Conformal geometric aspects of hyperplane sections of Lagrangian Grassmannians

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Politecnico di Torino

ongoing work with Jan Gutt and Giovanni Moreno

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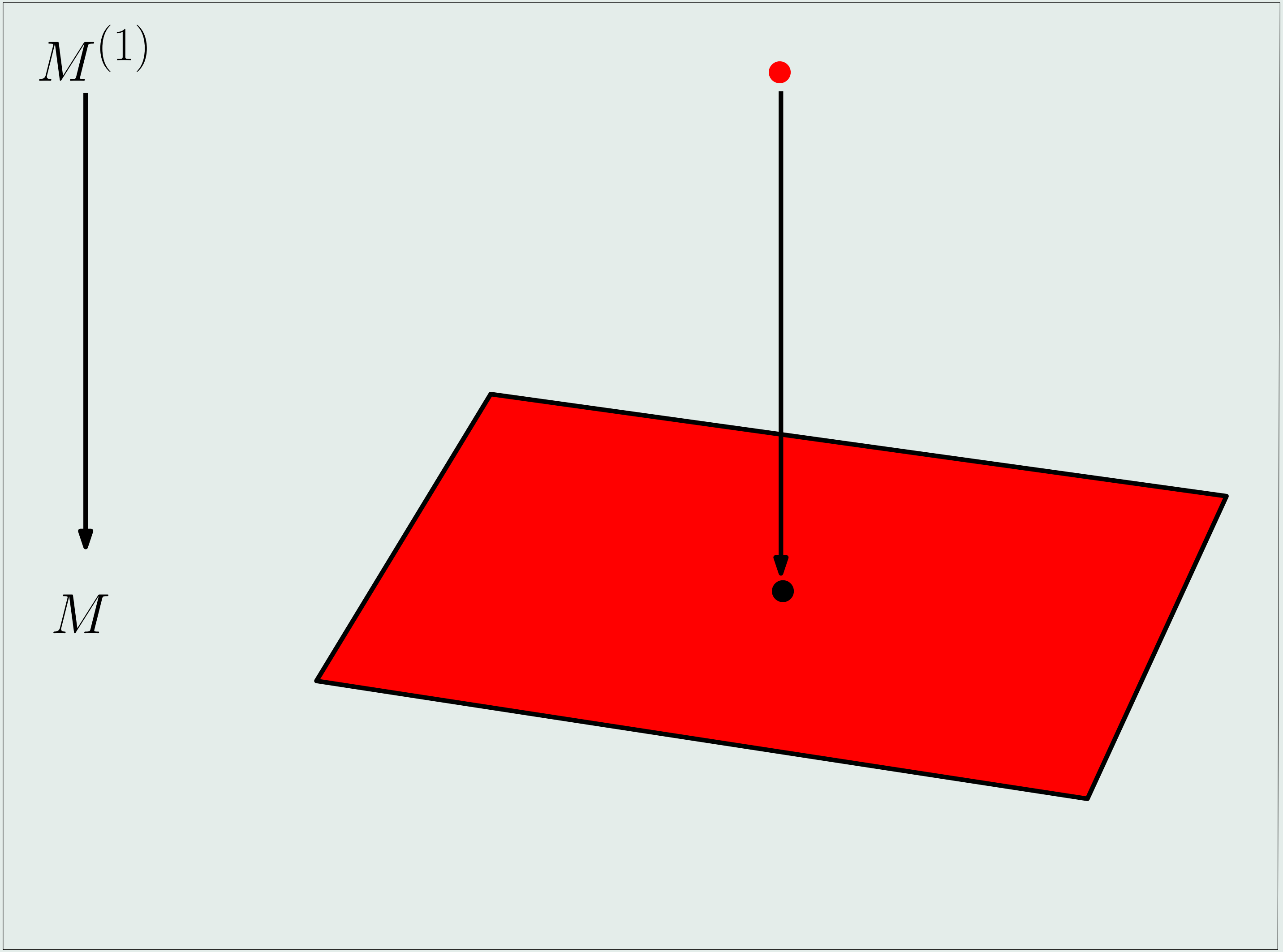
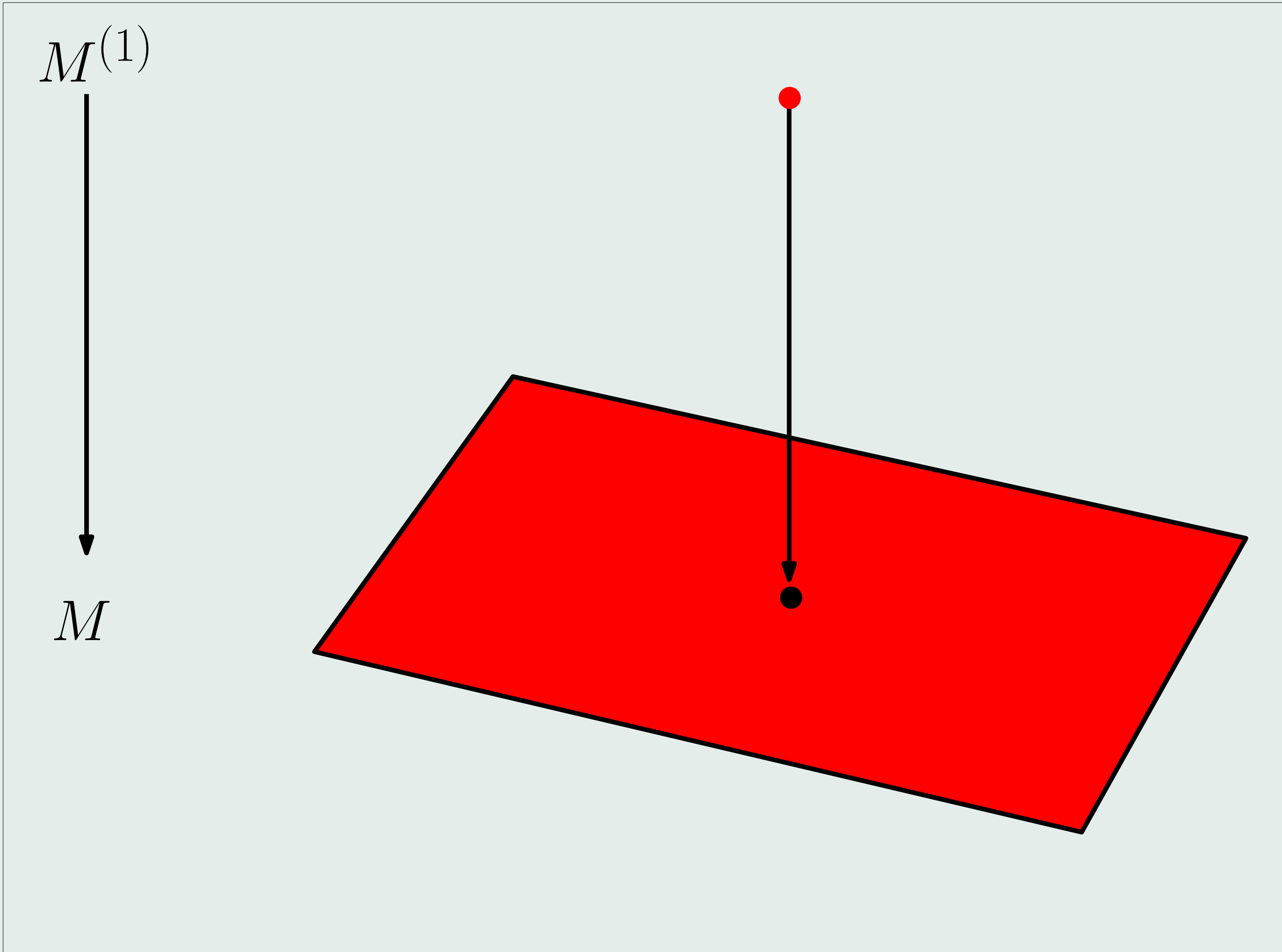


$M^{(1)}$
M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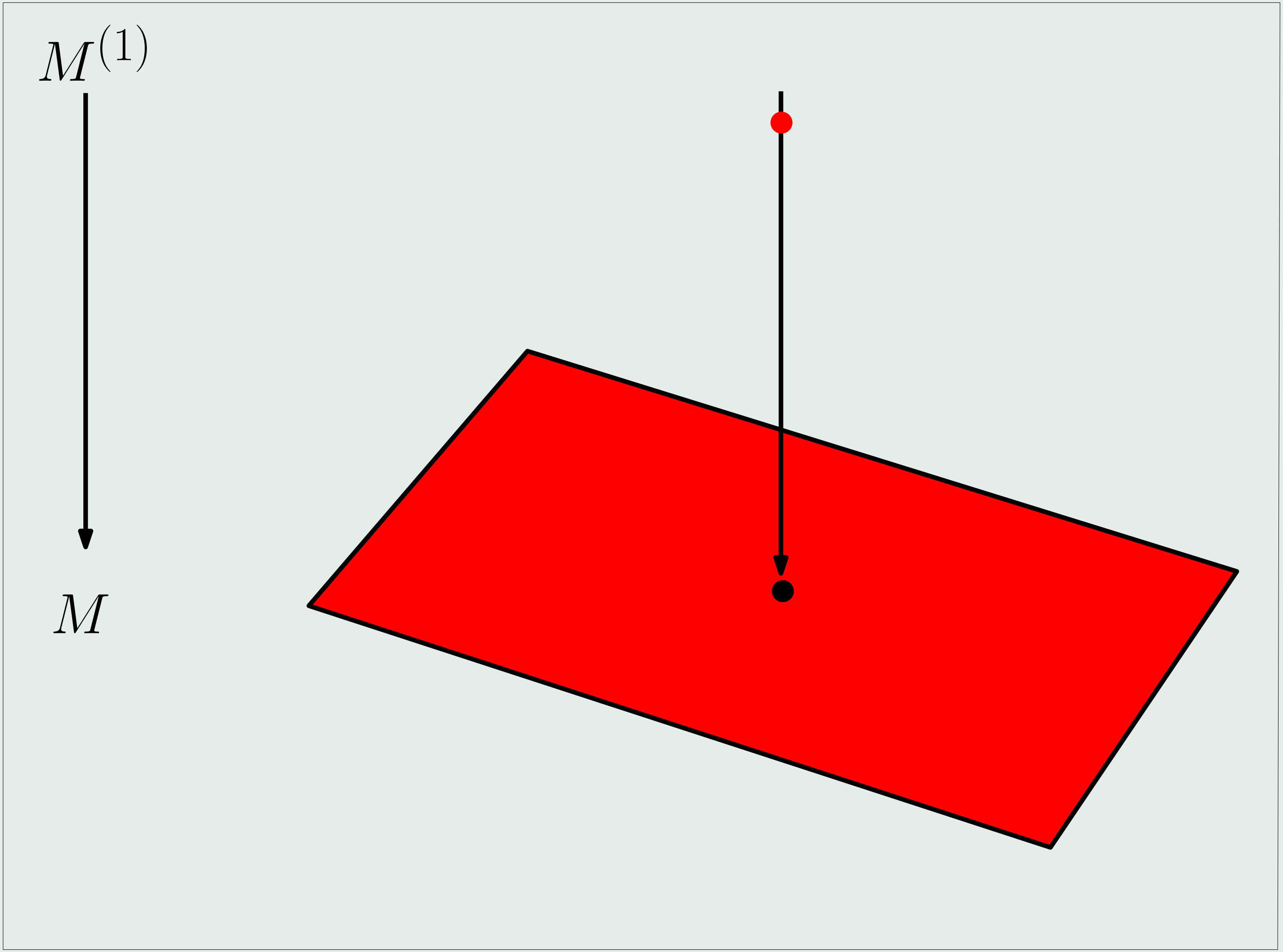
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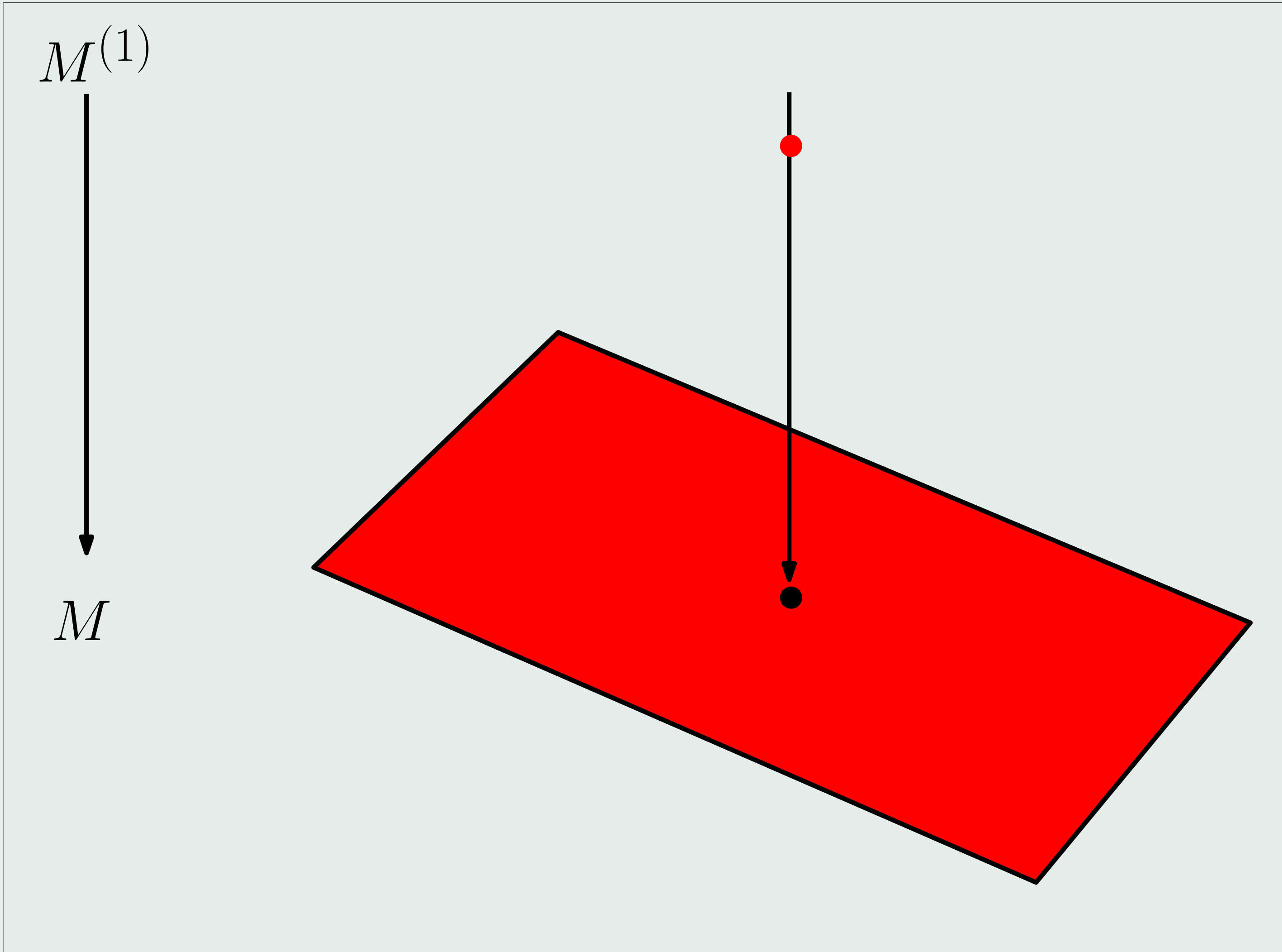

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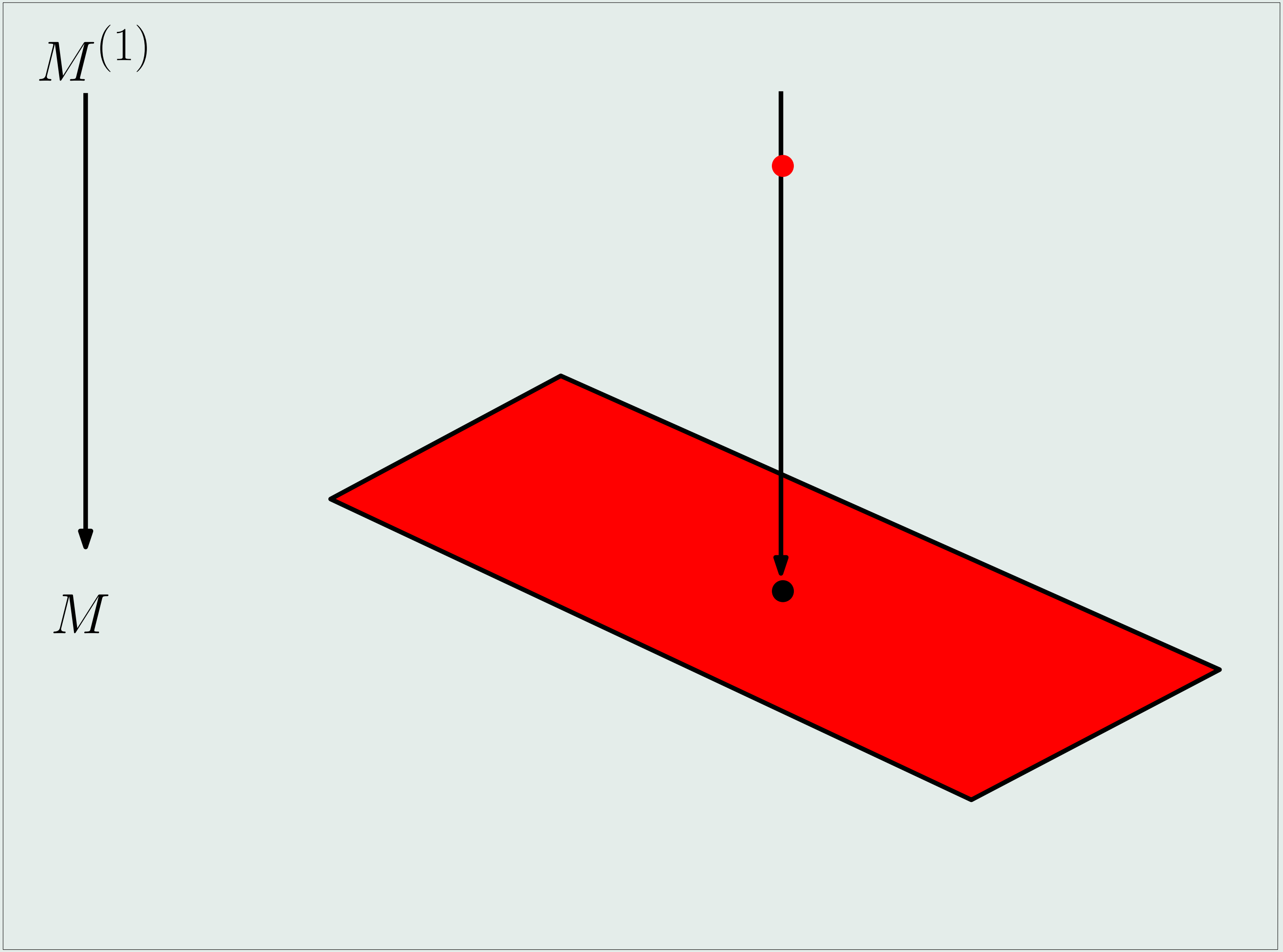

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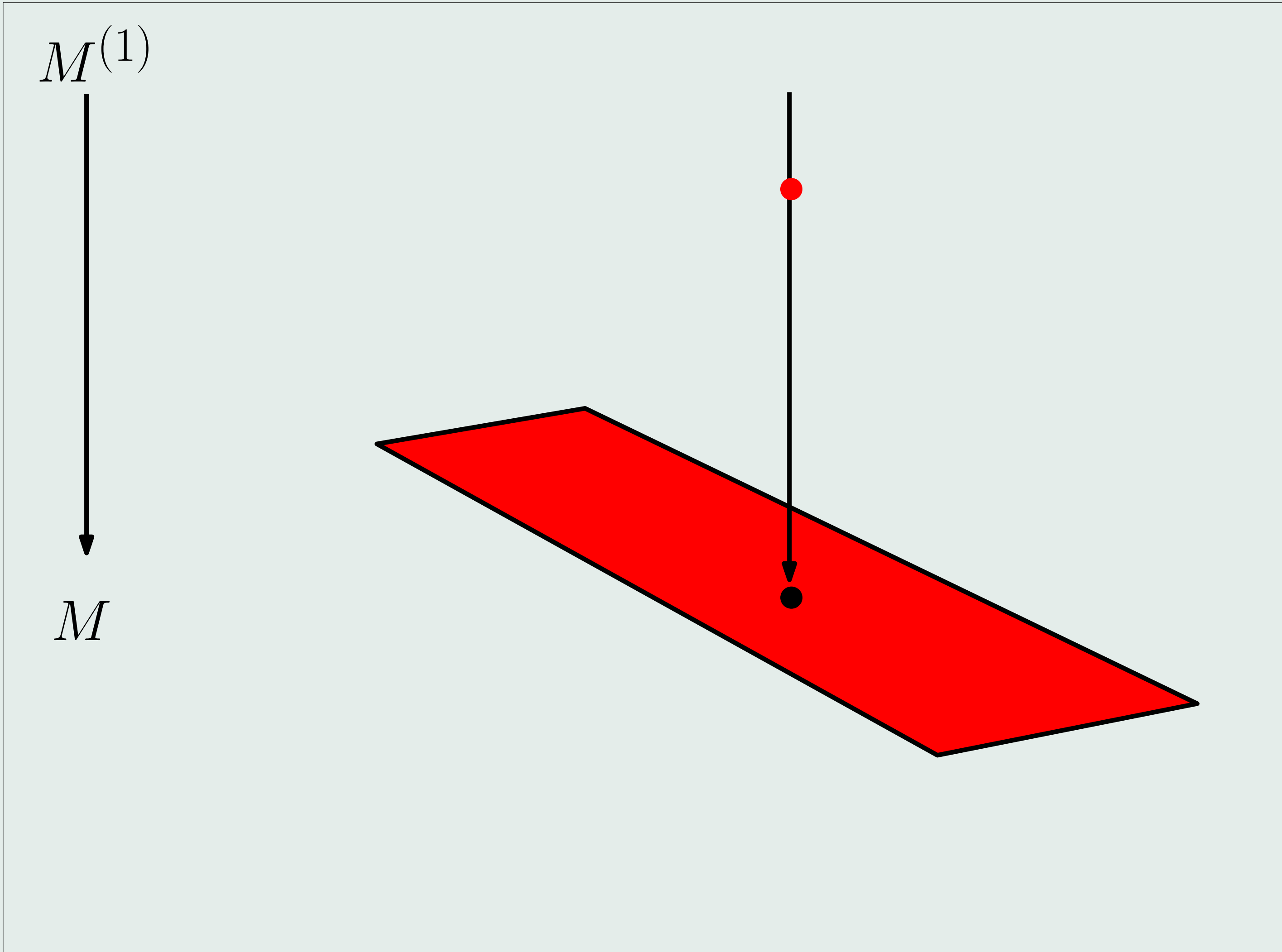
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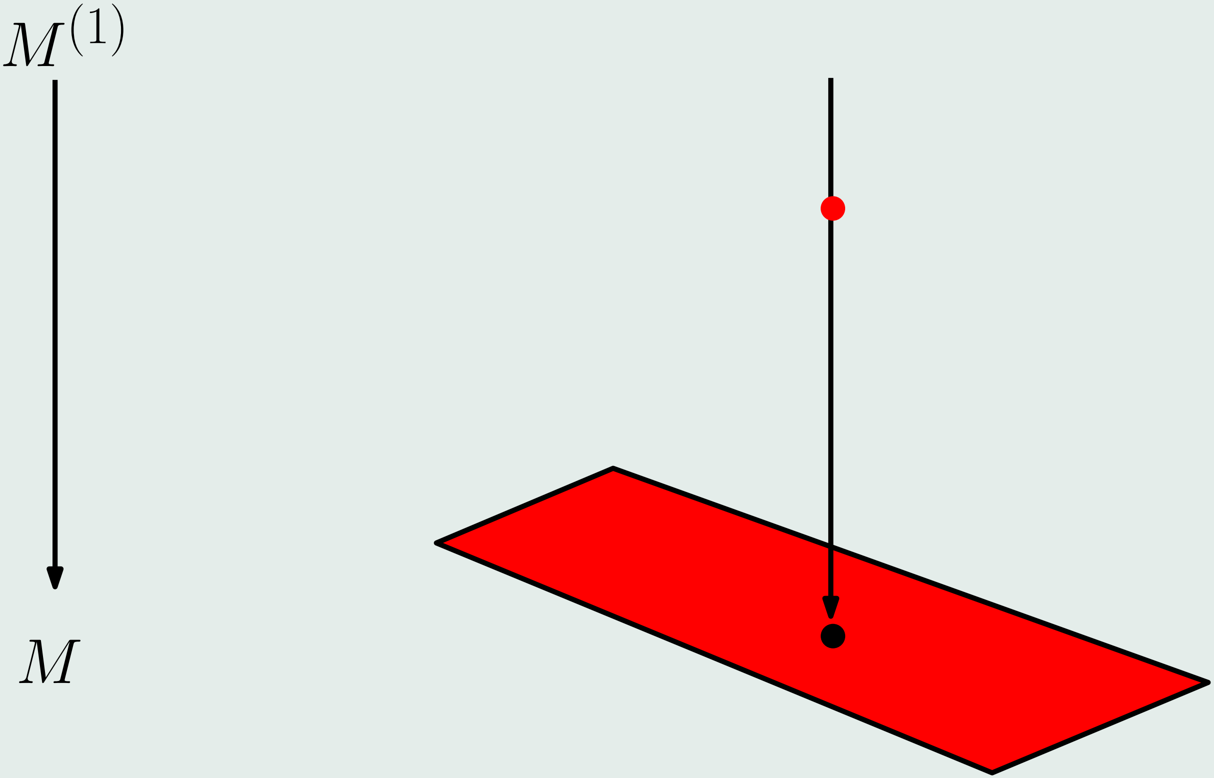

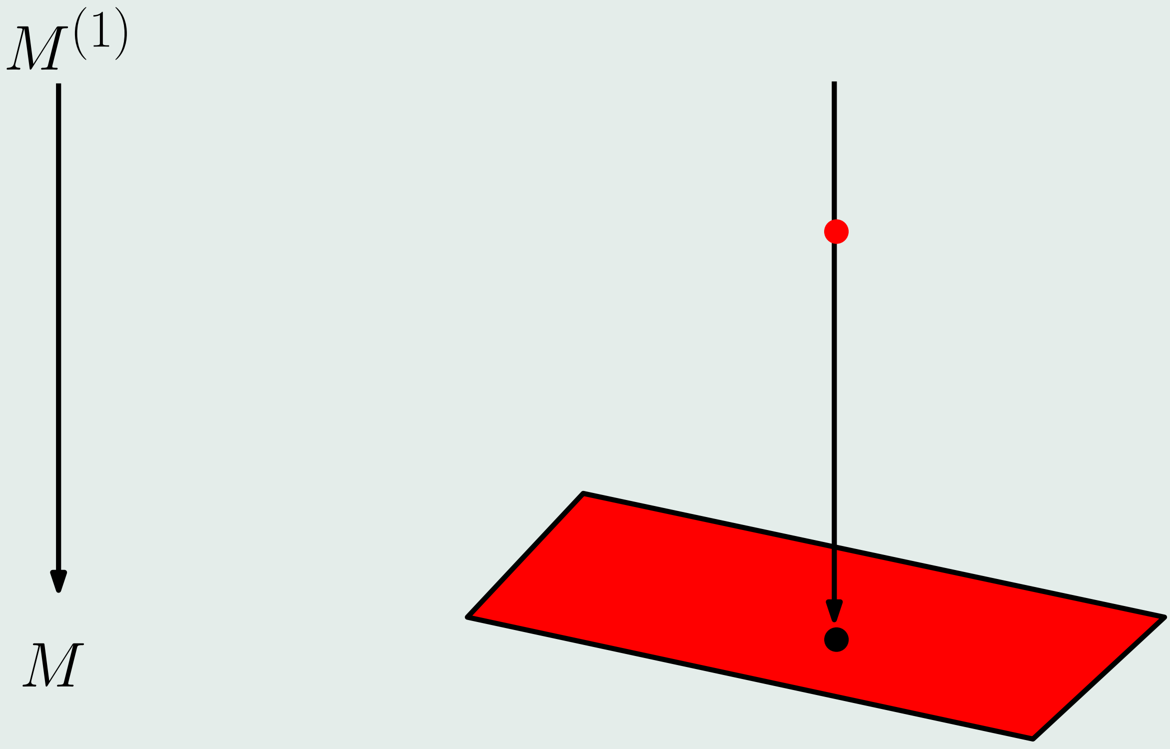
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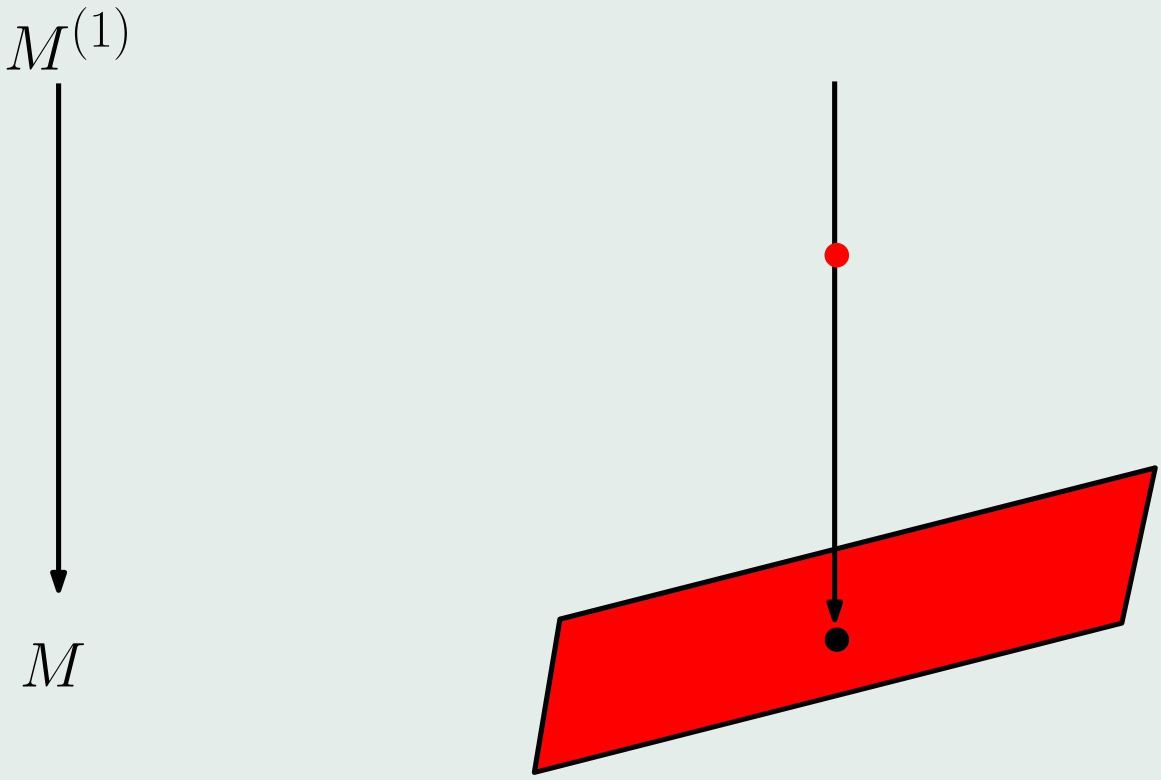
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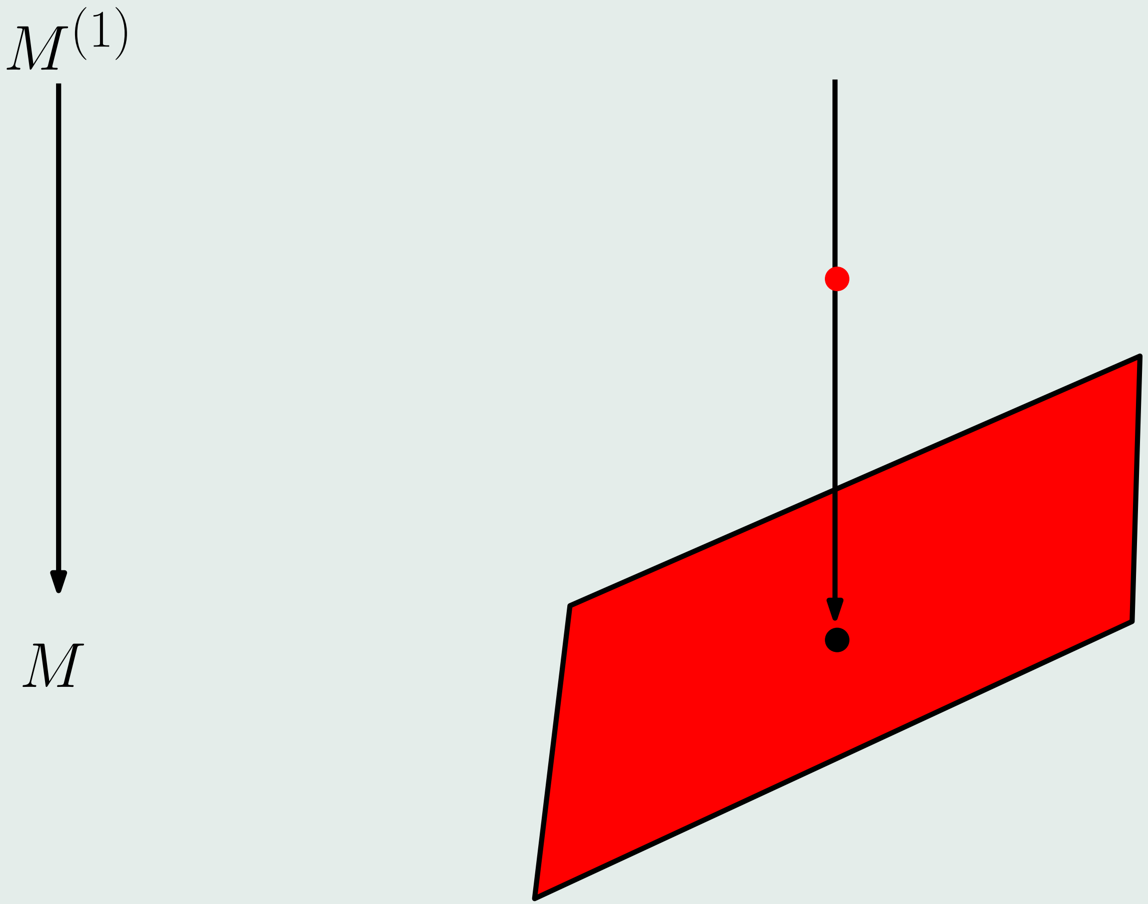

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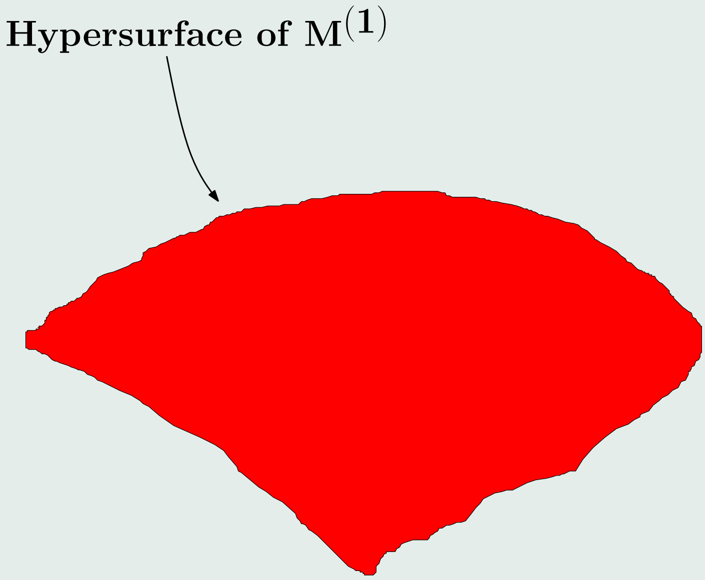
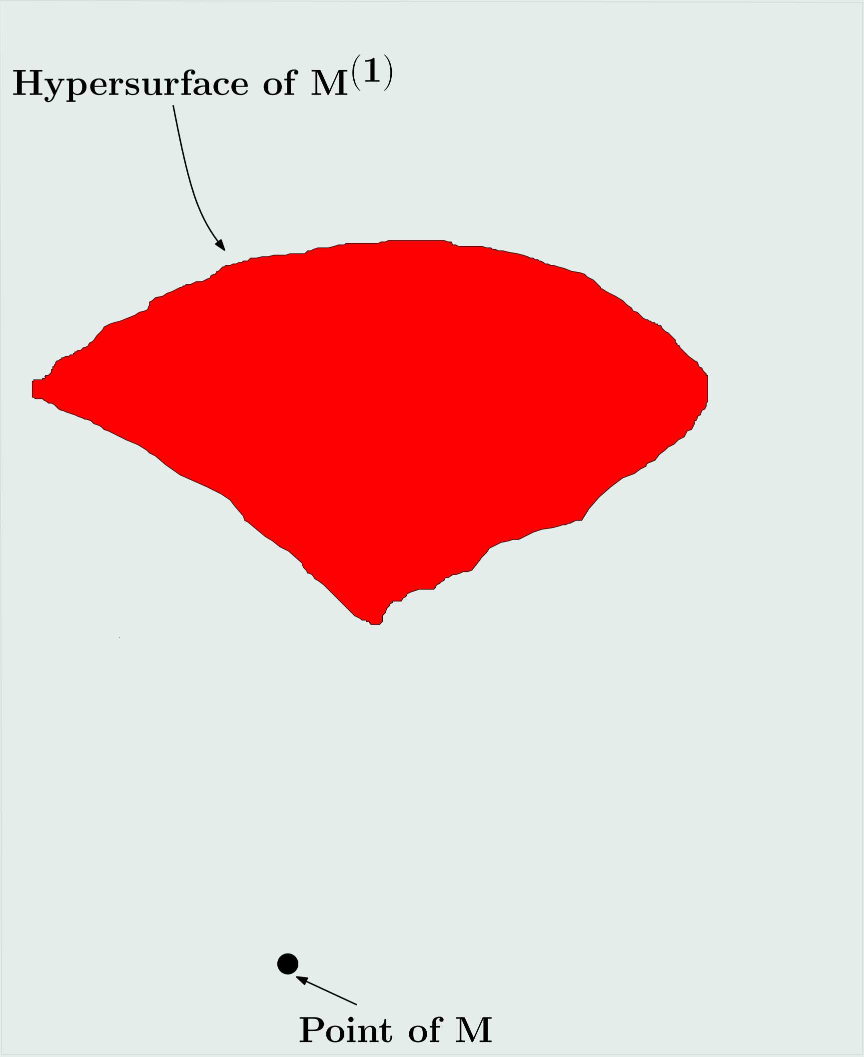
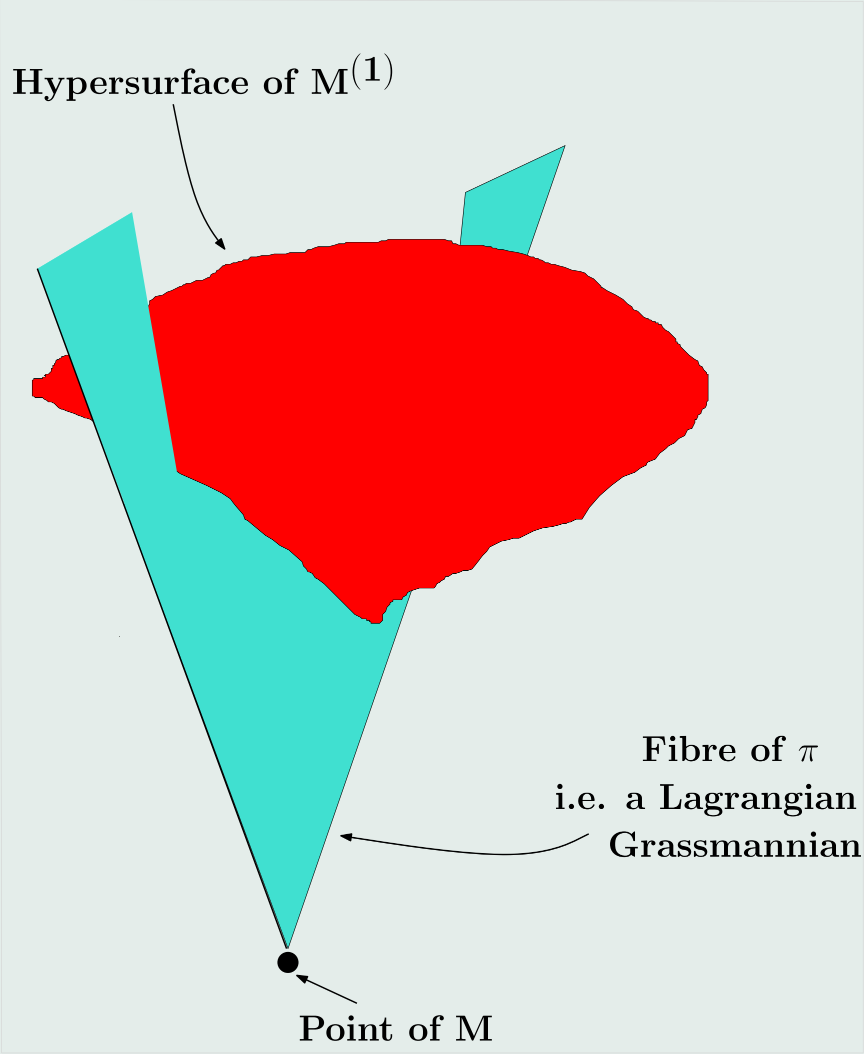
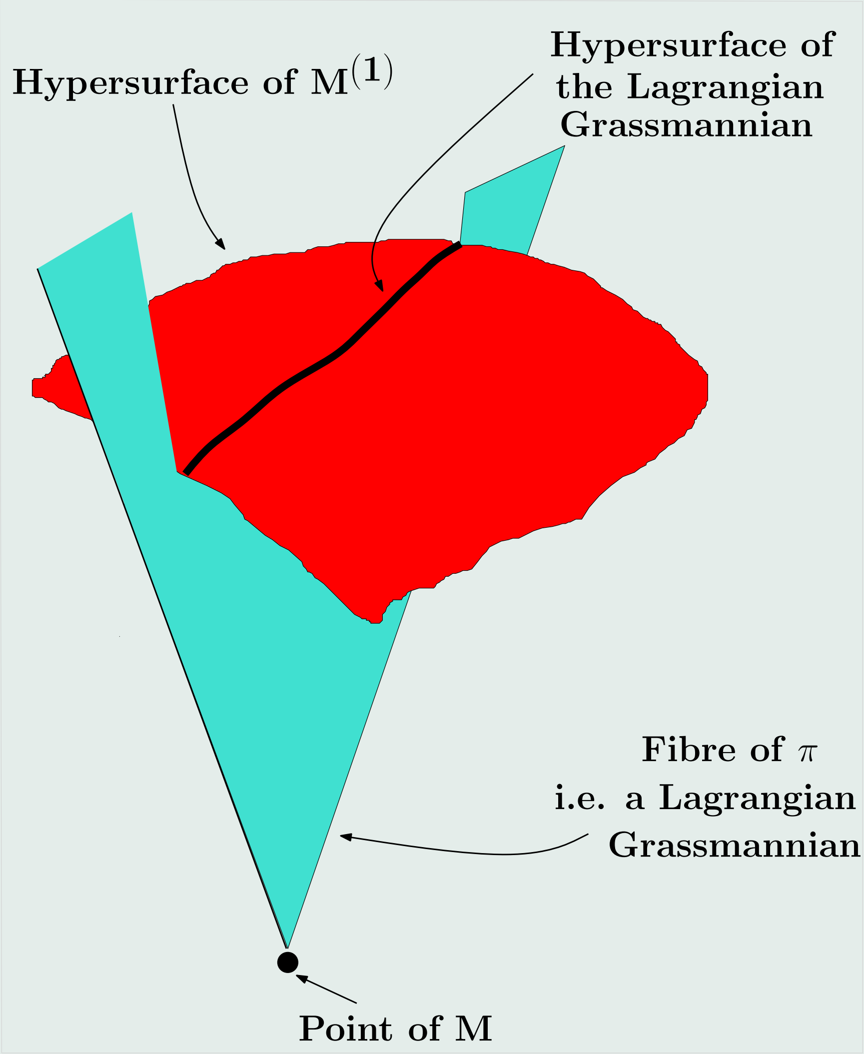
$$f(x, y, u, u_x, u_y) = 0\ , \qquad F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$


Hypersurface of $\mathbf{M}^{(1)}$


Hypersurface of $\mathbf{M}^{(1)}$
Point of M



Hypersurface of $M^{(1)}$
Fibre of π
i.e. a Lagrangian
Grassmannian
Point of M



Hypersurface of $M^{(1)}$
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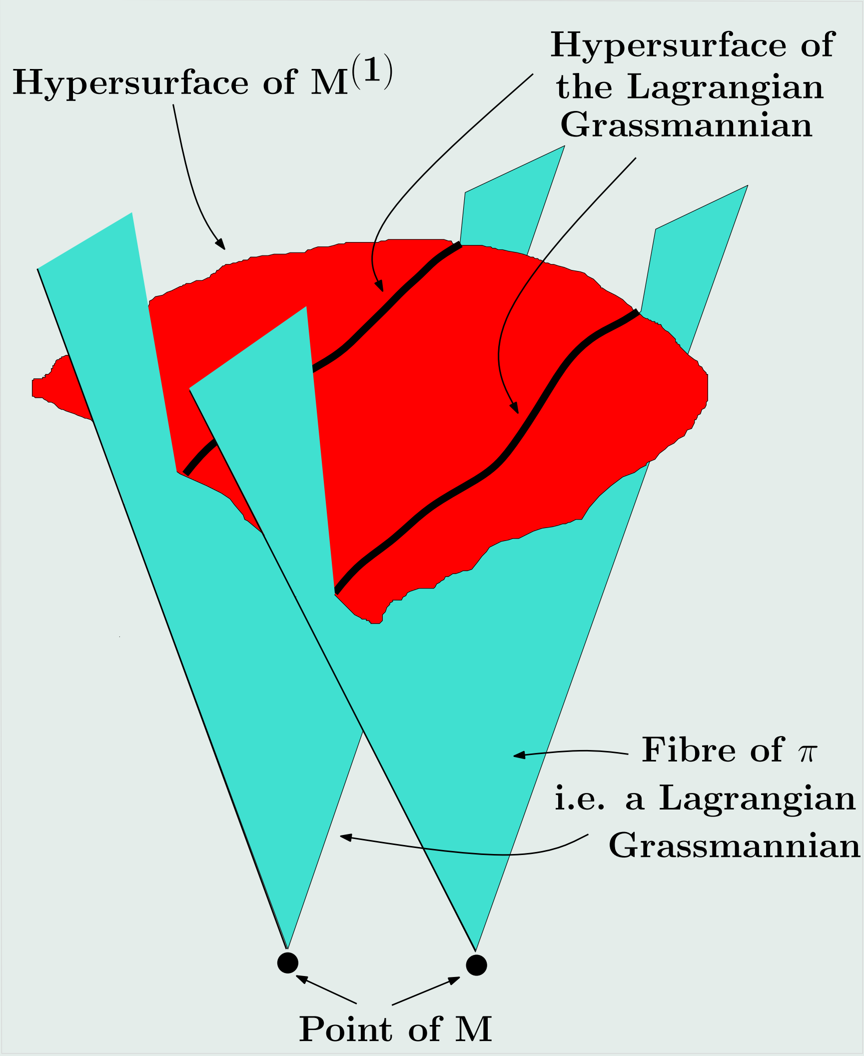
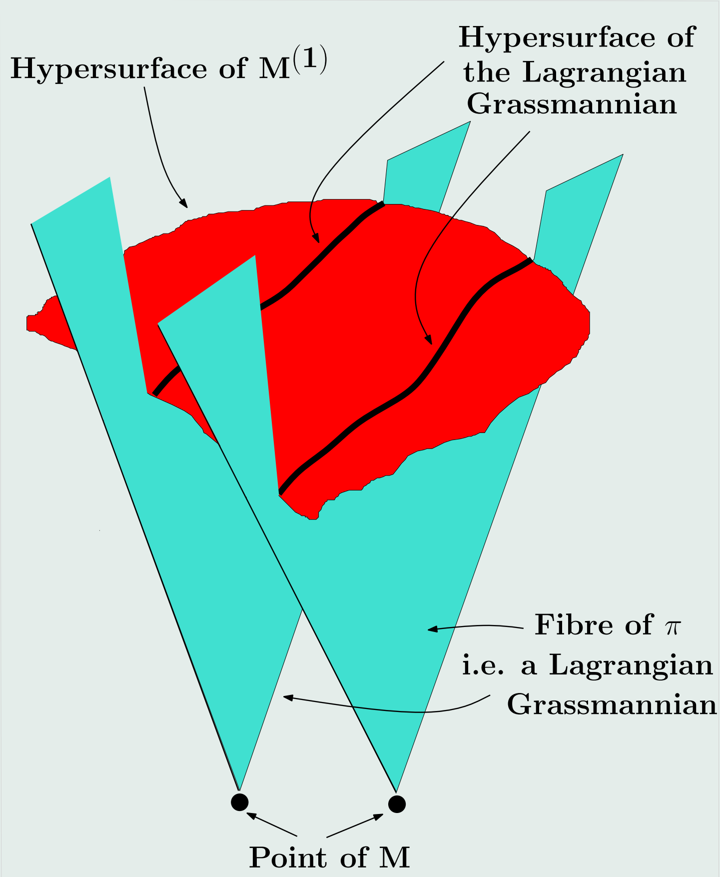

Hypersurface of $M^{(1)}$
Hypersurface of the Lagrangian Grassmannian
Fibre of π i.e. a Lagrangian Grassmannian
Point of M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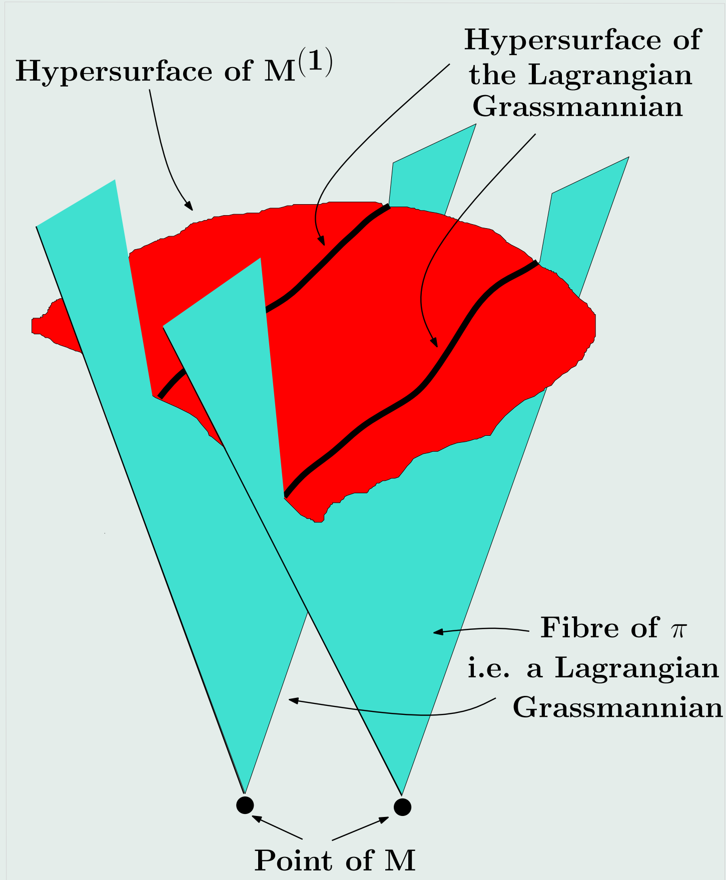
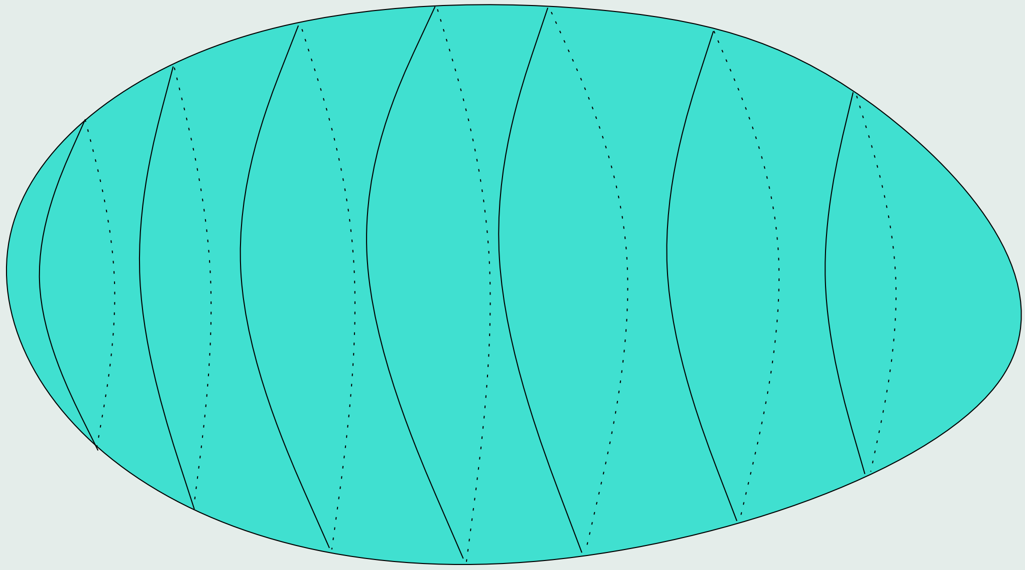
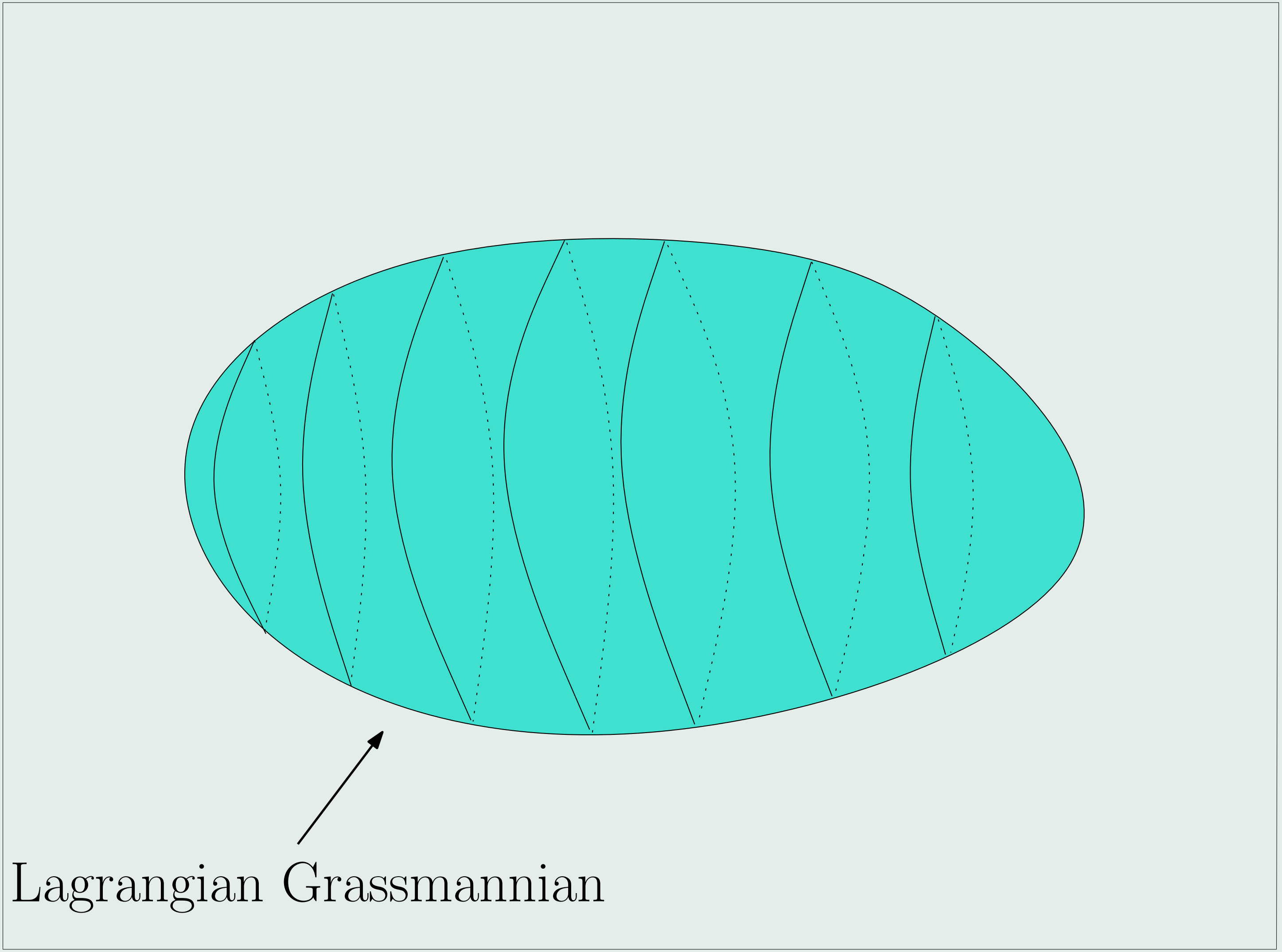
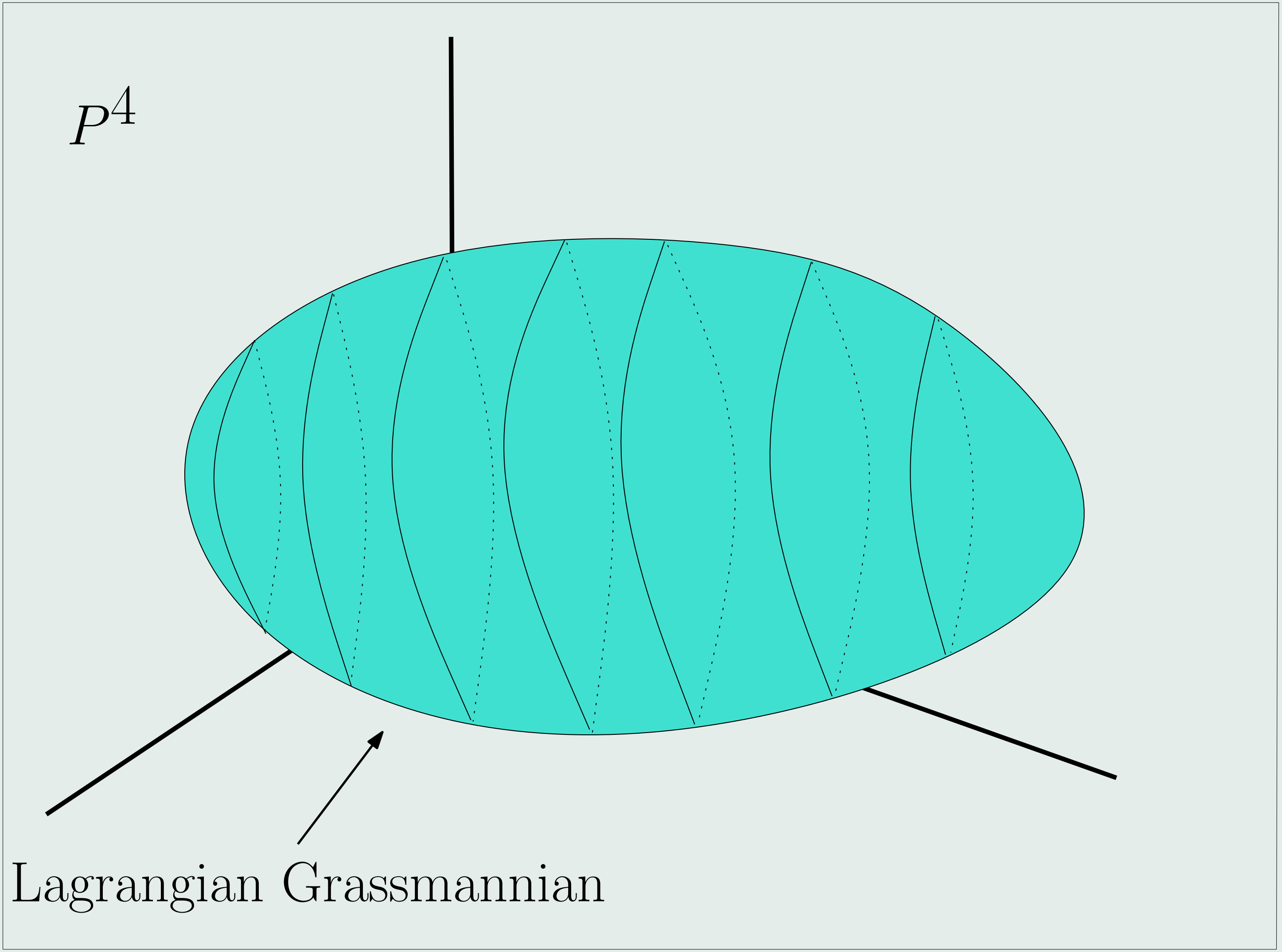


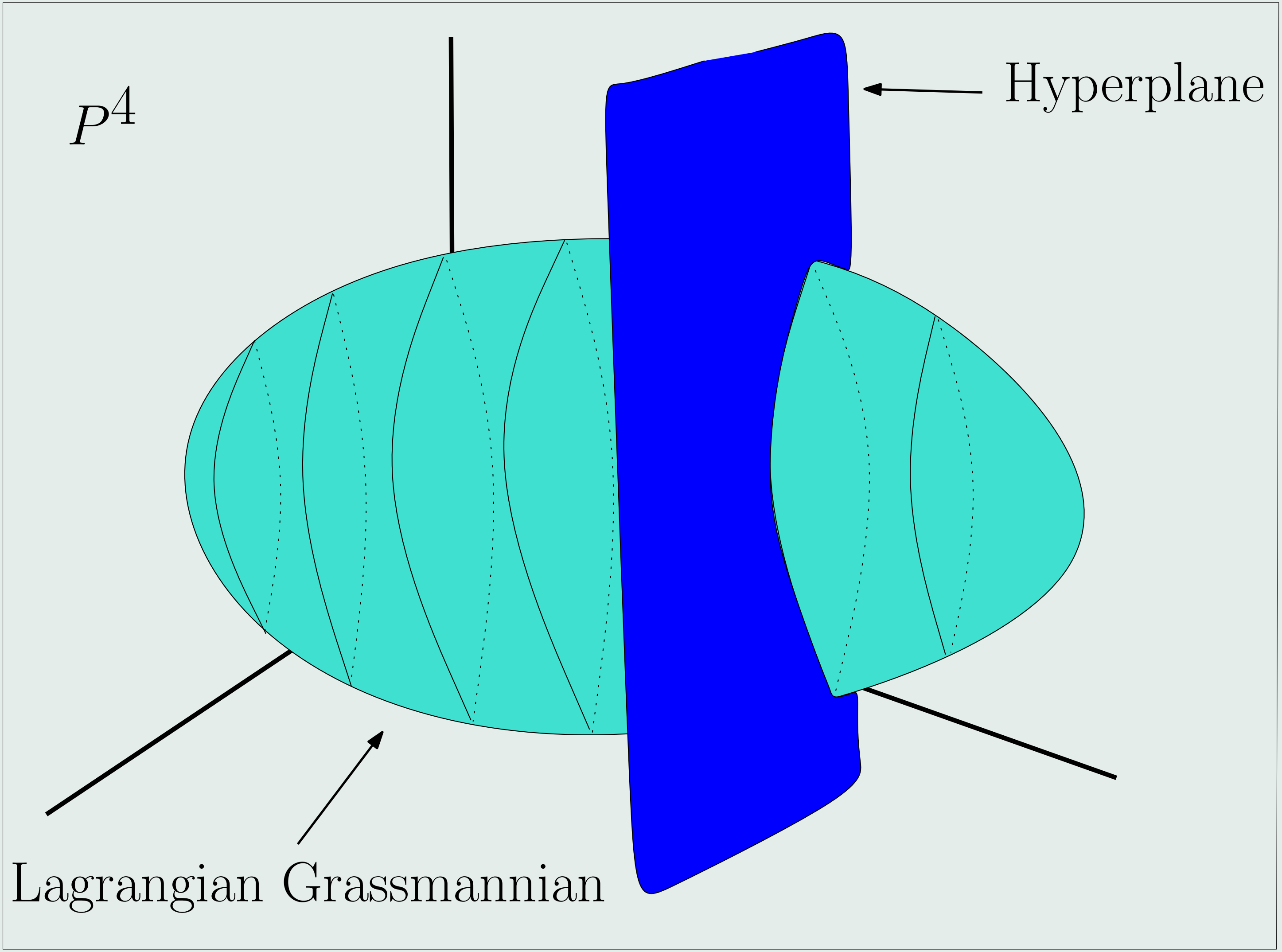
From now we fix the point $m \in M$, i.e. we work in a fibre.

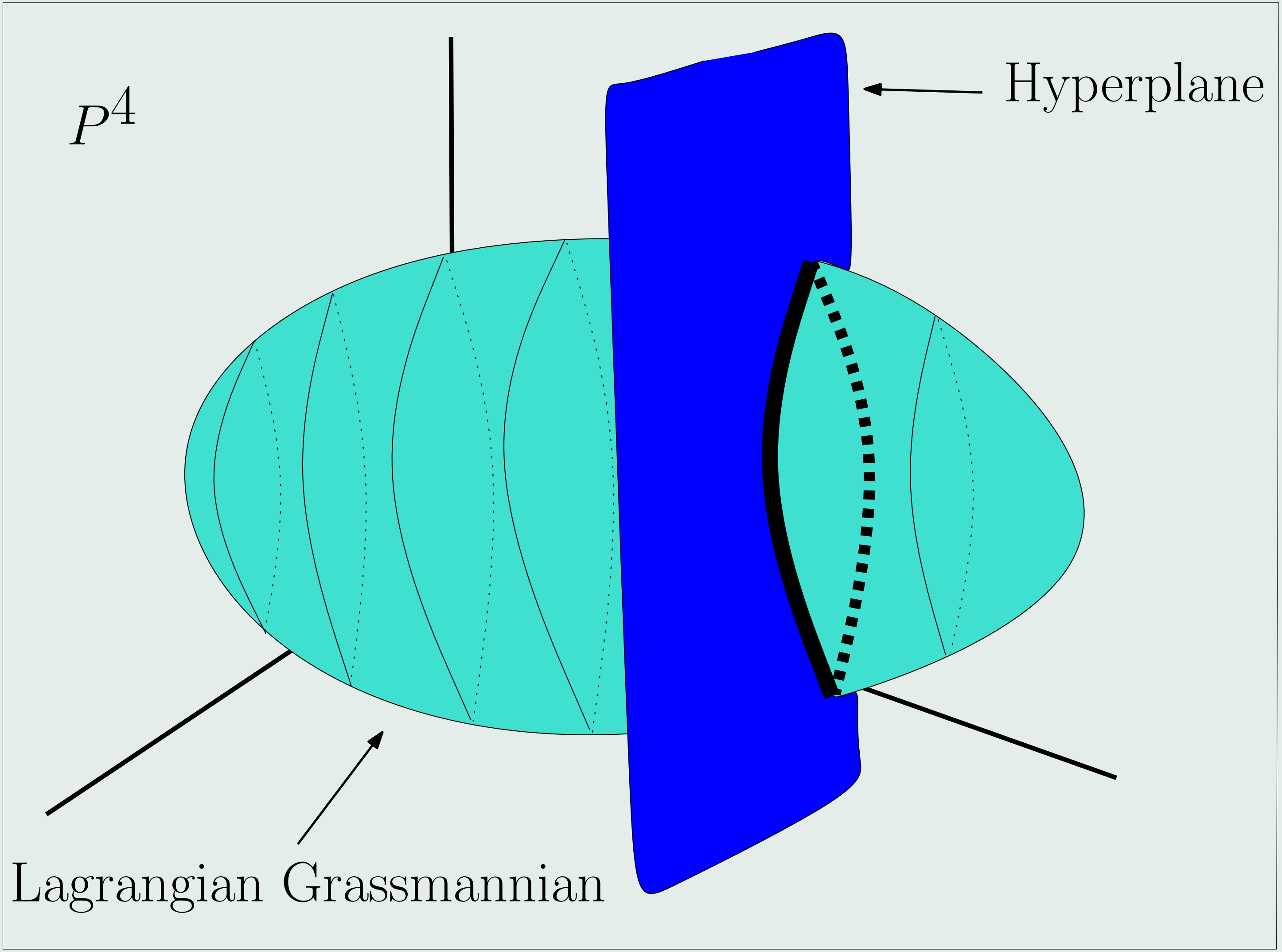



Lagrangian Grassmannian



P^4
Lagrangian Grassmannian


P^4
Hyperplane
Lagrangian Grassmannian



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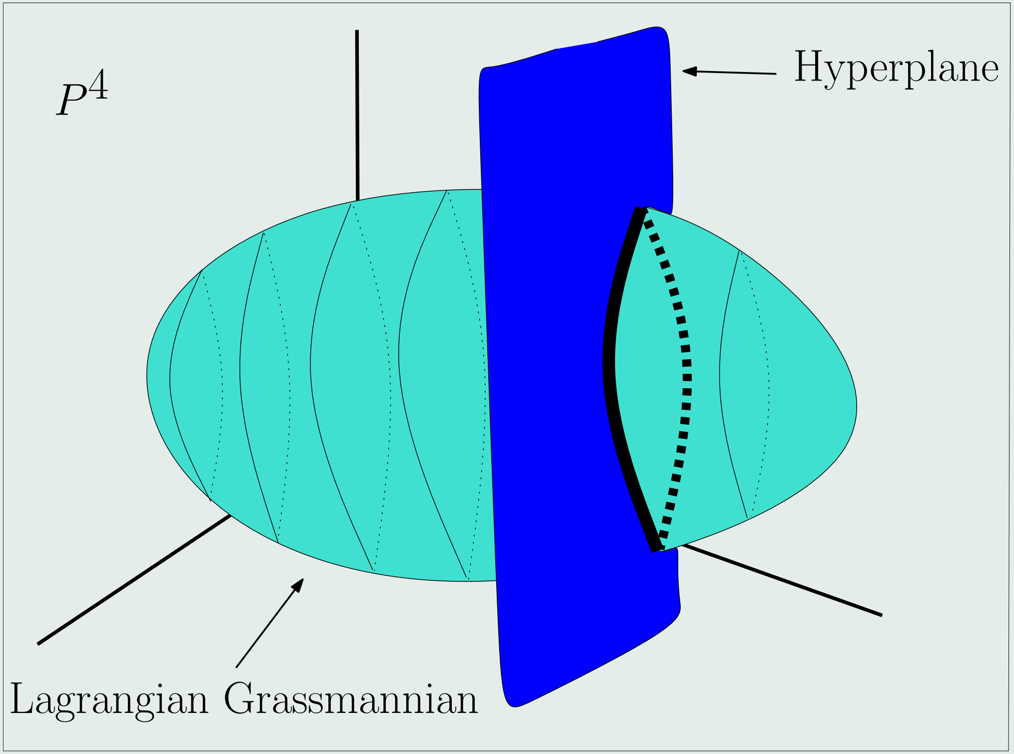
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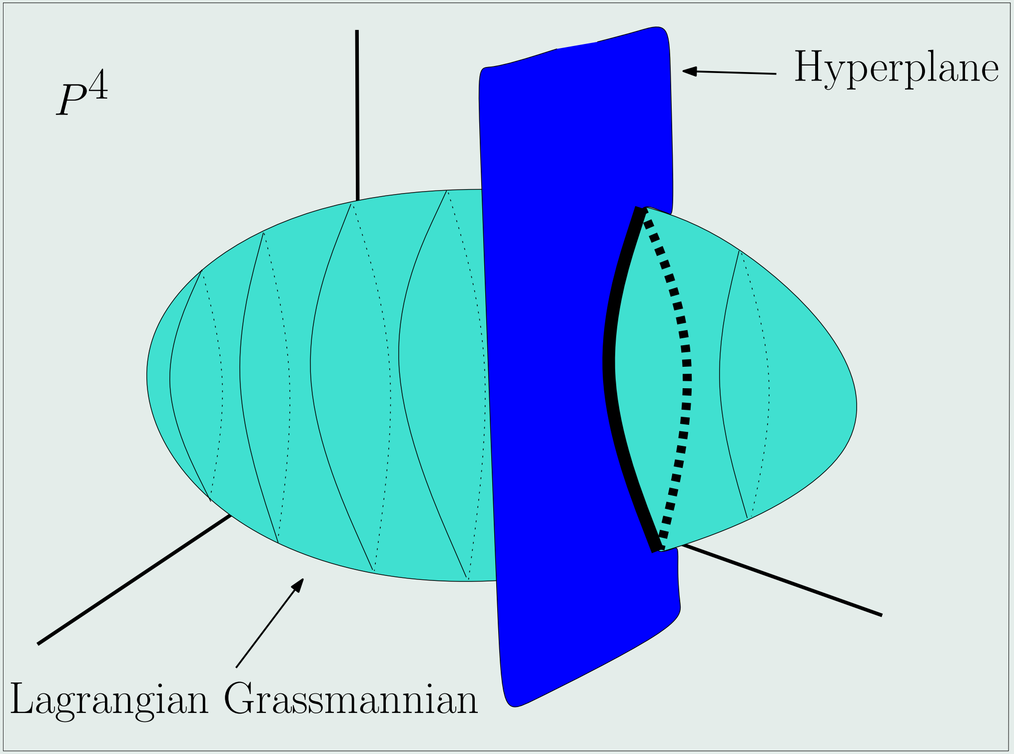


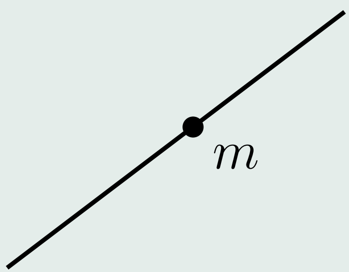
Hyperplane sections are Monge-Ampère equations:

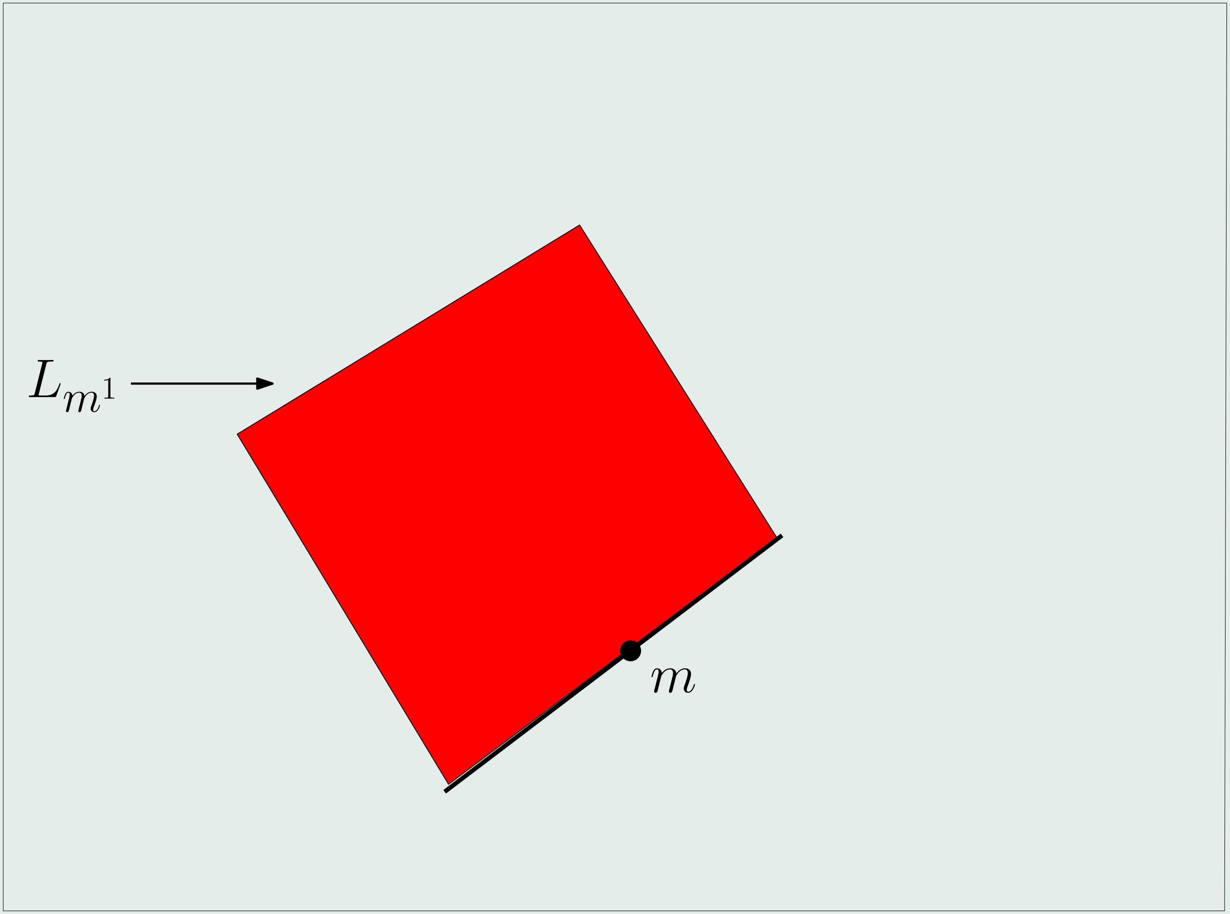
$$a_0 + a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4(u_{xx}u_{yy} - u_{xy}^2) = 0$$

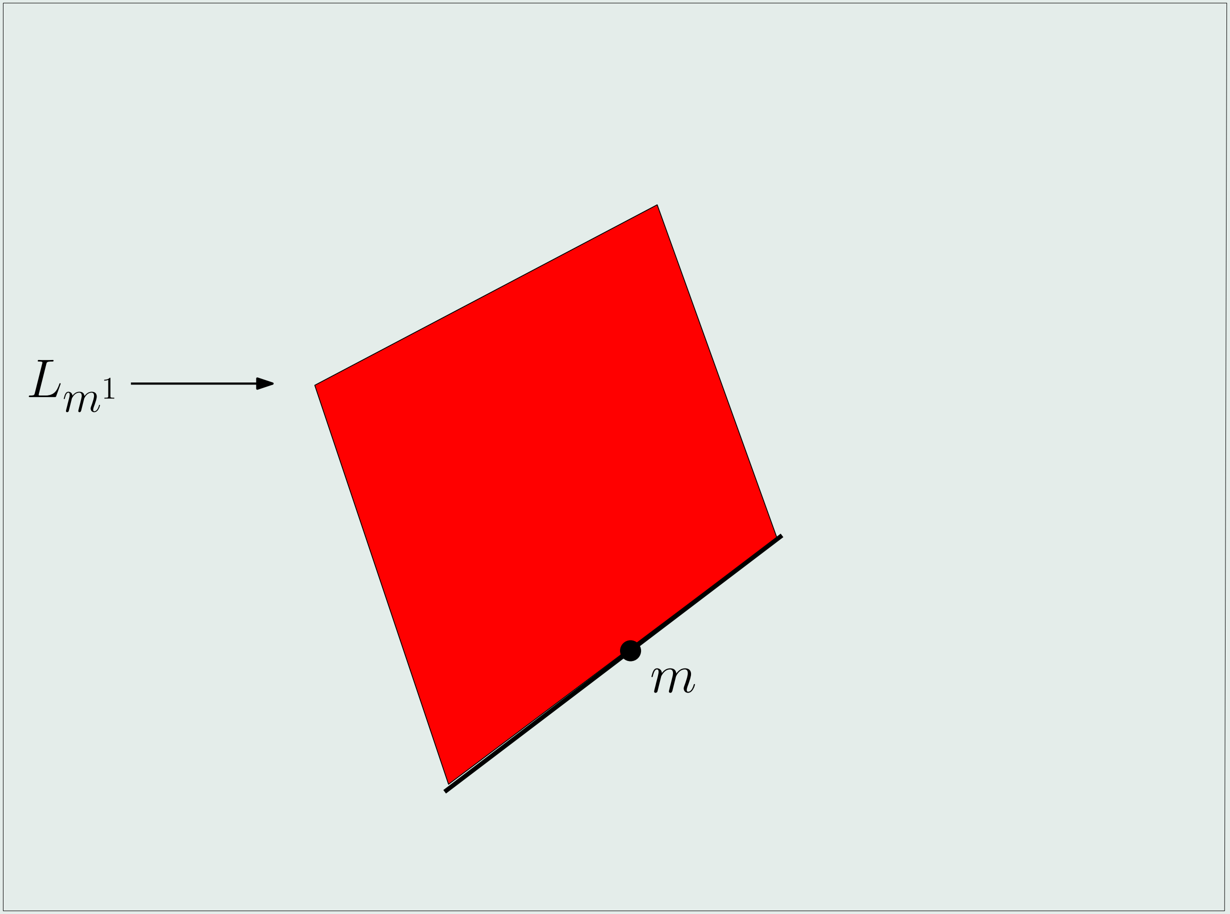
$$\text{Plücker} \ : \ (p_{11}, p_{12}, p_{22}) \ \hookrightarrow \ (1\ ,\ p_{11}\ ,\ p_{12}\ ,\ p_{22}\ ,\ p_{11}p_{22} - p_{12}^2)$$

Now we see how to define Monge-Ampère equations by using characteristics directions and how to characterize such PDEs in terms of some canonical conformal structures on Lagrangian Grassmannians

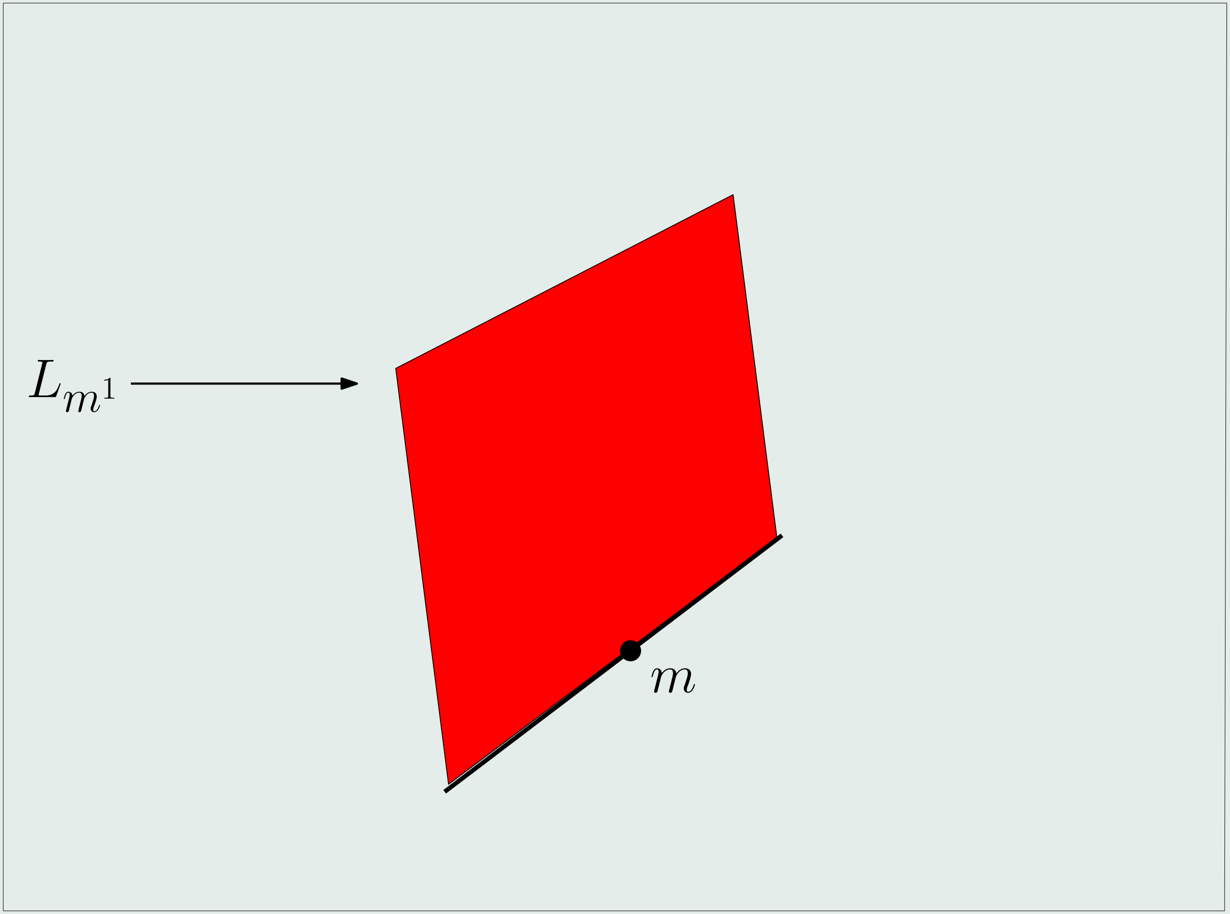

m


m


L_{m^1}
m



L_{m^1}
m



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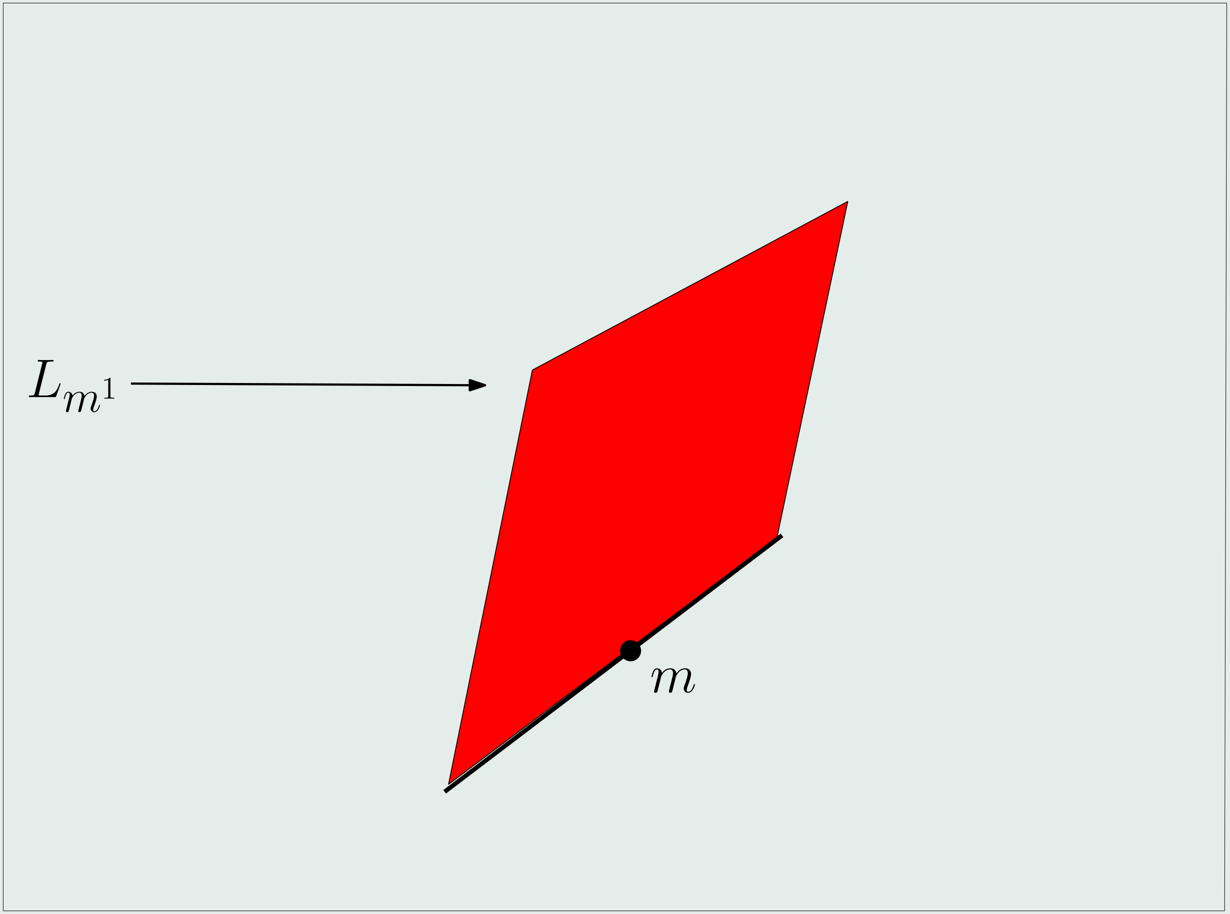

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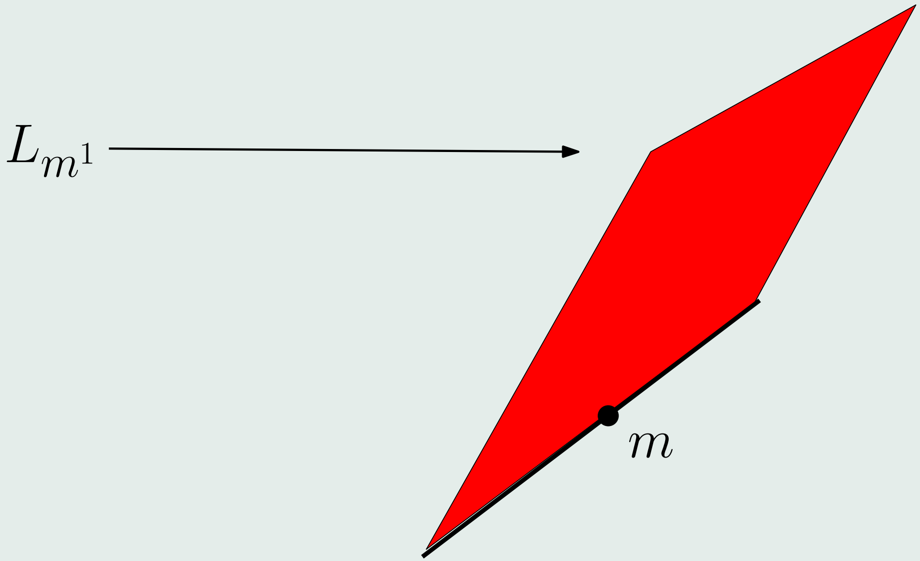
L_{m^1}
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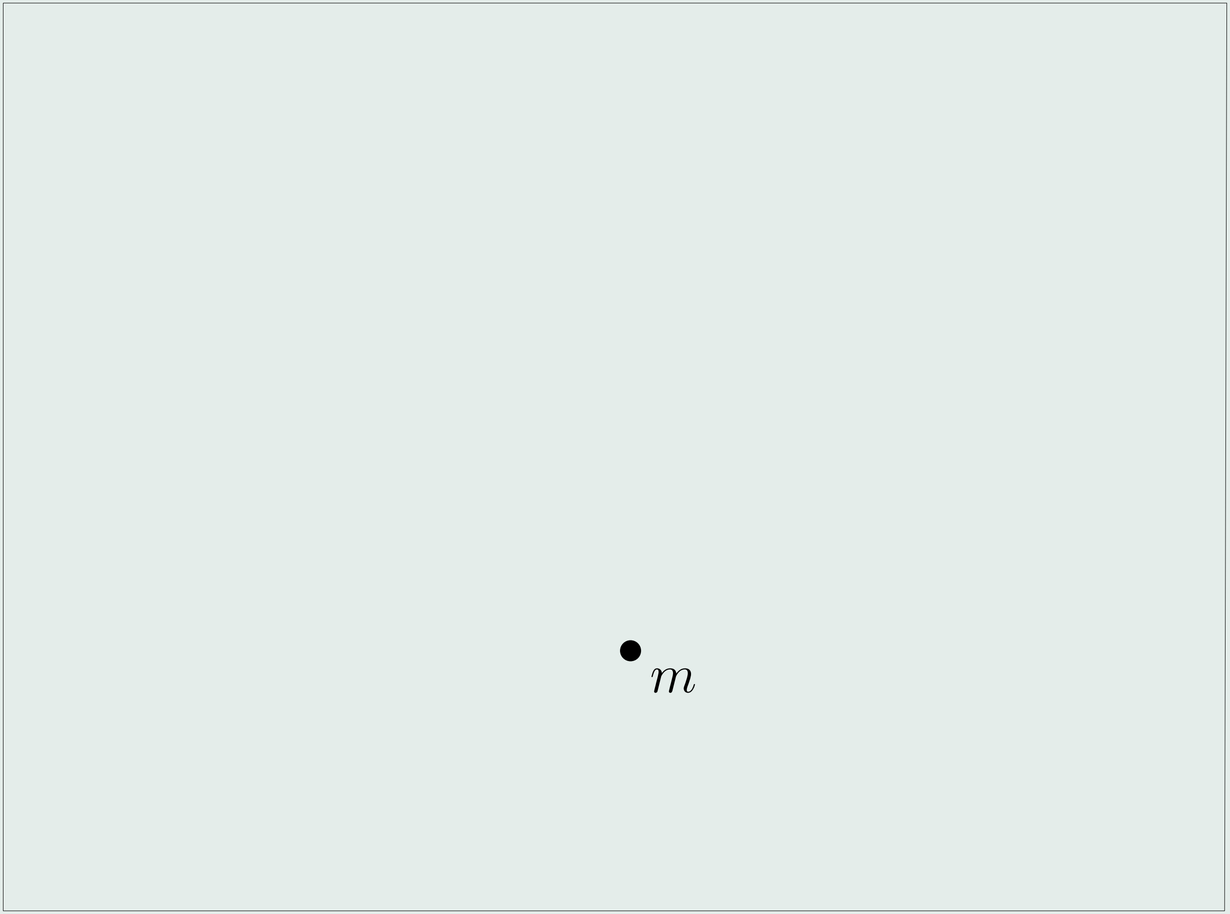
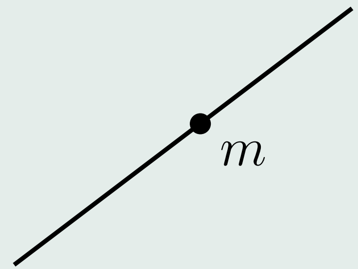
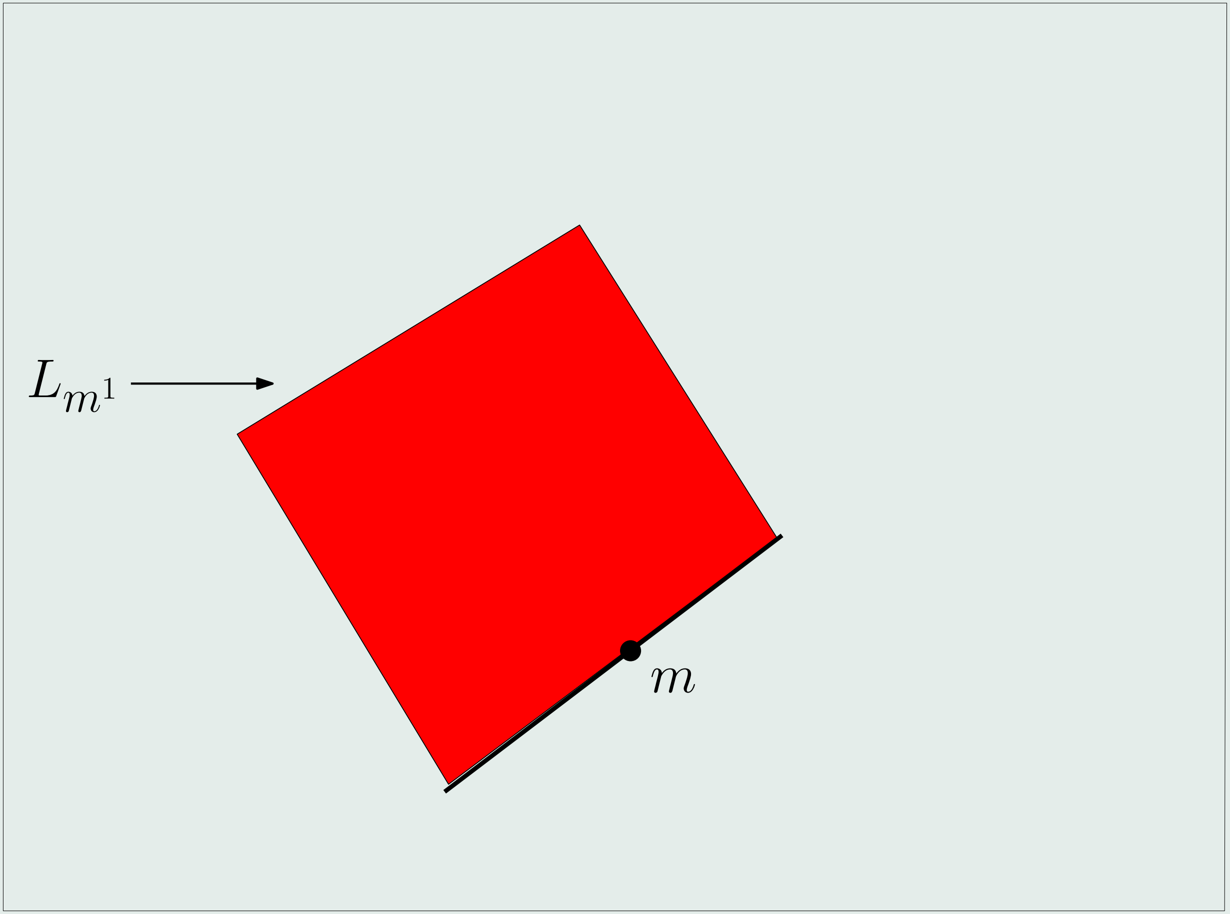

m


m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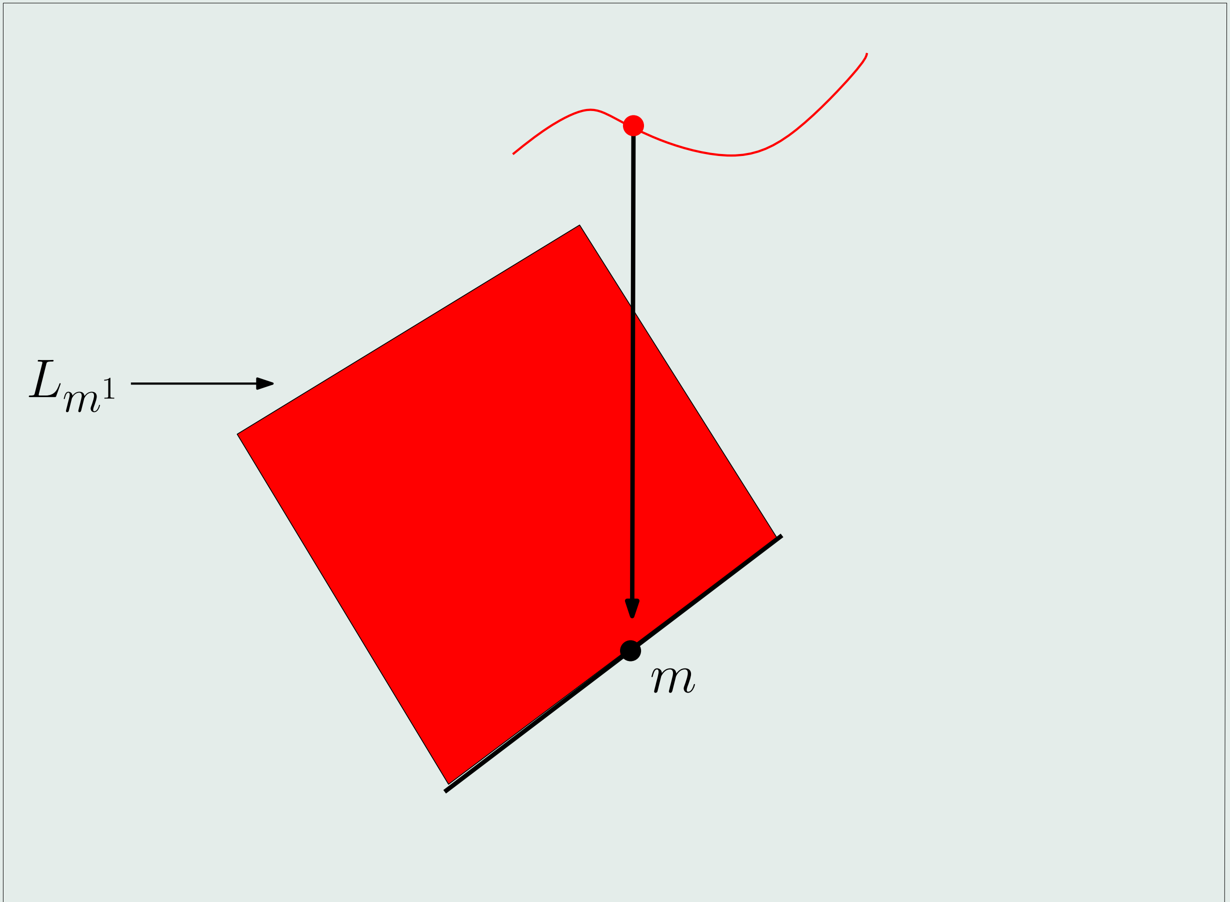
m



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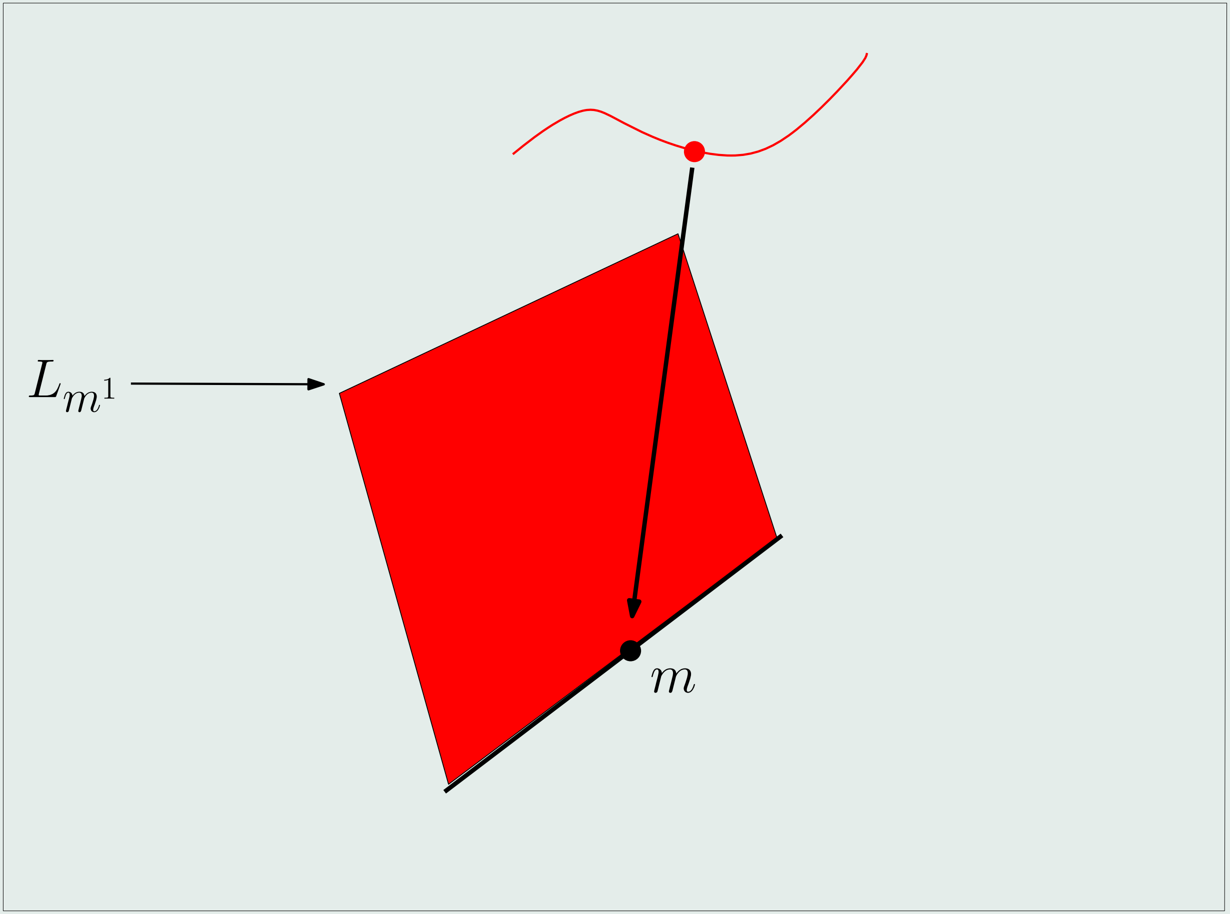
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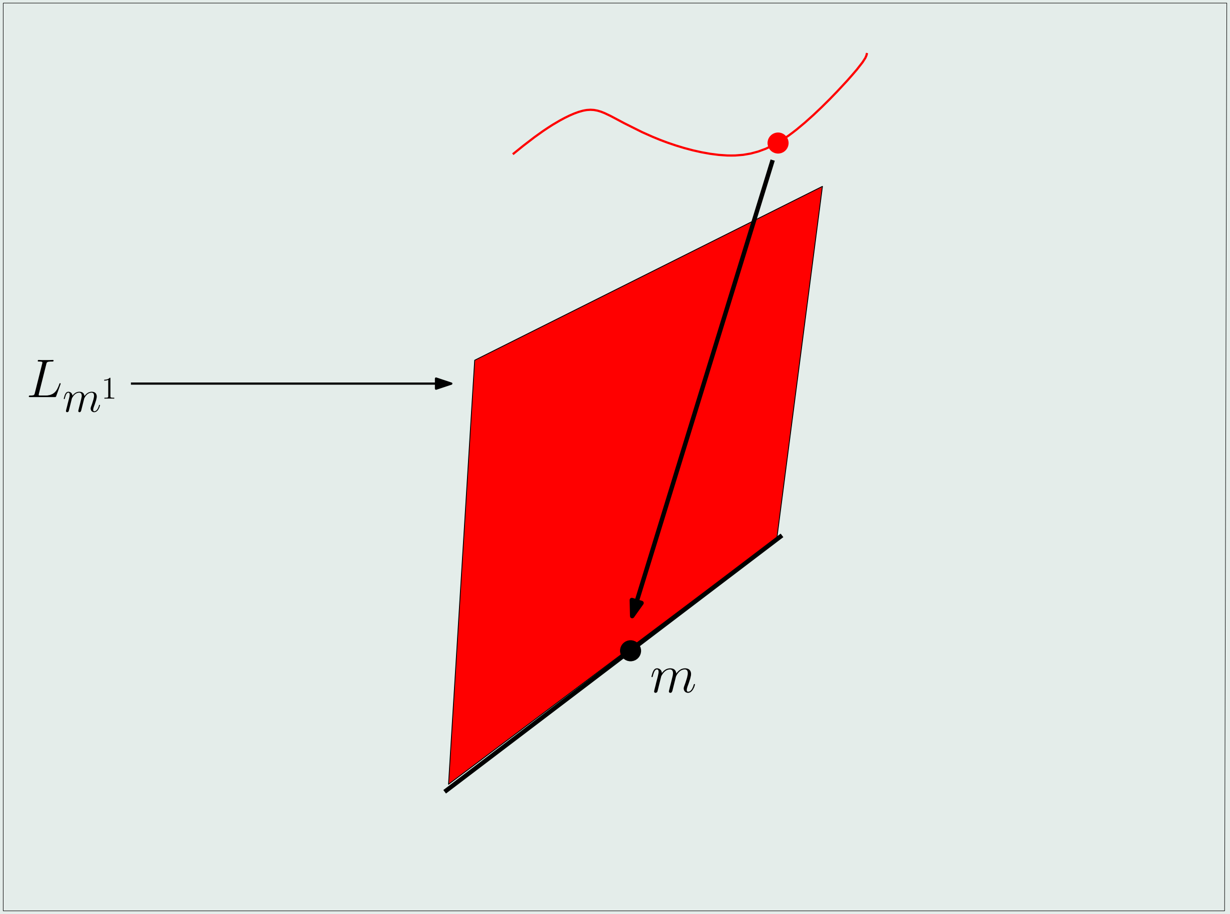
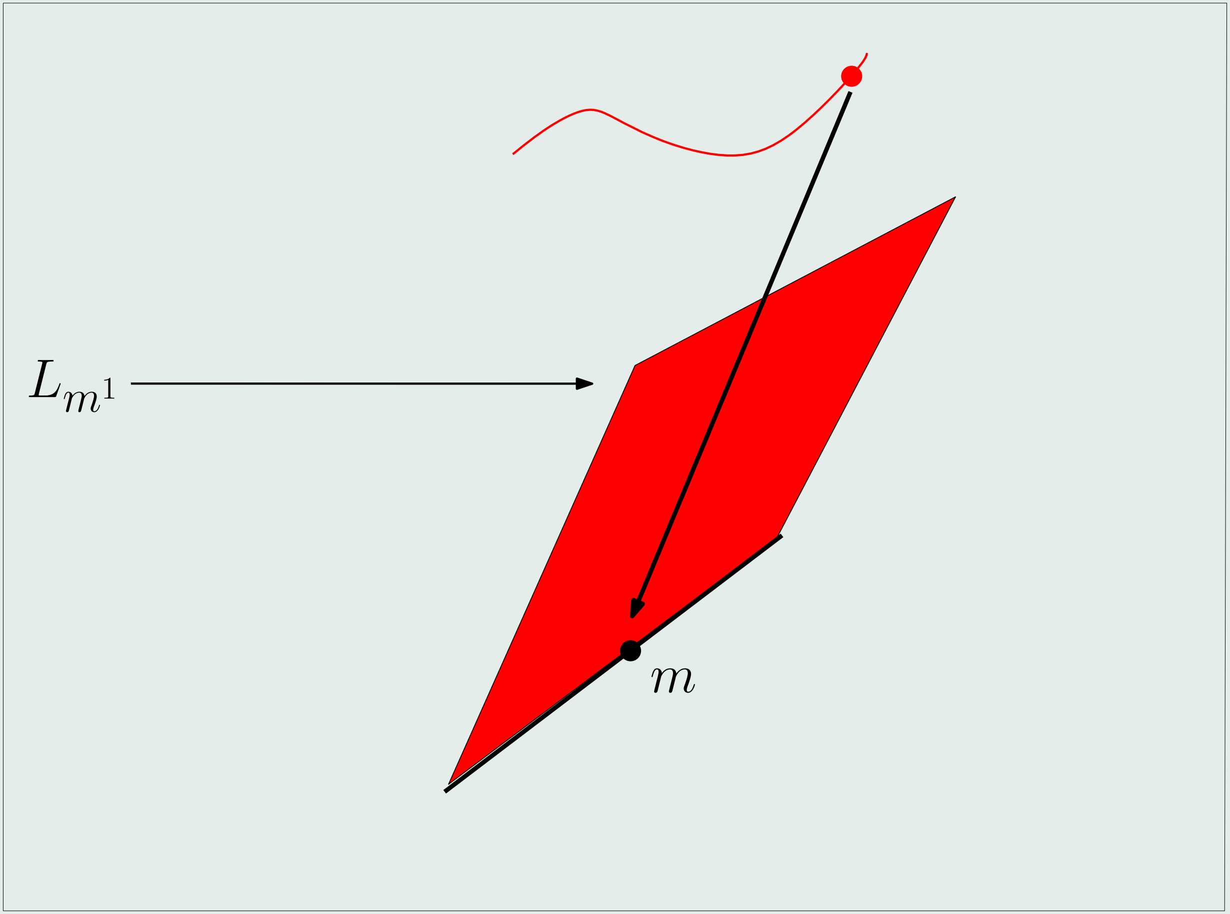
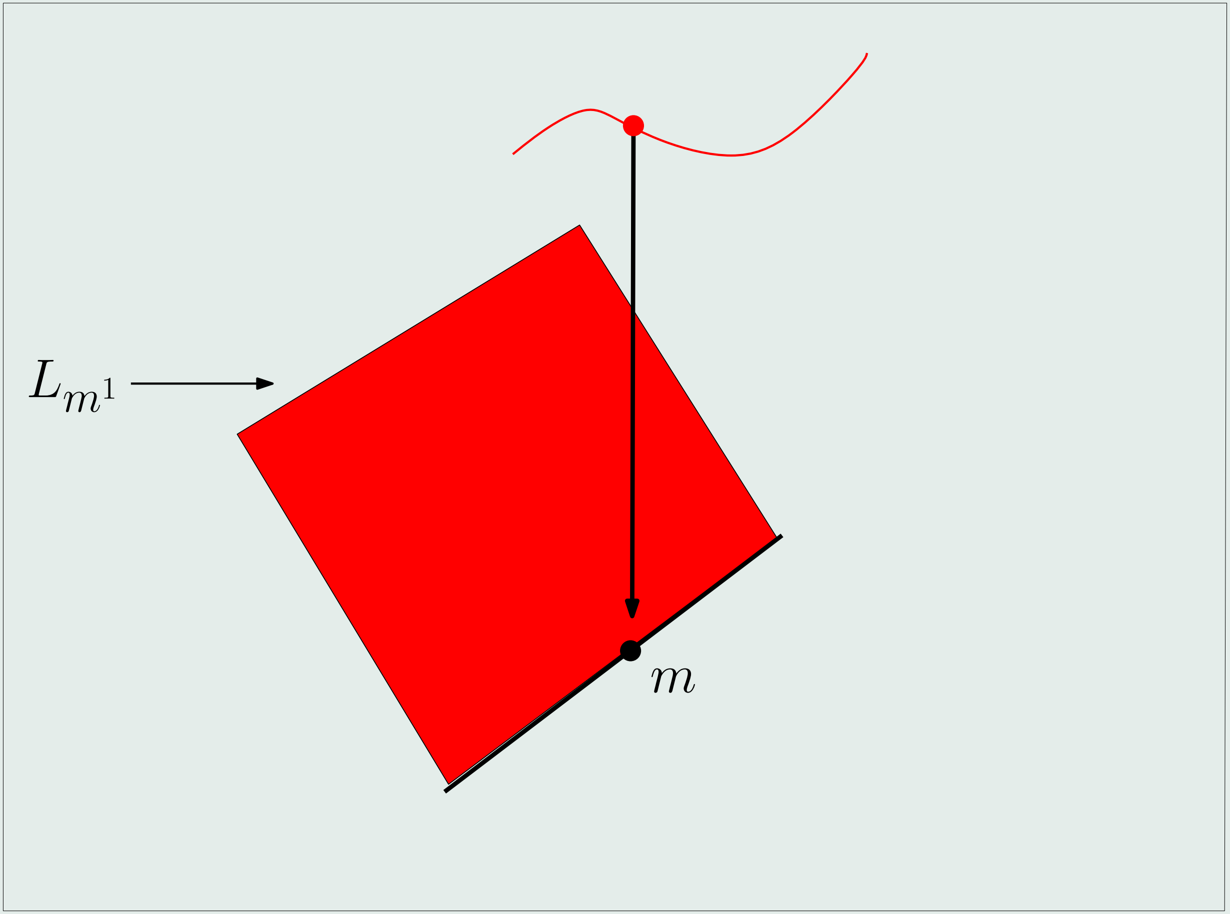


L_{m^1}
m


L_{m^1}
m


L_{m^1}
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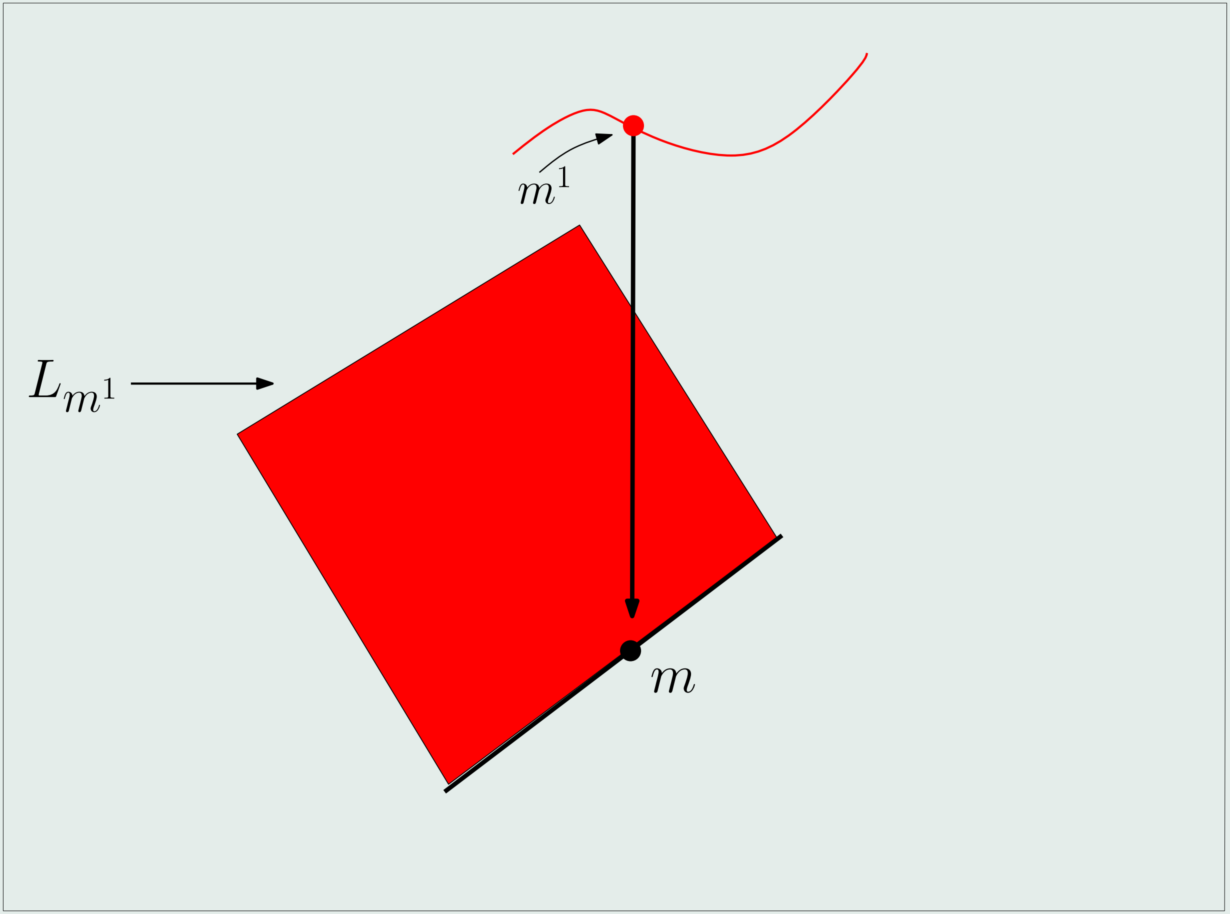
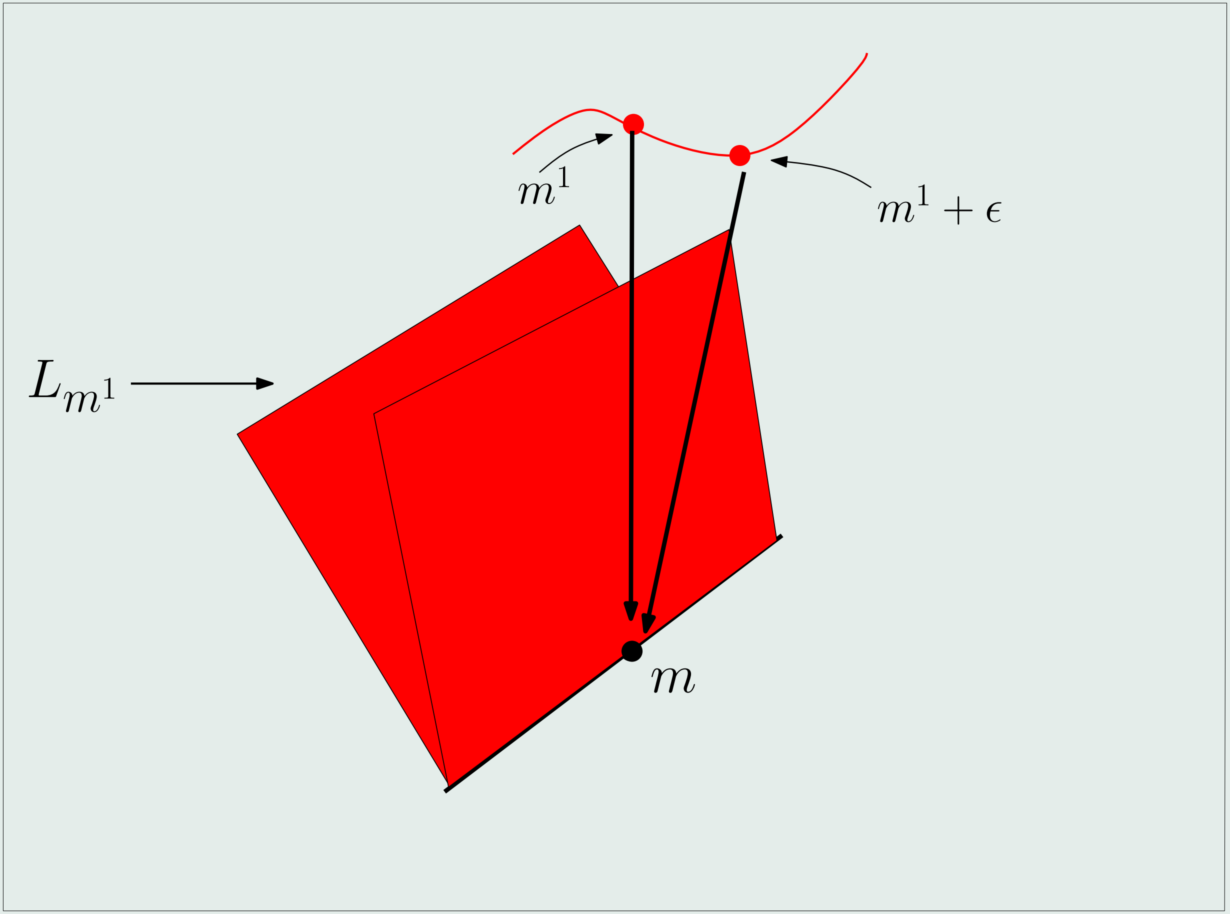
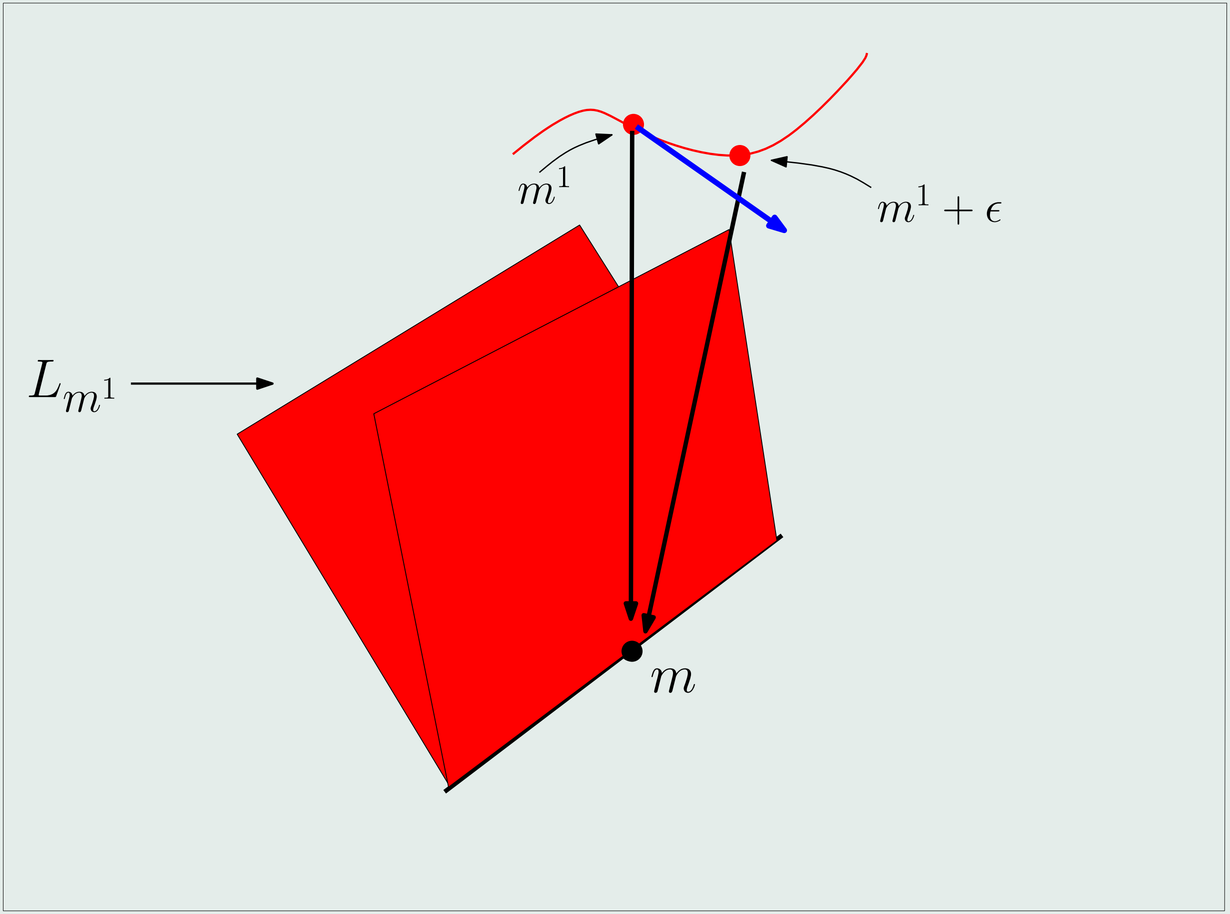
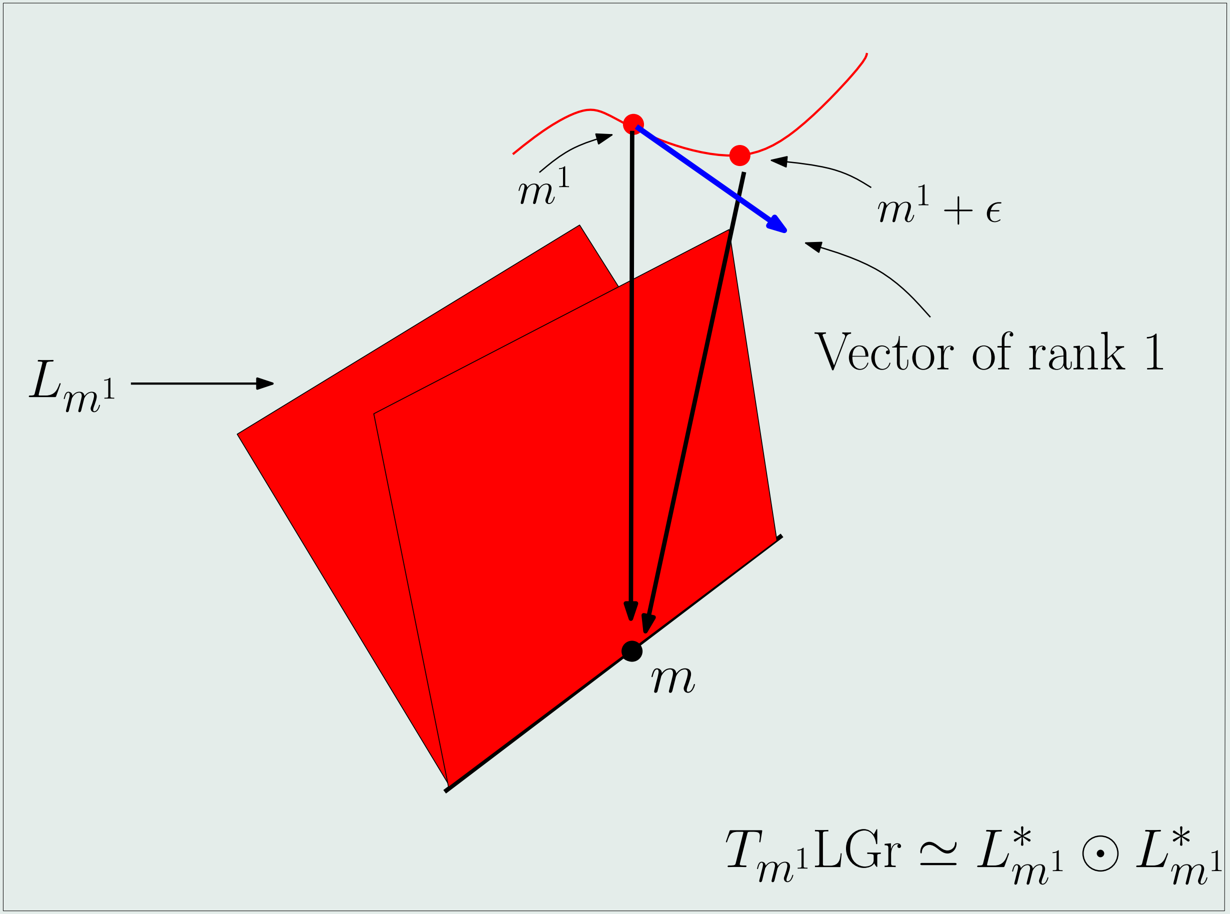


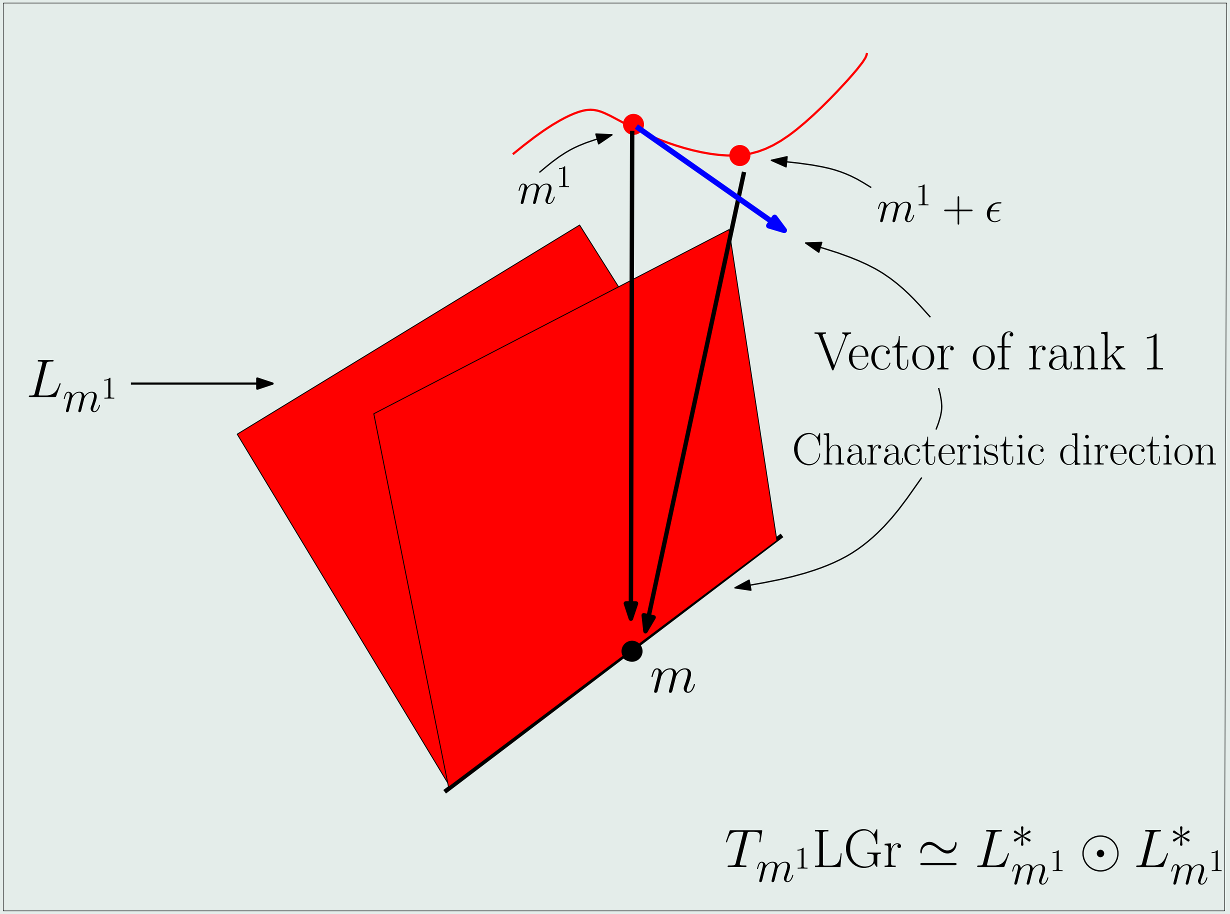
m^1
L_{m^1}
m


m^1
$m^1 + \epsilon$
L_{m^1}
m



m^1
$m^1 + \epsilon$
L_{m^1}
m


m^1
$m^1 + \epsilon$
Vector of rank 1
L_{m^1}
m
$T_{m^1}\mathrm{LGr} \simeq L^*_{m^1} \odot L^*_{m^1}$



m^1
$m^1 + \epsilon$
Vector of rank 1
L_{m^1}
Characteristic direction
m
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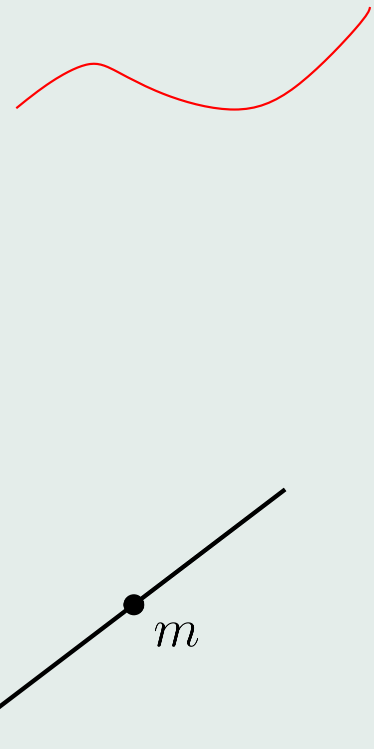
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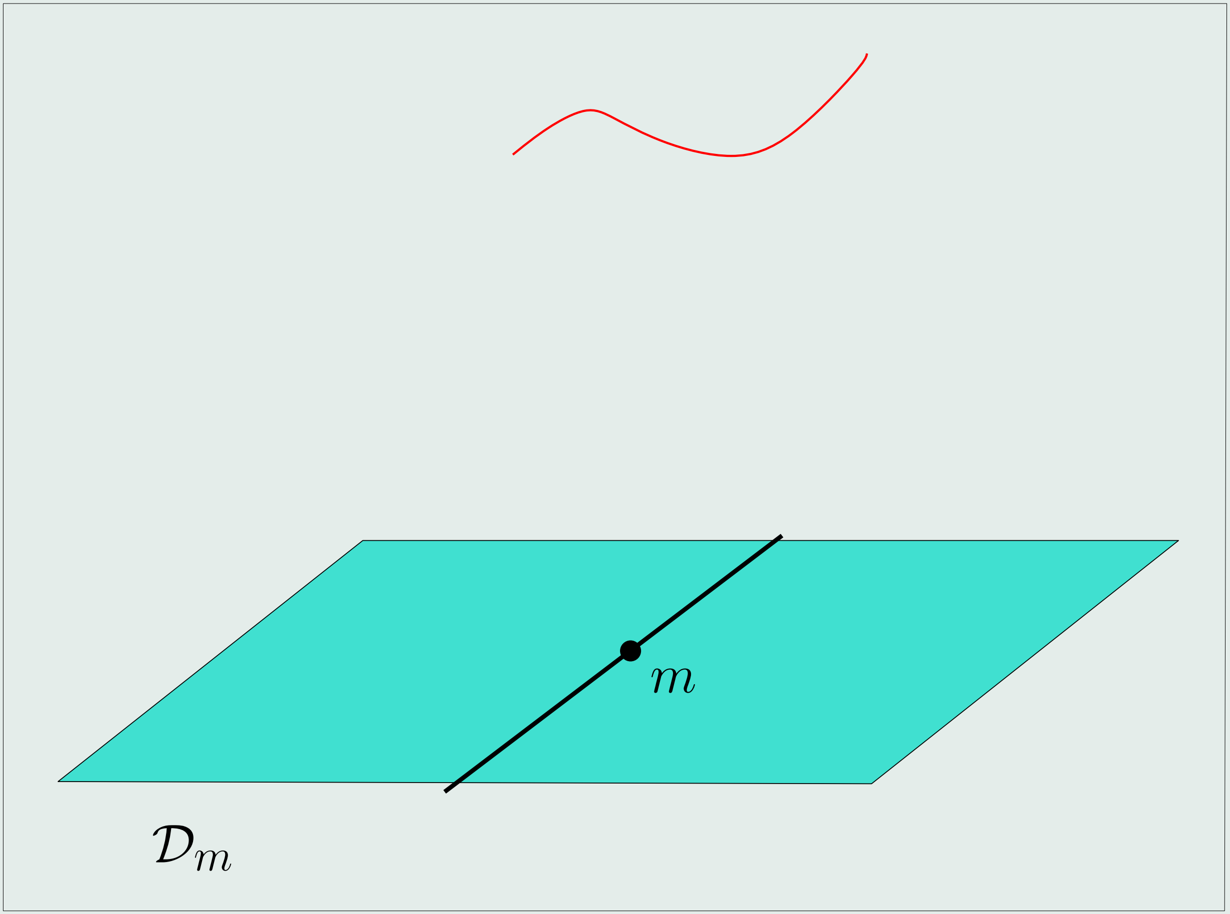

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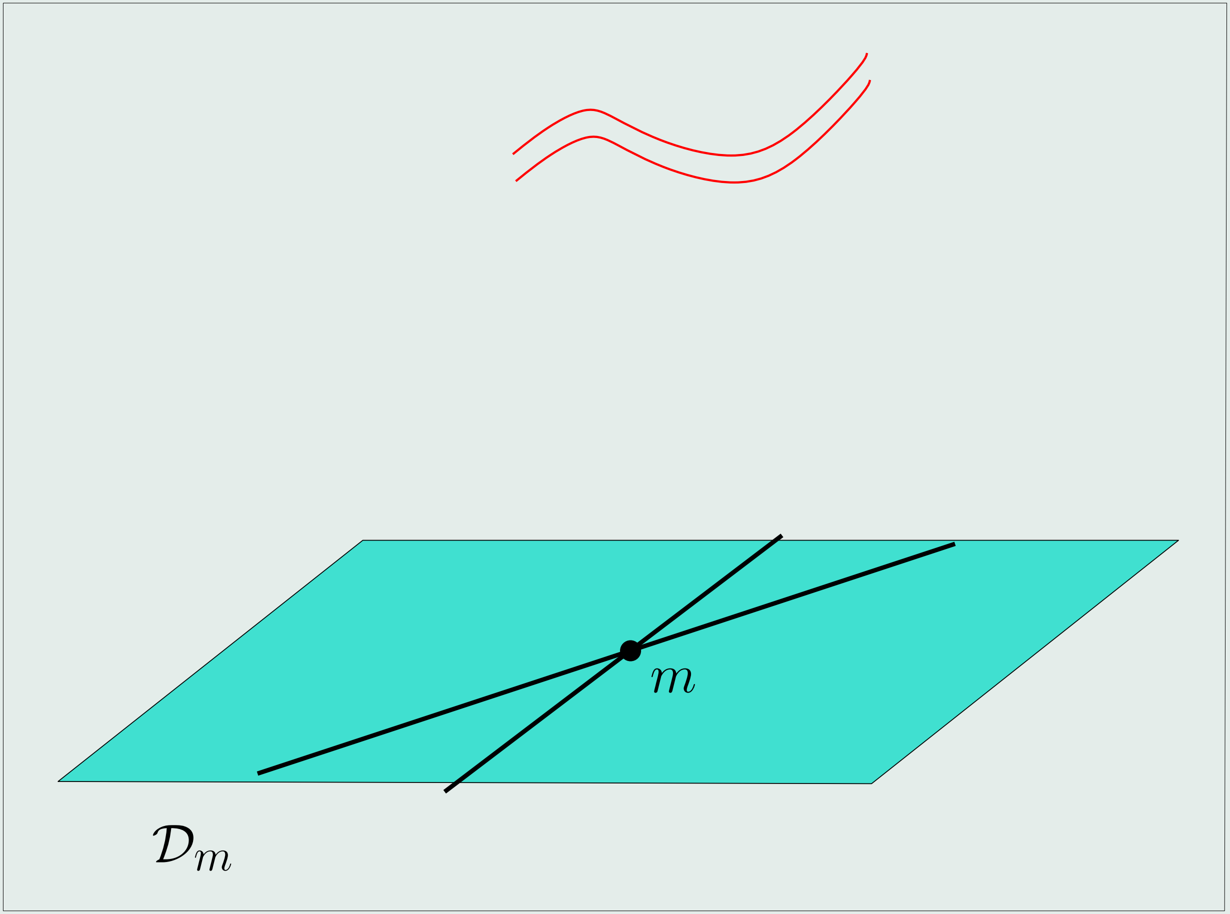

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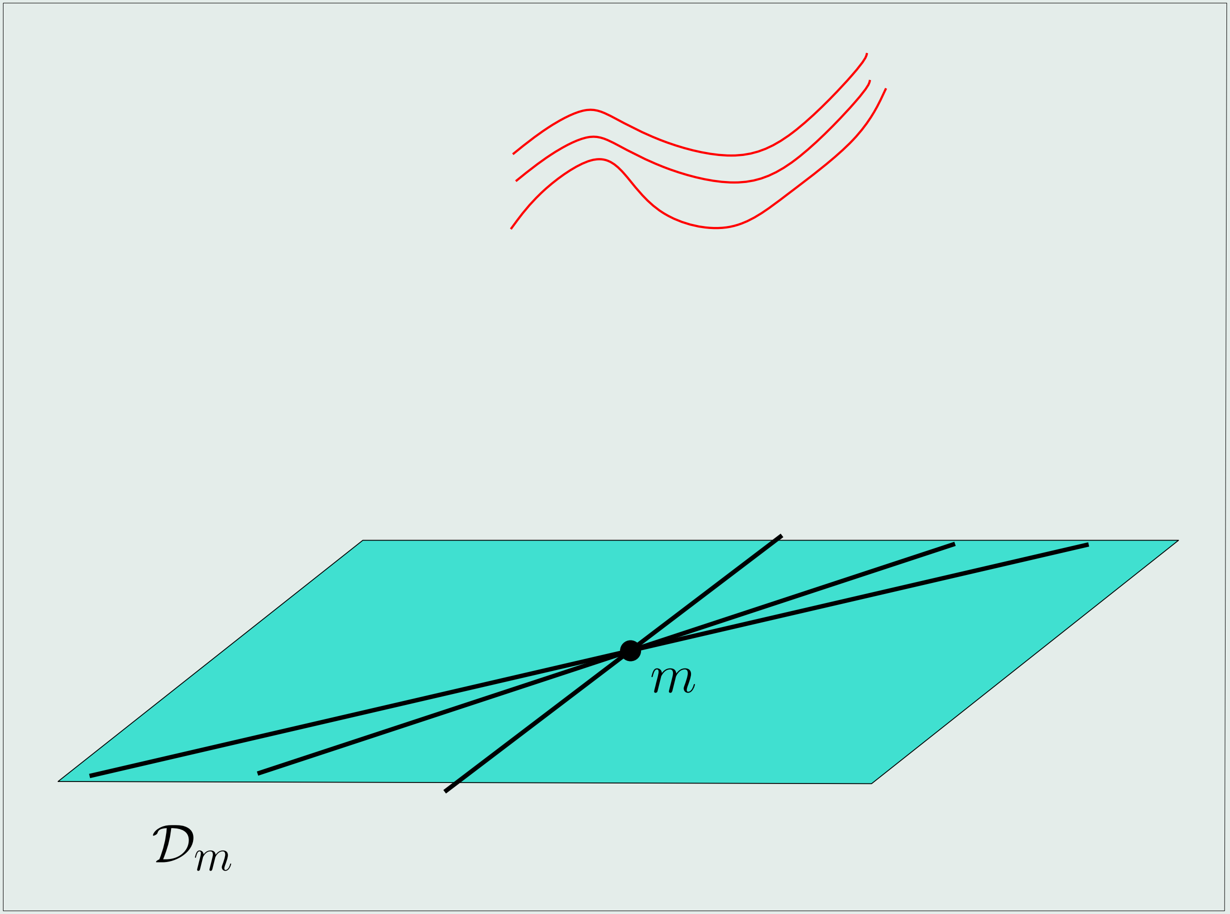

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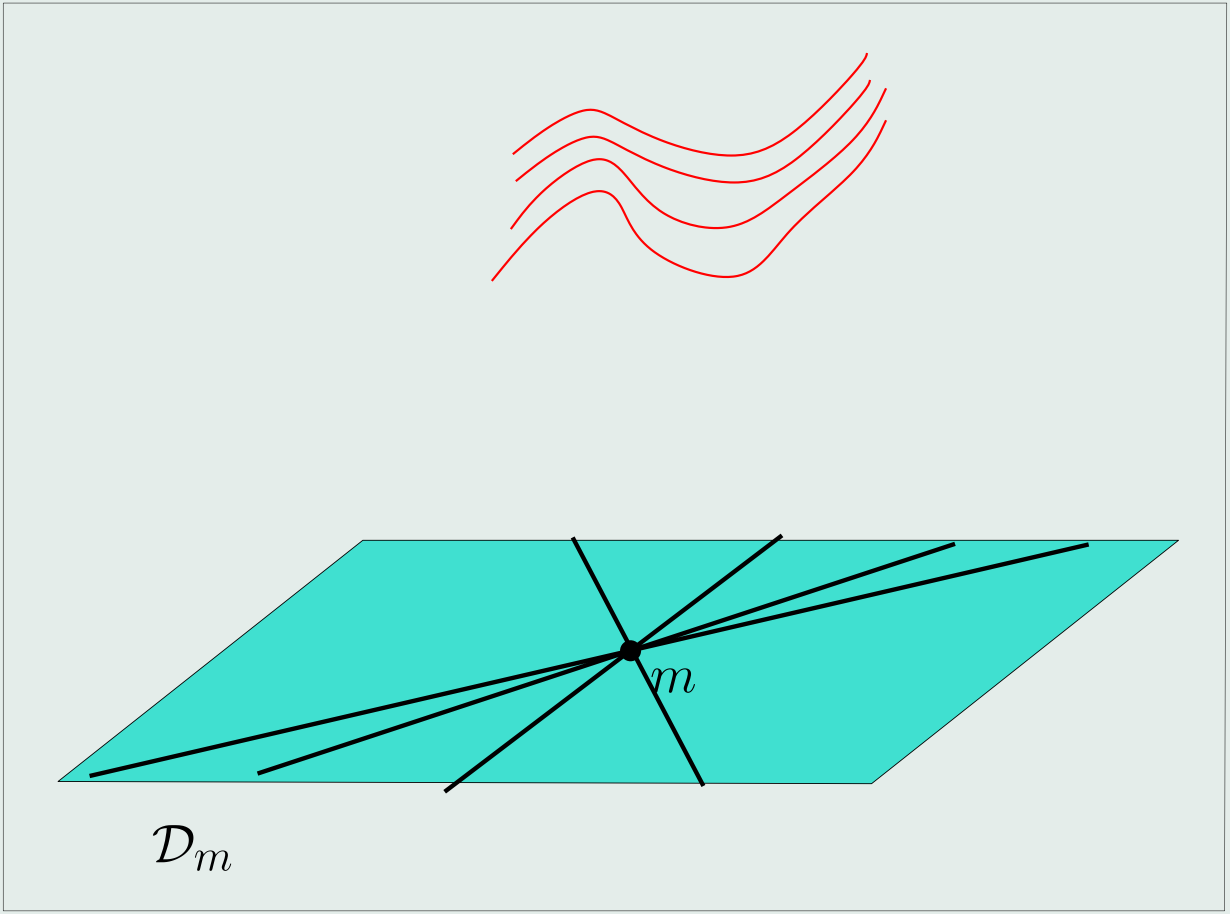

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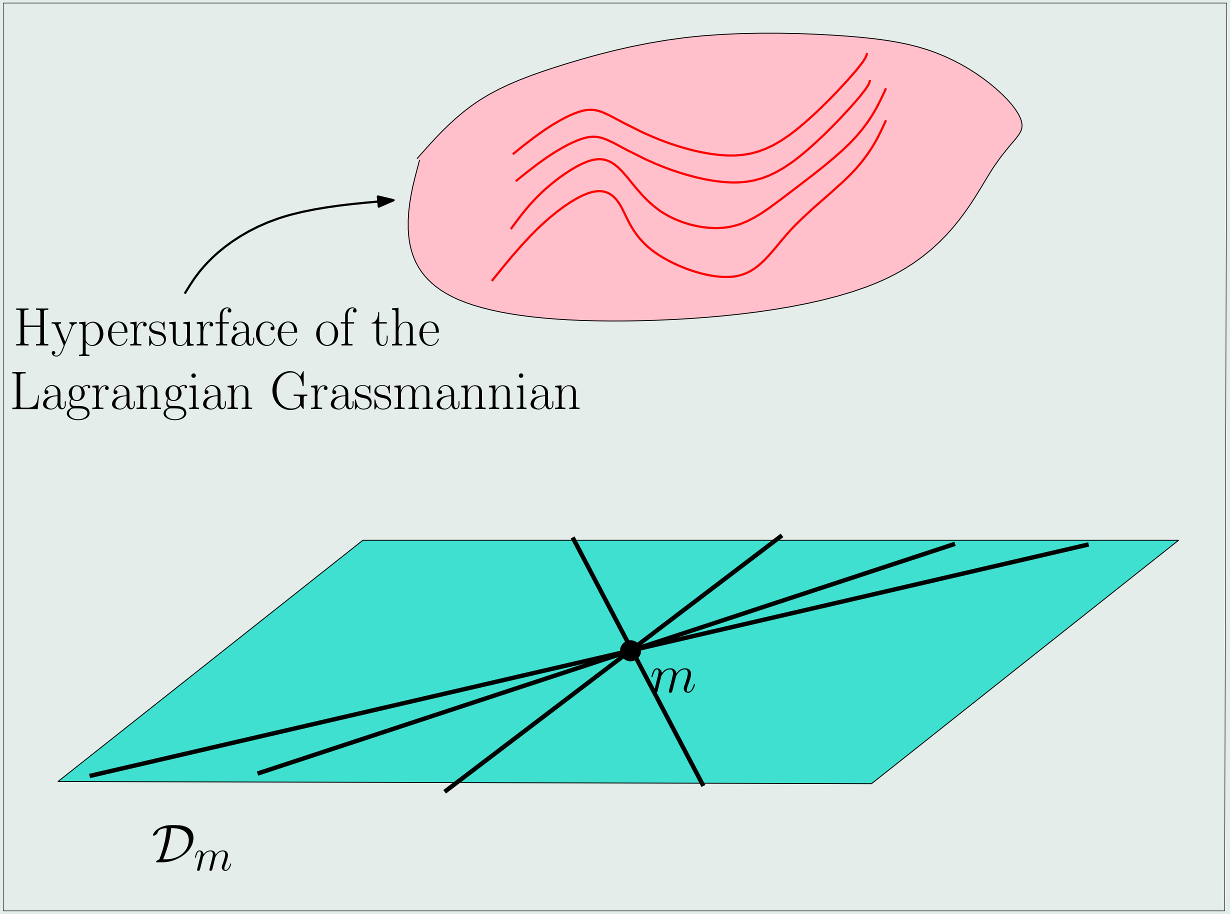
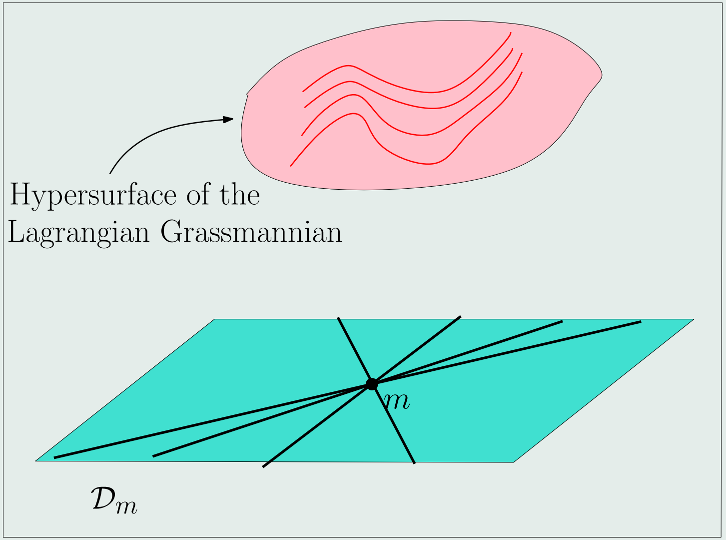
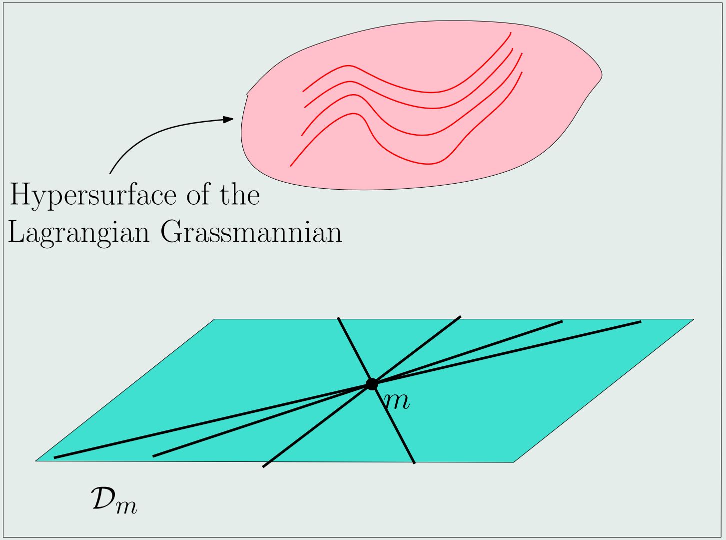



Hypersurface of the
Lagrangian Grassmannian
m
$\mathcal{D}_m$



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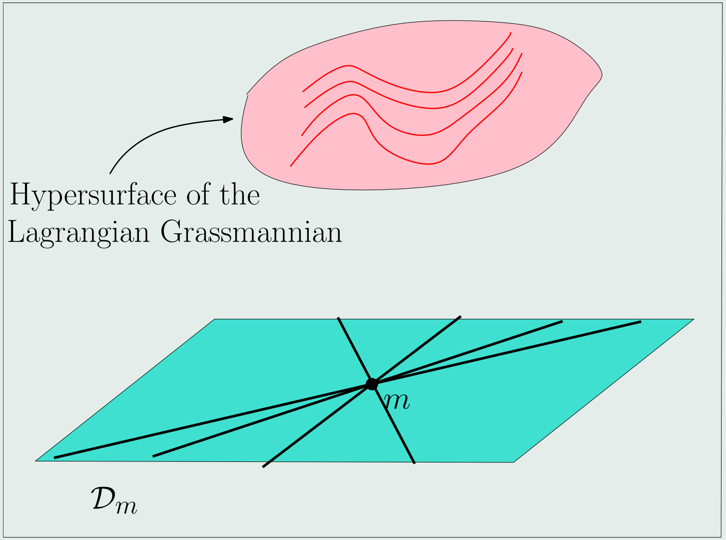

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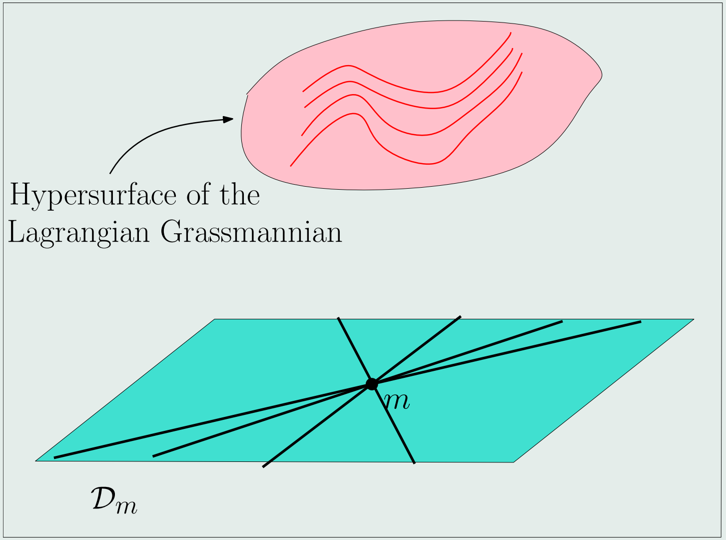

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Locally

$$\mathcal{E}_{\mathcal{D}} : \det \begin{pmatrix} u_{xx} - f_{11} & u_{xy} - f_{12} \\ u_{xy} - f_{21} & u_{yy} - f_{22} \end{pmatrix} = 0$$

$\mathcal{E}_{\mathcal{D}}$ is a parabolic una Monge–Ampère equation $\iff f_{12} = f_{21} \iff \mathcal{D} = \mathcal{D}^\perp$.

Rank 1 vectors of $\mathrm{LGr}(2,4)$ and its conformal structure

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So, we have a natural conformal structure on $\mathrm{LGr}(2,4)$

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- A straightforward computation shows that

$$\mathrm{II} - \mathrm{H}\cdot\mathrm{I}\,, \quad \mathrm{H} \text{ mean curvature}$$

is a conformal invariant.

$$\boxed{S \subset \mathrm{LGr}(2,4) \text{ is a hyperplane section} \iff \mathrm{II} - \mathrm{H}\cdot\mathrm{I} = 0}$$

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Hyperplane sections of $\mathrm{LGr}(3,6)$

Let us consider the symmetric 3-tensor T:

$$T = \det(dp_{ij})\,, \quad i,j = 1,\ldots,3$$

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Hyperquadric sections?