates" on the law of composition

The vector field Δ selects linear transformations out of diffeomorphisms

$$\Phi_*(\Delta) = \Delta$$

Linear vector fields $[X, \Delta] = 0$

Prop: $X = A^j_k x^k \frac{\partial}{\partial x^j}$

Translations $[X_a, \Delta] = X_a$

Prop: $X_a = a^j \frac{\partial}{\partial x^j}$

Lie algebras, tensorial version

$$\Lambda = c_{jk}^{l}\, x_l \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} = [x_j, x_k] \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

The linear map $V \to \mathfrak{X}(V^*)$

$u \mapsto \Lambda(d\hat{u}) = X_u \qquad \hat{u} \in Lin(V^*, \mathbb{R})$

is a homomorphism of Lie algebras iff the Jacobi identity is satisfied

equivalently : iff $L_{X_u}\Lambda = 0 \quad \forall u \in g$

Example

$M \equiv \{ x \equiv (x_1, x_2, x_3) \in \mathbb{R}^3 \}$ Consider $[x_j, x_k] = \varepsilon_{jkl}\, x_l$

$$\Lambda = \frac{1}{2} \varepsilon_{jkl} \frac{\partial}{\partial x_l}(x_1^2 + x_2^2 + x_3^2) \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} \qquad j,k,l = 1,2,3$$

in spherical coordinates (r, θ, φ), $x_1 = r \sin\theta \cos\varphi$, $x_2 = r \sin\theta \sin\varphi$, $x_3 = r\cos\theta$

$$\Lambda = \frac{1}{r \sin\theta}\left(\frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial\varphi} \right)$$

Additional examples

2) Consider $\Lambda = \omega^{jk} \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}$ $\quad \omega^{jk} \in \mathbb{R}$

the coordinate transformation $\xi_j = e^{x_j}$ gives a quadratic bracket

$$\Lambda = \sum_{j,k} \omega^{jk} \xi_j \xi_k \frac{\partial}{\partial \xi_j} \wedge \frac{\partial}{\partial \xi_k}$$

3) Consider $\Lambda = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$, the transformation $q = \ln x$, $p = p$

gives $\Lambda = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial p}$

Change of coordinates to "linearize" PB has been extensively studied. In Physics it is relevant when studying gauge theories with first class constraints

Symplectic realization of Poisson manifolds (Schwinger's problem in Q.M.)

Classification problem

Because of previous examples, the classification probelem or normal forms for $\Lambda = [x_j, x_k] \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$ should specify that we consider only linear transformations.

We shall first give the example of 3-dim Lie algebras.

On $\mathbb{R}^3$ we consider the volume form by means of a contravariant tensor

$$\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$$

a bivector field is in one-to-one correspondence with a one-form α

$$\Lambda_\alpha = \alpha\left(\frac{\partial}{\partial x_1}\right) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \alpha\left(\frac{\partial}{\partial x_3}\right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \alpha\left(\frac{\partial}{\partial x_2}\right) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1}$$

clearly all vector fields X_u associated with linear functions $\hat{u}$ are in the kernel of α.

From $L_{X_u} \Lambda_\alpha = 0$ we derive $L_{X_u} \alpha = 0$, therefore $i_{X_u} d\alpha = 0$ -

Thus the Jacobi identity is satisfied iff $\alpha \wedge d\alpha = 0$, i.e., the characteristic distribution is involutive!

3-dimensional Lie algebras

A generic one-form may be written as

$$\alpha = (a_1 x + b_1 y + c_1 z)\,dz + (a_2 x + b_2 y + c_2 z)\,dx + (a_3 x + b_3 y + c_3 z)\,dy$$

we may distinguish

i) $d\alpha = 0$, $\alpha = dF$

ii) $d\alpha \neq 0$

In the first case, by using invertible linear transformations we reduce the quadratic function F to its normal form

$$F = ax^2 + by^2 + cz^2 = (X, Y, Z)\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}\begin{vmatrix} X \\ Y \\ Z \end{vmatrix}$$

We have first

rank one $[Y, Z] = aX$, $[Y, X] = 0$, $[Z, X] = 0$ H. W.

rank two : $ab > 0$, $ab < 0$, $[X, Y] = 0$, $[Y, Z] = aX$, $[Z, X] = bY$ $SO(2) \times \mathbb{R}^2$ $SO(1,1) \times \mathbb{R}^2$

rank three

$ab > 0$, $bc > 0$, $ac > 0$ $SO(3)$

$abc < 0$ $SO(2,1) \approx SL(2, \mathbb{R})$

unimodular Lie algebras

3 dim Lie algebras - continued -

We consider now the case $d\alpha \neq 0$

From $\alpha \wedge d\alpha = 0$, if we reduce $d\alpha$ to a normal form, say $d\alpha = m\, dx\, dy$ $\quad \alpha = m x\, dy + d\underbrace{(p x^2 + n y^2 + q y x)}_{F}$

$$F = (x, y) \begin{pmatrix} p & q/2 \\ q/2 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The term $m x\, dy$ gives $\Lambda = m x \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}$
the Lie algebra of the affine group

The exact part gives combinations of the H.W. algebra because the quadratic function may be reduced to

$$F = r x^2 \pm s y^2$$

We conclude that:

1) Poisson Brackets associated with monomials x^2, y^2, z^2 are pairwise compatible and each one gives a H.W. algebra

2) monomials x^2, y^2 give algebras compatible with the one defined by $x\, dy$ (affine Lie algebra)

In conclusion: All three dimensional Lie algebras are combinations of H.W. algebras and affine algebras

Vinogradov approach

Remark: An application

By using a different parametrization of α, say

$$\alpha = \frac{1}{2} d(N_1 x_1^2 + N_2 x_2^2 + N_3 x_3^2) + \frac{a}{2}(x_2 dx_3 - x_3 dx_2)$$

$$\alpha \wedge d\alpha = 0 \iff a N_1 = 0$$

These quadratic one-forms give rise to the classification proposed by Bianchi

Bianchi A, α exact, i.e., $a = 0$

Bianchi B $d\alpha \neq 0$

This classification gives rise to Bianchi universes which give an approach to find exact integration of equations of motion in the Friedmann-Robertson-Walker approach to Einstein equations

Nother symmetries in Bianchi universes
S. Capozziello, G.M., C. Rubano, P. Scudellaro
Int. Jour. Mod Phys D 6(4) (1997) 491-503

From Poisson tensors to differential forms

We consider a volume form on the n-dimensional orientable manifold M

If $\Lambda = X_1 \wedge X_2 \wedge \dots \wedge X_k$ we consider $\Psi_\Lambda = i_\Lambda \Omega$

This is an isomorphism between k-vector fields and $(n-k)$ forms

$$(i_\Lambda \Omega)(Y_1, Y_2, \dots, Y_{n-k}) = \Omega(X_1, X_2, \dots, X_k, Y_1, \dots, Y_{n-k})$$

For a bivector field $\Lambda = c^{jk}\partial_j \wedge \partial_k$, $\Omega = dx^1 \wedge \dots \wedge dx^n$

$$\Psi_\Lambda = 2\sum_{j<k} (-1)^{j+k} c^{jk} dx^1 \dots \overset{\vee}{j} \dots \overset{\vee}{k} \dots \wedge dx^n$$

$\overset{\vee}{j}$ stands for omission

For vector fields $i_{[X,Y]} = i_X i_Y d - d\, i_{X\wedge Y} + i_X d\, i_Y - i_Y d\, i_X$

and similarly for bivector fields

$$i_{[\Lambda_1, \Lambda_2]} = -i_{\Lambda_1} i_{\Lambda_2} d - d\, i_{\Lambda_2 \wedge \Lambda_1} + i_{\Lambda_1} d\, i_{\Lambda_2} + i_{\Lambda_2} d\, i_{\Lambda_1}$$

Thm. A bivector field is Poisson iff

$$2\, i_\Lambda d \Psi_\Lambda = d \Psi_{\Lambda\wedge\Lambda}$$

Poisson Λ_1, Λ_2 are compatible iff

$$d\Psi_{\Lambda_1 \wedge \Lambda_2} = i_{\Lambda_1} d \Psi_{\Lambda_2} + i_{\Lambda_2} d \Psi_{\Lambda_1}$$

Poisson → differential forms

Using d, L_X, i_X we can state properties of the Schouten brackets in terms of differential forms

1) $$L_X \Omega = \operatorname{div} X \, \Omega = d\, i_X \Omega$$

2) $$d(i_{X\wedge Y}\Omega) = i_{[Y,X]}\Omega + i_X\, d\, i_Y \Omega - i_Y\, d\, i_X \Omega$$

defining $D = \Psi^{-1} \circ d \circ \Psi$ we get

$$D(X) = \operatorname{div} X$$

$$D(X \wedge Y) = [Y, X] + \operatorname{div}(Y) X - \operatorname{div}(X) Y$$

for vector fields X and Y,

$$D(\Lambda_1 \wedge \Lambda_2) = [\Lambda_1, \Lambda_2] + D(\Lambda_1) \wedge \Lambda_2 - \Lambda_1 D(\Lambda_2)$$

for bivector fields. Thus the Jacobi identity becomes

$$D(\Lambda \wedge \Lambda) = 2 \Lambda \wedge D(\Lambda)$$

A Poisson tensor is closed if $D(\Lambda) = 0$ $(\Rightarrow D(\Lambda \wedge \Lambda) = 0)$

This is equivalent with

Ψ_Λ (and hence $\Psi_{\Lambda\wedge\Lambda}$) is closed

This definition is volume dependent

Starting with a given Ω we may decompose Ψ_Λ into a closed part plus a remaining term.

The remainder is usually associated with a rank 2 Poisson tensor

From $\Psi_\Lambda = i_\Lambda \Omega$ we may define $i_{X_\Lambda}\Omega = d\Psi_\Lambda$

we have $\operatorname{div} X_\Lambda = 0$ $\quad L_{X_\Lambda}\Lambda = 0$

Having constructed $d\Psi$ we look for a 1-form such that

$$\Omega = \theta \wedge d\Psi$$

$$\theta(X_\Lambda) = 1 \qquad d\theta = 0$$

define $\quad X_\theta = \Lambda(\theta)$

The bivector field $X_\theta \wedge X_\Lambda - \Lambda$ is closed

Remark: θ is defined only up to a closed 1-form which is a constant of the motion for X_Λ and such that $\theta \wedge \Psi_\Lambda = 0$

X_θ and X_Λ define an involutive distribution

$L_{X_\theta}\Lambda = 0$

$\Rightarrow X_\theta \wedge X_\Lambda$ Poisson

$L_{X_\theta}\Lambda = 0$ compatibility

Theorem

If Λ is Poisson, there is an X defined on the support of X_Λ such that
$$\Lambda - X \wedge X_\Lambda$$
is a closed Poisson tensor

When the Poisson tensor has leaves of dimension at most two, then $\Lambda \wedge \Lambda = 0$ and we have $i_\Lambda d\psi_\Lambda = 0$.

Viceversa:

Theorem. If $f, f_1, f_2, \dots f_{n-2}$ are smooth functions on an open and dense subset of an n-dimensional manifold M with a given volume form Ω such that $f df_1 df_2 \dots df_{n-2}$ is a smooth $(n-2)$-form on M corresponding to a bivector Λ then Λ is Poisson with orbits of dimension at most two for which $f_1, \dots, f_{n-2}$ are Casimir functions and the bracket is defined by
$$\{g, h\}\,\Omega = f\, dg \wedge dh \wedge df_1 \wedge \dots \wedge df_{n-2}$$

By using previous theorem and noticing that

$x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$ (affine subalgebra, (dyon)

$x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$ H. W. (triadon)

have leaves of dimension two we can use the disassembling techniques proposed by Vinogradov in the papers I mentioned at the beginning

Outlook

Poisson Brackets are essential ingredient to describe evolution in Classical Mechanics

In Quantum Mechanics, on the space of observables, a similar role is provided by Lie-Jordan algebra

We recall that on a vector space V (over $\mathbb{R}$) a Jordan product is a binary, bilinear symmetric product $a \odot b = b \odot a$, not associative, but satisfying the identity

$$(a \odot b) \odot (a \odot a) = a \odot (b \odot (a \odot a)) \qquad \forall\, a, b \in V$$

The Lie bracket $[\ ,\ ] : V \times V \to V$ and the symmetric brackets are compatible in the following sense

$$[a, b \odot c] = [a, b] \odot c + b \odot [a, c]$$

$$a \odot (b \odot c) - (a \odot b) \odot c = \mu^2 \left([a, [b, c]] - [[a, b], c] \right) \qquad a, b, c \in V,\ \mu^2 \in \mathbb{R}$$

The observables of a quantum system provide an example

$$[a, b] = -\frac{i}{2}(ab - ba) \qquad a \odot b = \frac{1}{2}(ab + ba)$$

If we consider expectation-value functions, say on the manifold of quantum (pure)-states we may define

$$\Lambda = \frac{1}{2}[X_j, X_k]\frac{\partial}{\partial X_j} \wedge \frac{\partial}{\partial X_k} \qquad R = (x_j \odot x_k)\frac{\partial}{\partial X_j} \otimes_s \frac{\partial}{\partial X_k}$$

They define a Lie-Jordan algebra on the space of expectation-value functions

How to classify them?

This would be (almost) equivalent to the classification of associative algebra (considered by Carton and Study)

Extension of the classification of Lie algebras to infinite dimensions (Hilbert manifolds)