Metrics admitting projective symmetries

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Politecnico di Torino

Joint results with

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Diffieties, Cohomological Physics, and Other Animals

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This and the encouragement to think “simply” are the most important teachings I got from him.

Basic definitions

Definition. A vector field is called ***projective*** w.r.t. a metric g, if its local flow sends (unparametrized) geodesics into (unparametrized) geodesics.

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The class of connections sharing the same geodesics of a connection Γ is called ***projective connection*** (associated to Γ).

Problems of Sophus Lie

Problem (Sophus Lie)

Untersuchungen über geodätische Curven (Math. Ann. 20, 1882)

I. Es wird verlangt, die Form eines Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Curven eine infinitesimale Transformation gestatten.

II. Man soll die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Curven mehrere infinitesimale Transformationen gestatten.

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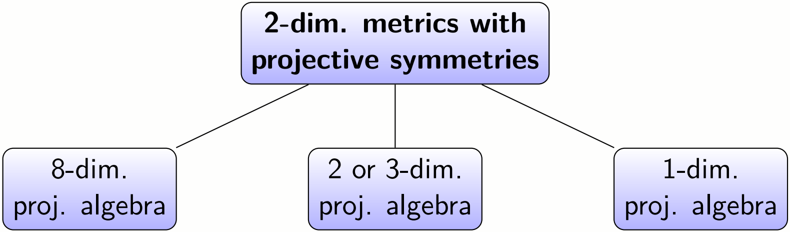
The problem has a natural multi-dimensional generalization, that we shall discuss later

Starting point for a classification of 2-dim. metrics with projective symmetries

Theorem (Sophus Lie)

The Lie algebra of projective vector fields on a 2-dimensional manifold can be 0, 1, 2, 3 *and* 8 *dimensional.*

2-dim. metrics with projective symmetries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8-dim. proj. algebra
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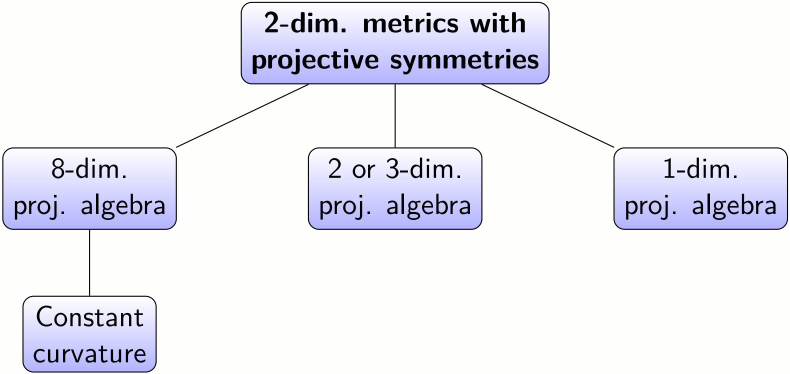
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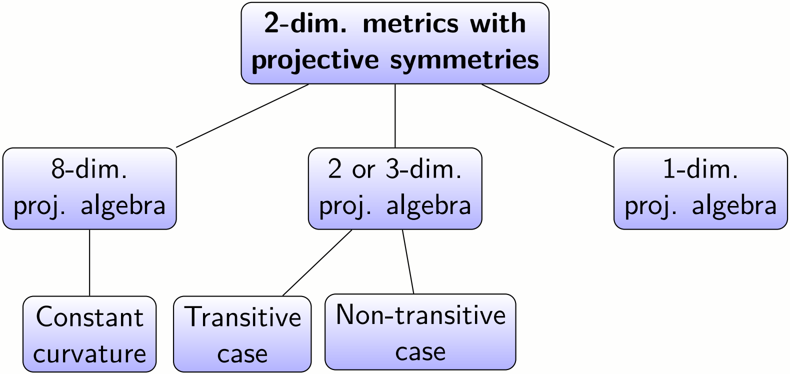
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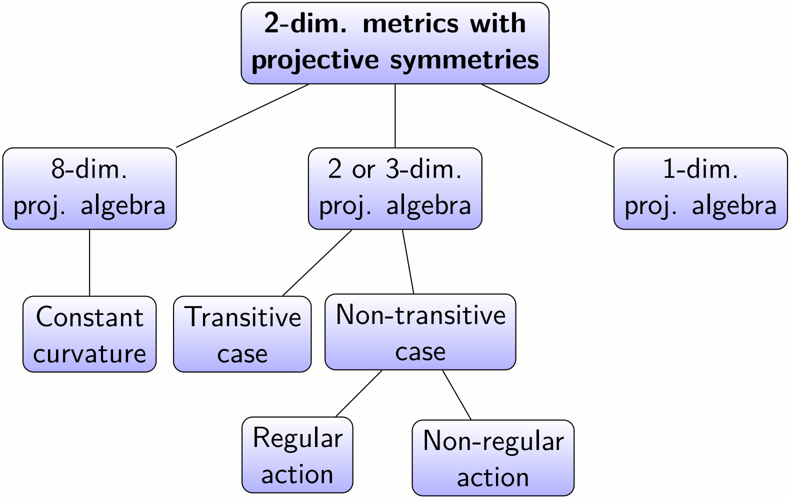
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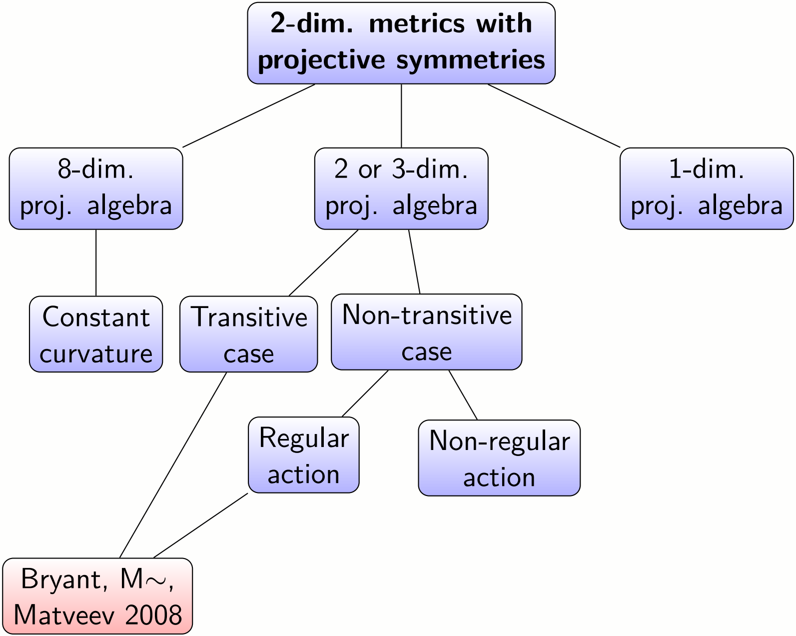
Non-transitive case
Regular action
Non-regular action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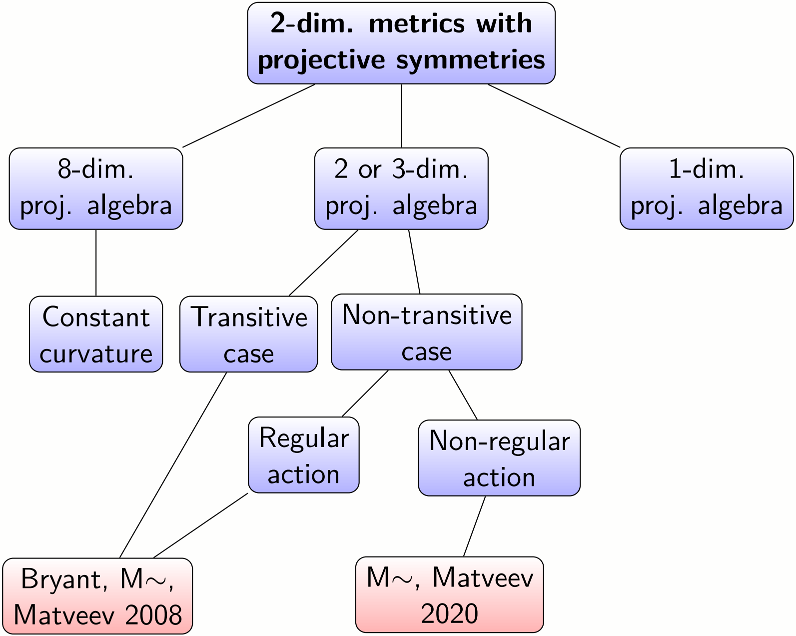
Bryant, M∼, Matveev 2008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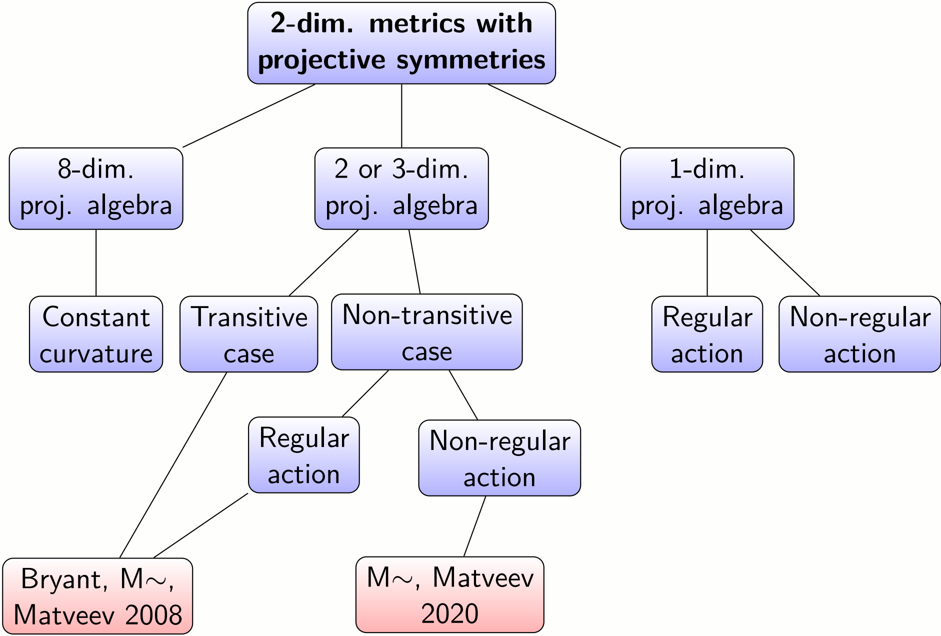
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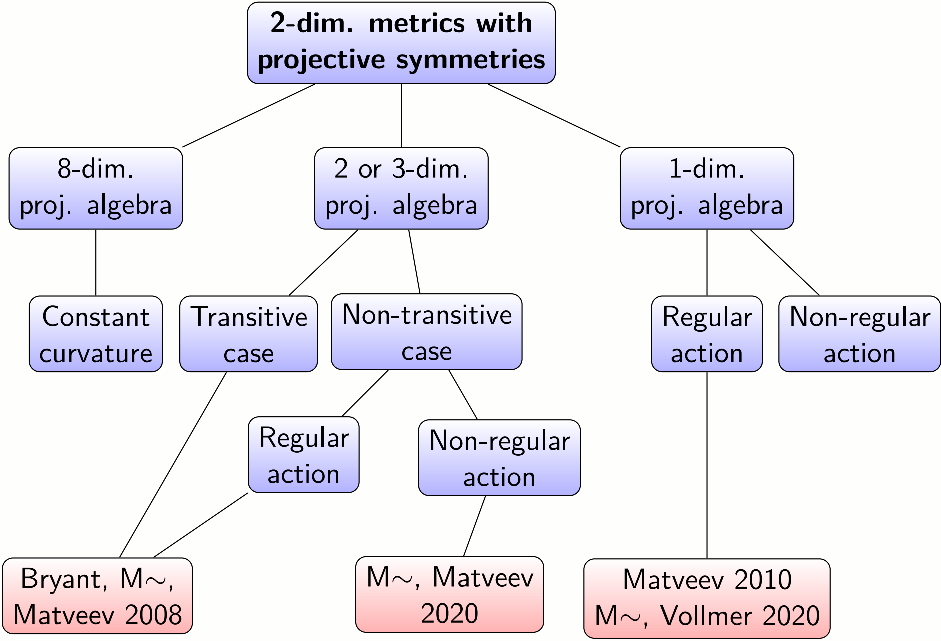
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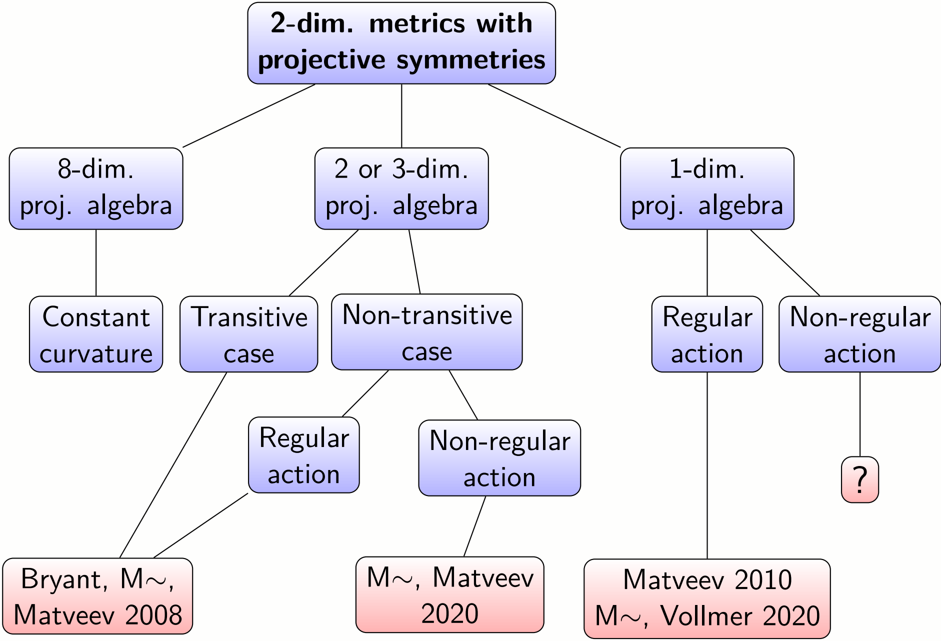
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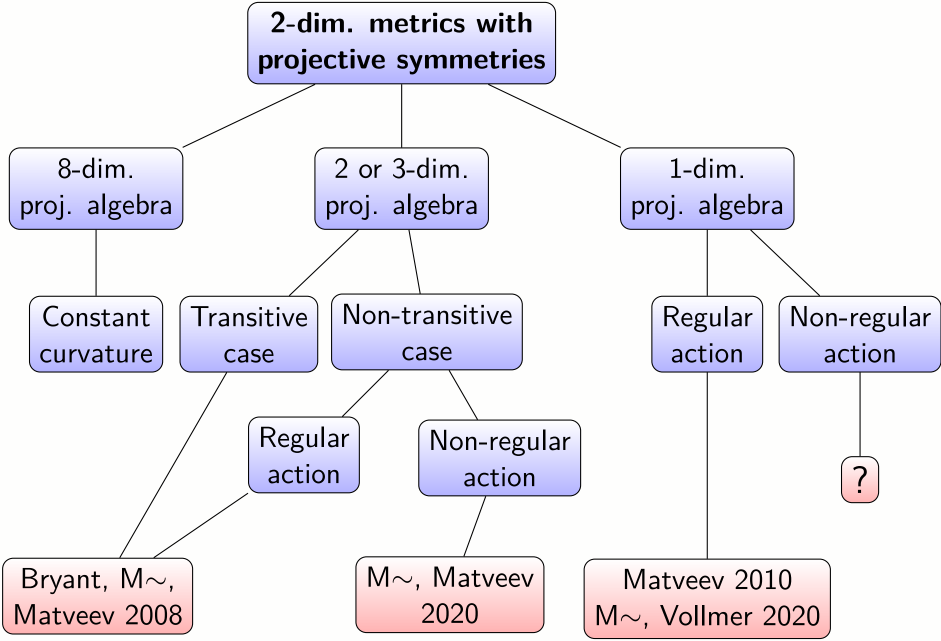
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We shall give more details later.

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5. 3-dimensional case.
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Step 1: the Lie problem as a symmetry problem of ODEs

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Equation of geodesics:

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$$y'' = F_0(x,y) + F_1(x,y)y' + F_2(x,y)y'^2 + F_3(x,y)y'^3 \tag{2}$$

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$$\left(\begin{array}{c} \textbf{Existence} \\ \textbf{of a metric} \\ g = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2 \end{array} \right) \Leftrightarrow \left(\begin{array}{c} \textbf{Solvability of the} \\ \textbf{following system} \\ \textbf{(Metrizability Equations)} \end{array} \right)$$

$$\left. \begin{array}{rcl} F_0 & = & \frac{1}{2\det(g)}\left(g_{12}g_{11x} - 2g_{11}g_{12x} + g_{11}g_{11y}\right) \\ F_1 & = & \frac{1}{2\det(g)}\left(3g_{12}g_{11y} - 2g_{11}g_{22x} + g_{22}g_{11x} - 2g_{12}g_{12x}\right) \\ F_2 & = & \frac{1}{2\det(g)}\left(2g_{12}g_{12y} - 3g_{12}g_{22x} - g_{11}g_{22y} + 2g_{22}g_{11y}\right) \\ F_3 & = & \frac{1}{2\det(g)}\left(2g_{22}g_{12y} - g_{22}g_{22x} - g_{12}g_{22y}\right) \end{array} \right\}$$

Step 2: metrizability of projective connections

$$y'' = -\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12})y' - (\Gamma^2_{22} - 2\Gamma^1_{12})y'^2 + \Gamma^1_{22}y'^3 \qquad (5)$$

$$y'' = F_0(x,y) + F_1(x,y)y' + F_2(x,y)y'^2 + F_3(x,y)y'^3 \qquad (6)$$

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Non–linear!

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Theorem (Eastwood, Matveev (2009))

An N-dim. metric g lies in a given projective class if and only if the following system has solution

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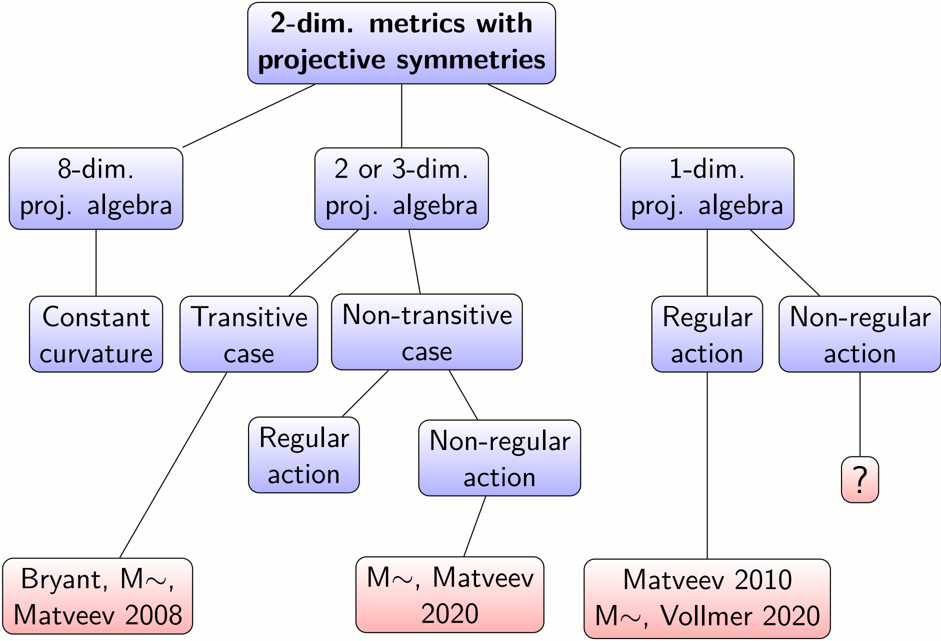
Theorem (Bryant, Eastwood, Dunajski (2009))

There exists a differential invariant of order six which decides if a 2-dimensional projective connection is metrizable.


2-dim. metrics with projective symmetries
8-dim. proj. algebra
2 or 3-dim. proj. algebra
1-dim. proj. algebra
Constant curvature
Transitive case
Non-transitive case
Regular action
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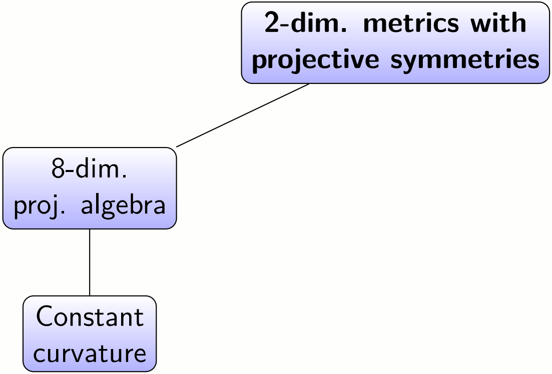
?
Bryant, M∼, Matveev 2008
M∼, Matveev 2020
Matveev 2010 M∼, Vollmer 2020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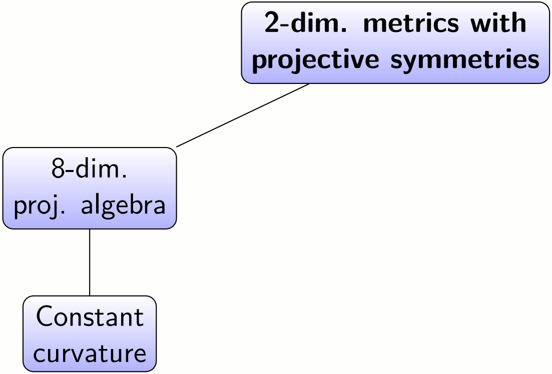

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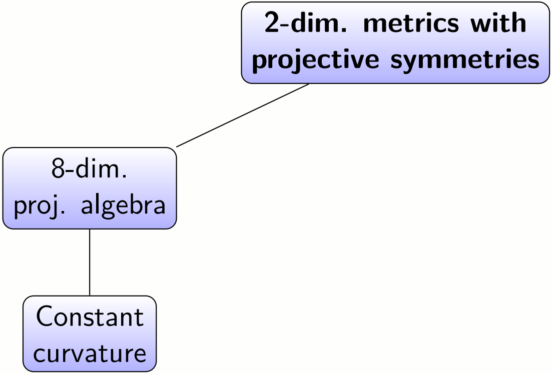
$$y'' = 0$$



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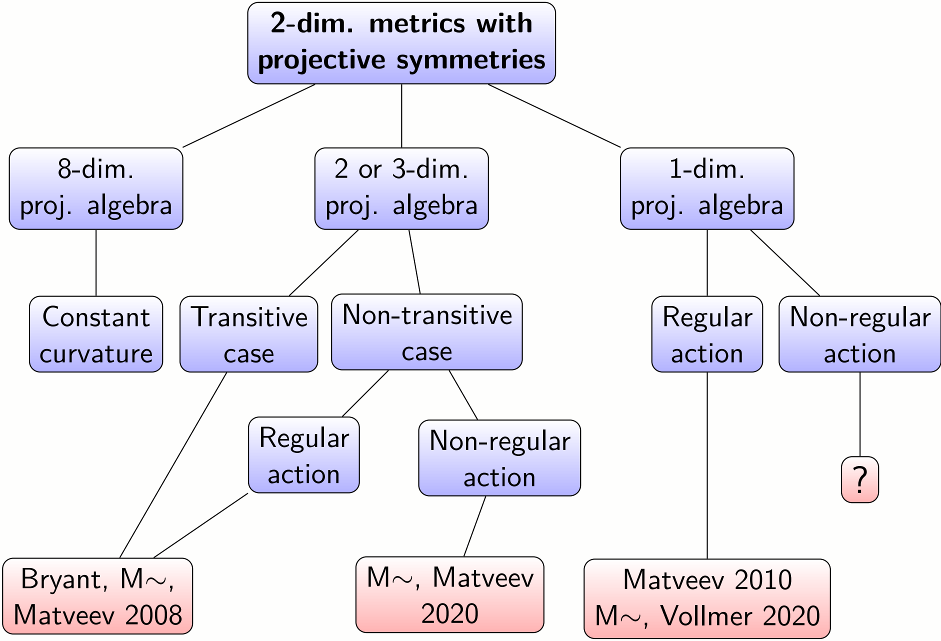
and any metric admitting the previous equation as the equation of unparametrized geodesics (projective connection) has constant curvature.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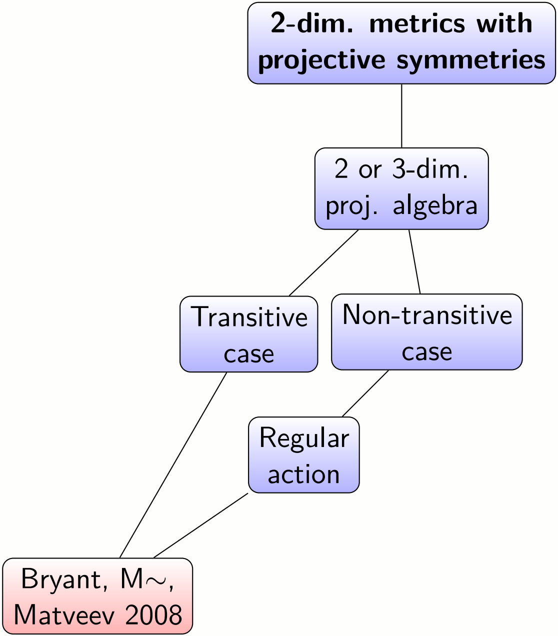
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Normal forms of projective connection with 2 symmetries

Recall how to solve the Lie problem.

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(a) Lie algebras which can be the symmetry algebras for a projective connection (Lie, 1882).

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(b) All possible realizations of the 2-dim. non commutative Lie algebra $\mathfrak{s}$ as vector fields on $\mathbb{R}^2$ (Lie 1882) .

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We now focus on the case when the projective algebra $[X, Y] = X$ is realized as follows:

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Reformulation of Lie problem: Find all metrics with projective connection

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by studying the (linear) metrizability equations with the help of the following

Theorem (Koenigs)

Let $\mathcal{I}(g) := \{$quadratic integrals of the geodesic flow of $g\}$. Then $\dim \mathcal{I}(g) \in \{1, 2, 3, 4, 6\}$ and $\dim \mathcal{I}(g) = 6 \longleftrightarrow g$ is of constant curvature.

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we obtain the following

Theorem

The projective connection (9) *comes from a metric iff*

$$\boxed{y'' = C_1 y' \pm e^{-2x} y'^3}$$

Metrics with at least 2 projective vector fields: regular case

Theorem (R. Bryant, G. Manno, V. Matveev (2008))

Let a 2-dim. metric g of nonconstant curvature admit two projective vector fields which are linearly independent at $p \in D^2$.

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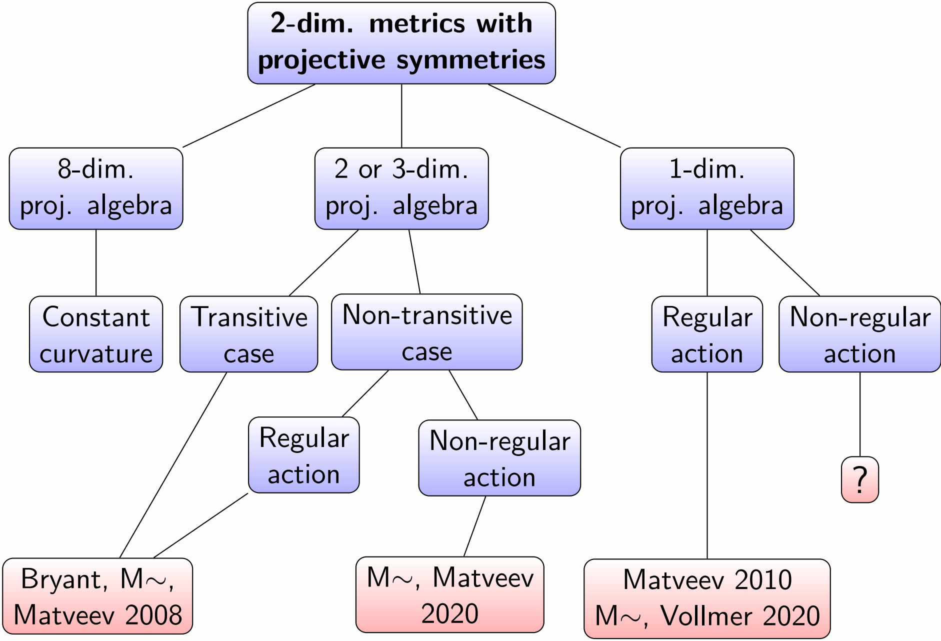
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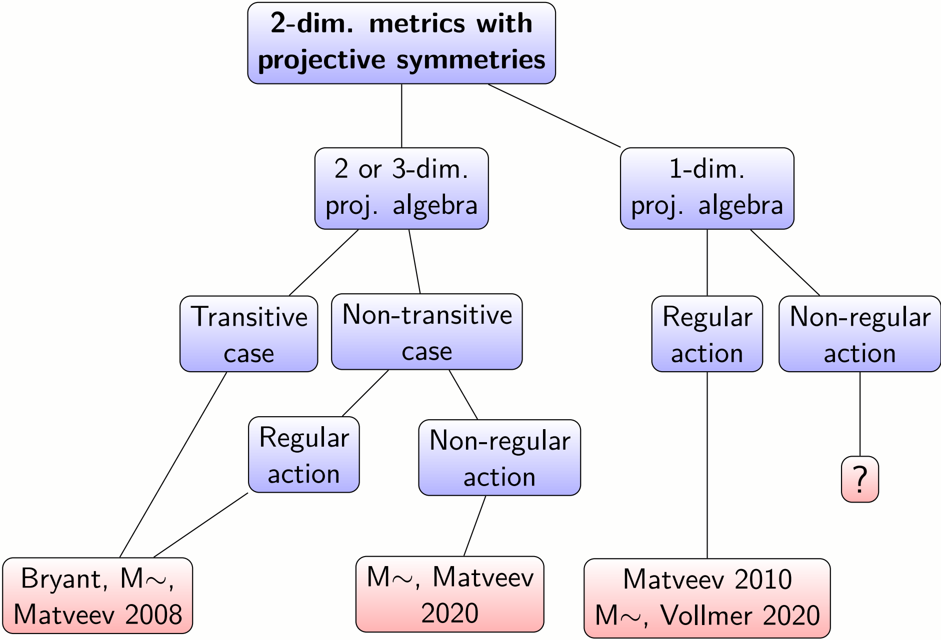
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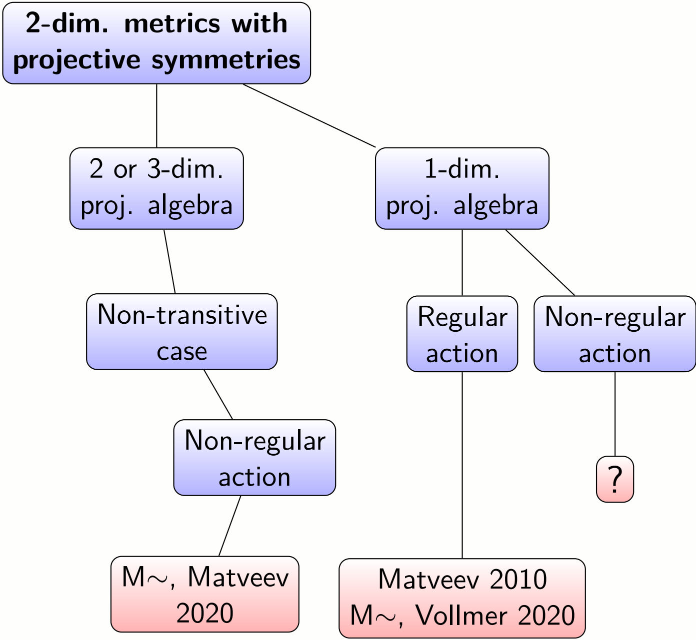

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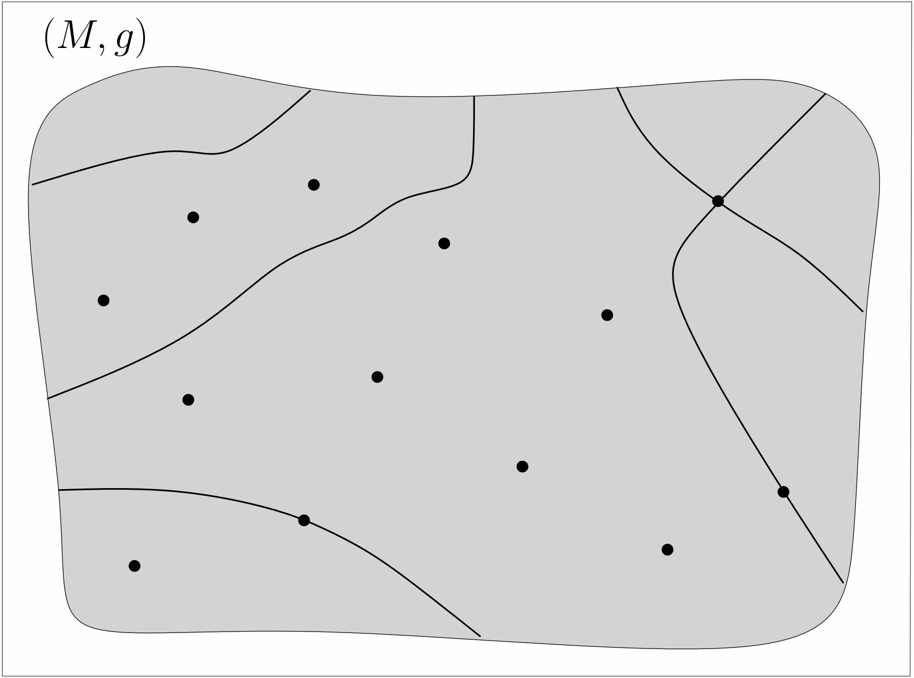
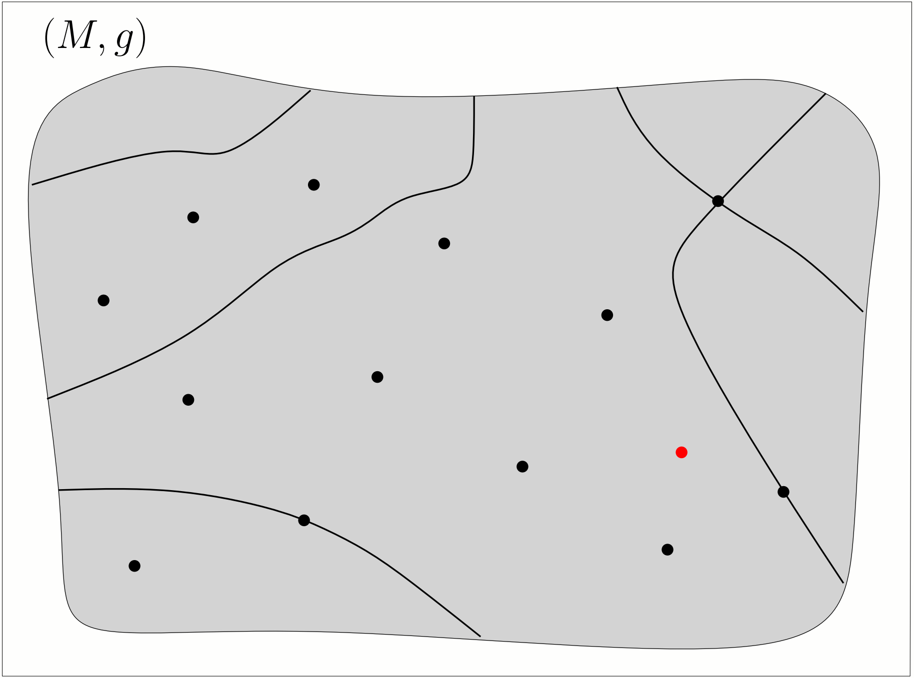

(M, g)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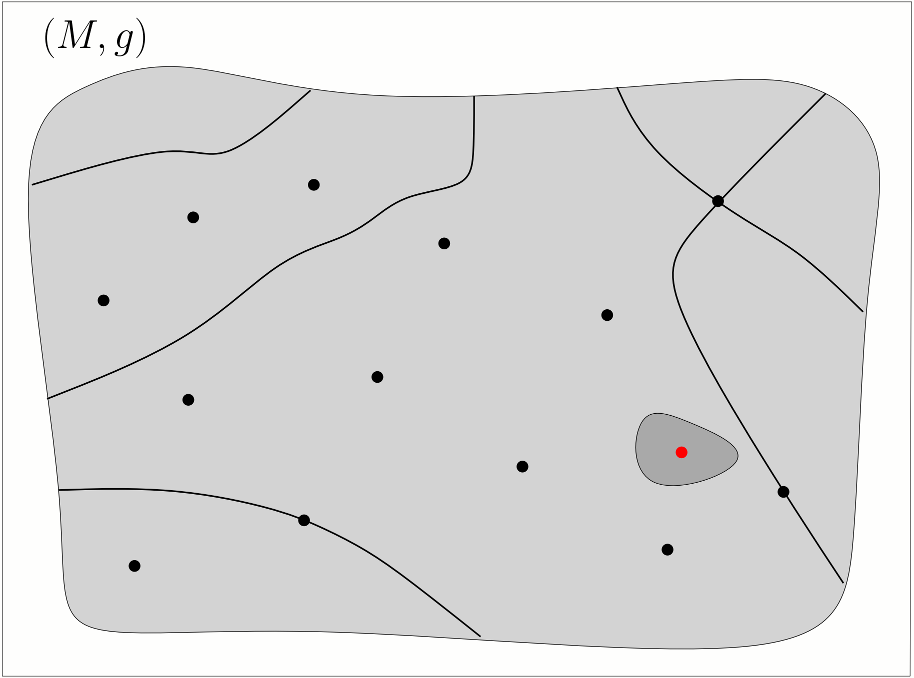
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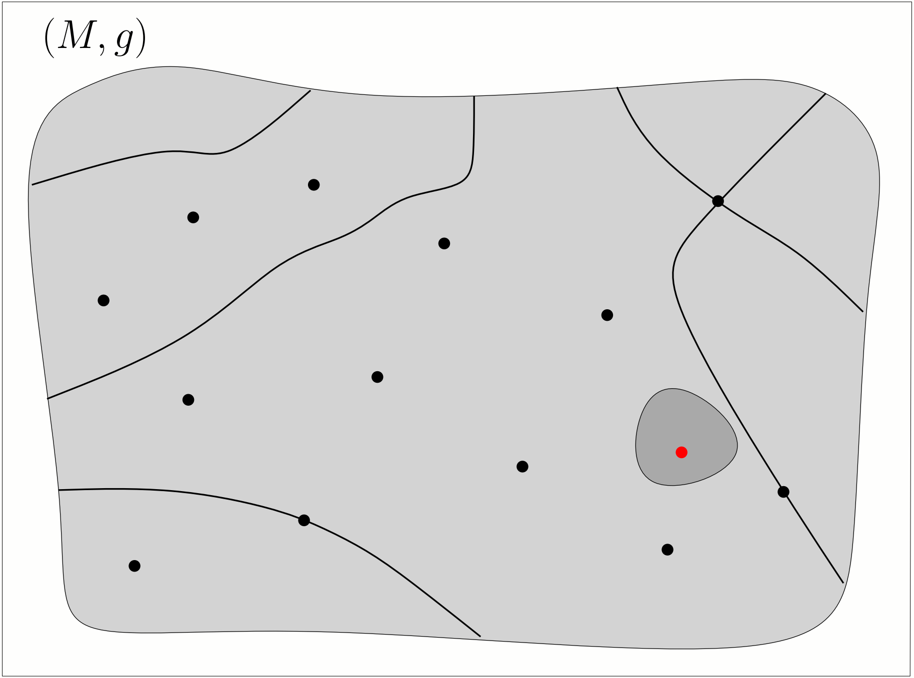
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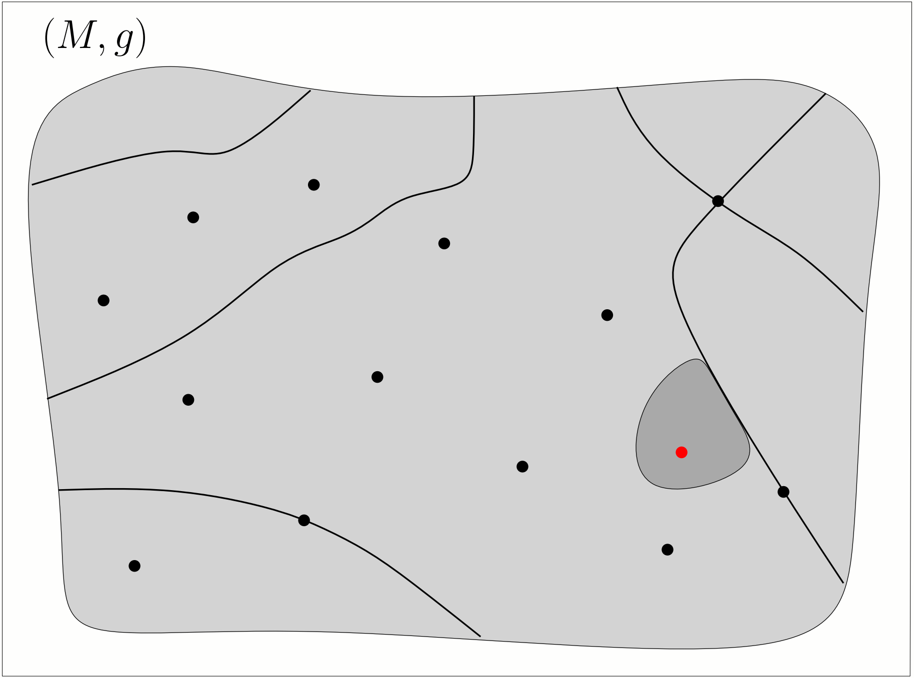
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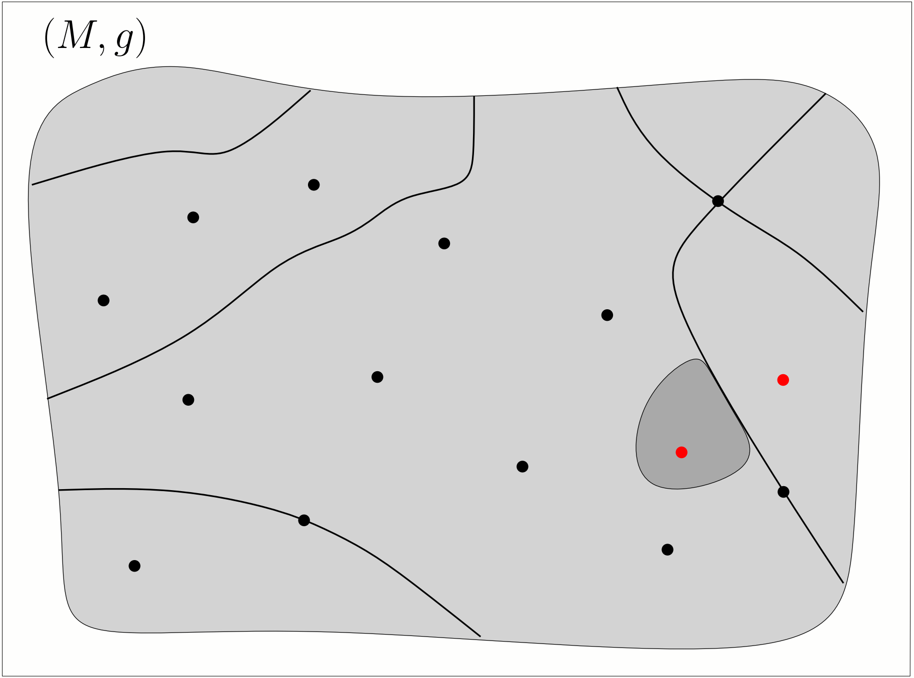
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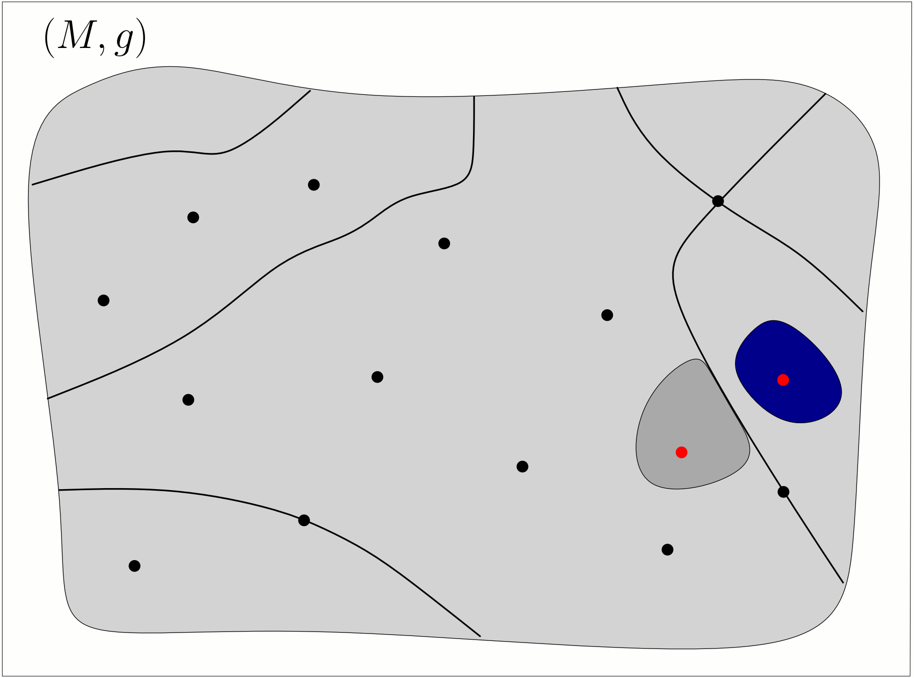
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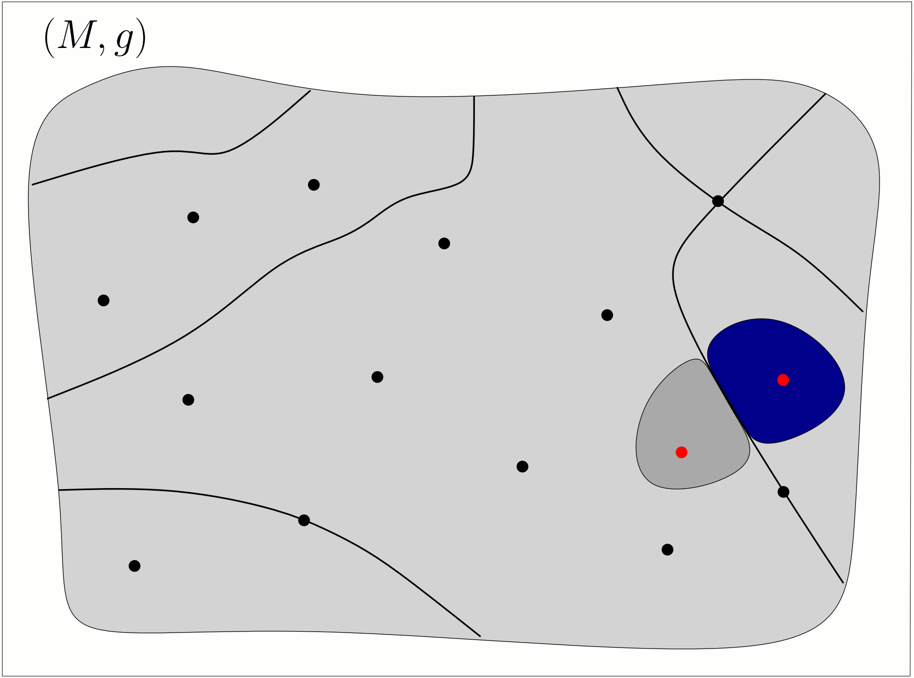
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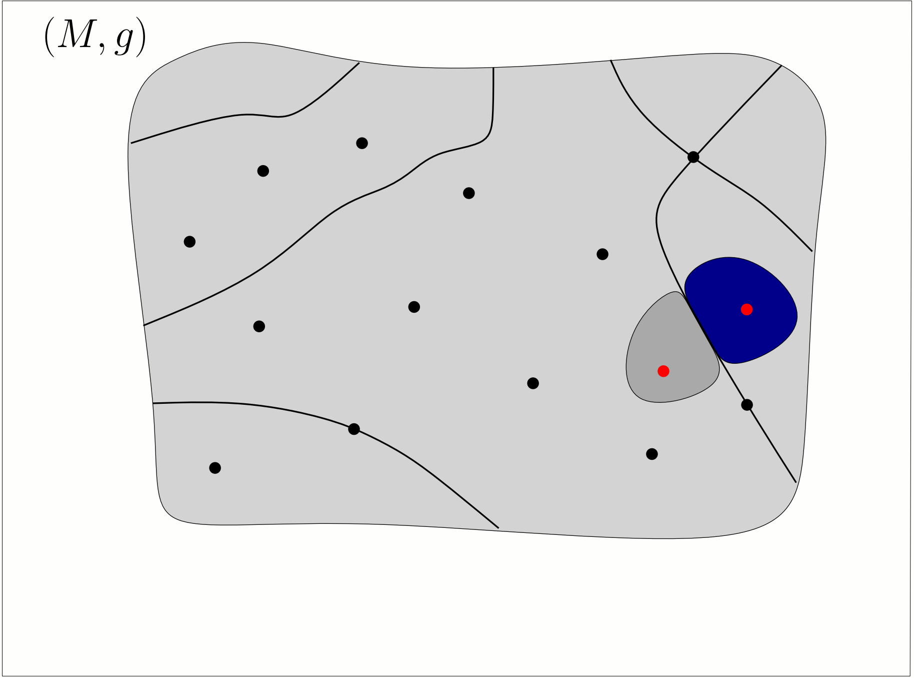
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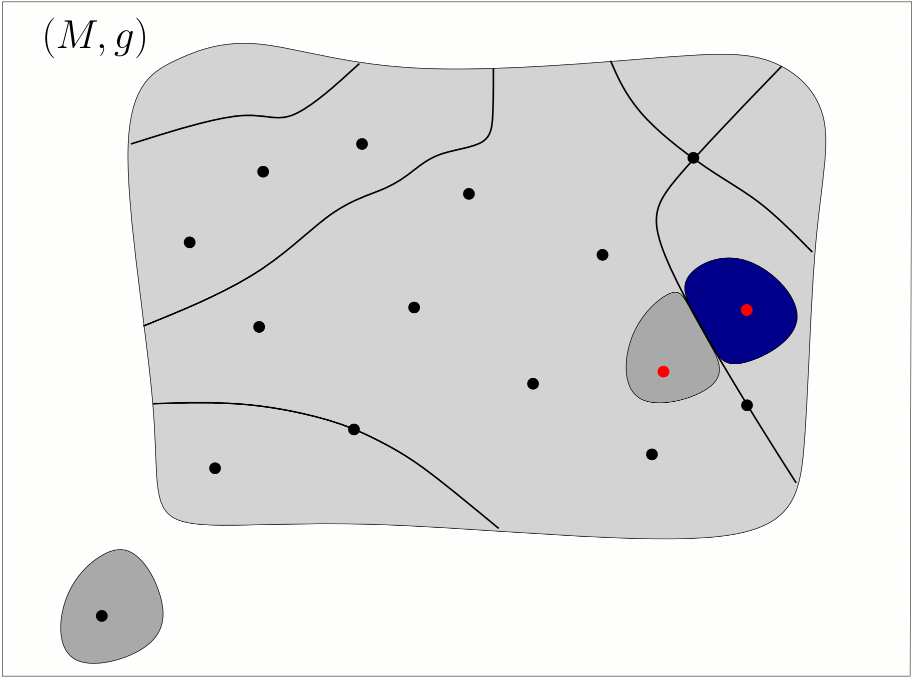
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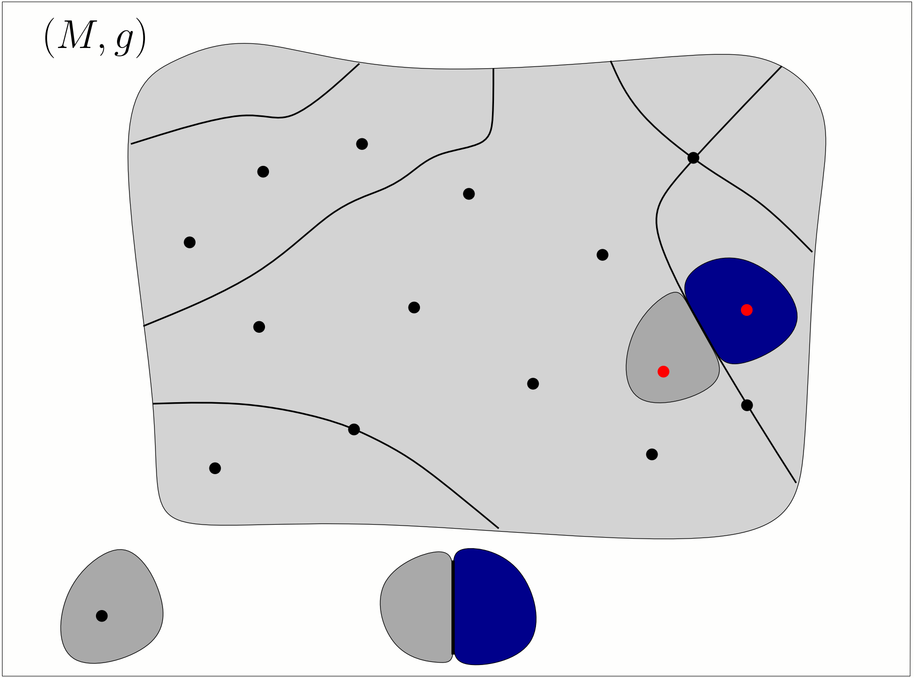
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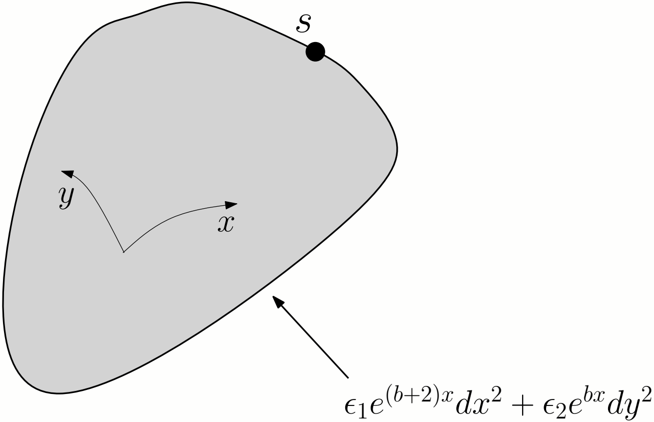
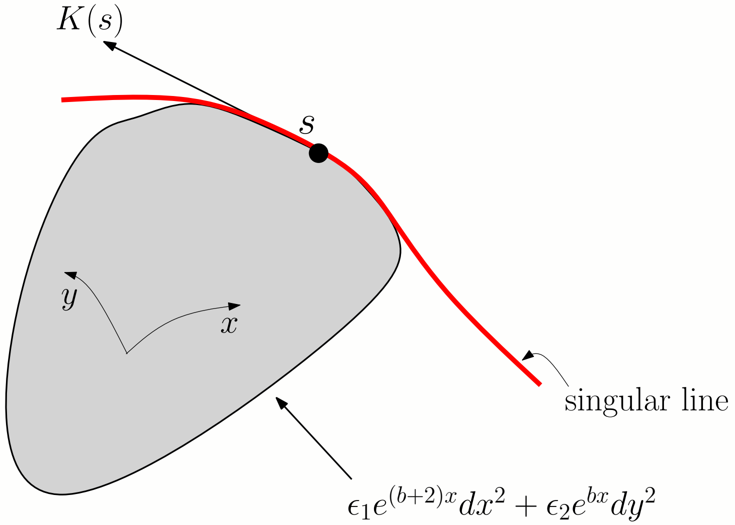

s
y
x
$\epsilon_1 e^{(b+2)x} dx^2 + \epsilon_2 e^{bx} dy^2$


$K(s)$
s
y
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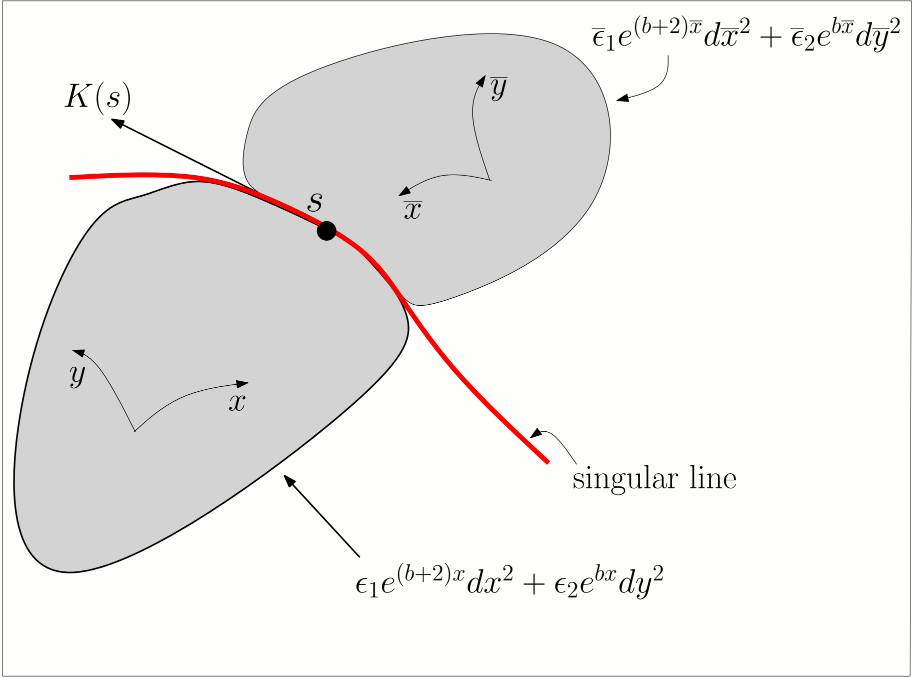
$\overline{\epsilon}_1 e^{(b+2)\overline{x}} d\overline{x}^2 + \overline{\epsilon}_2 e^{b\overline{x}} d\overline{y}^2$
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s
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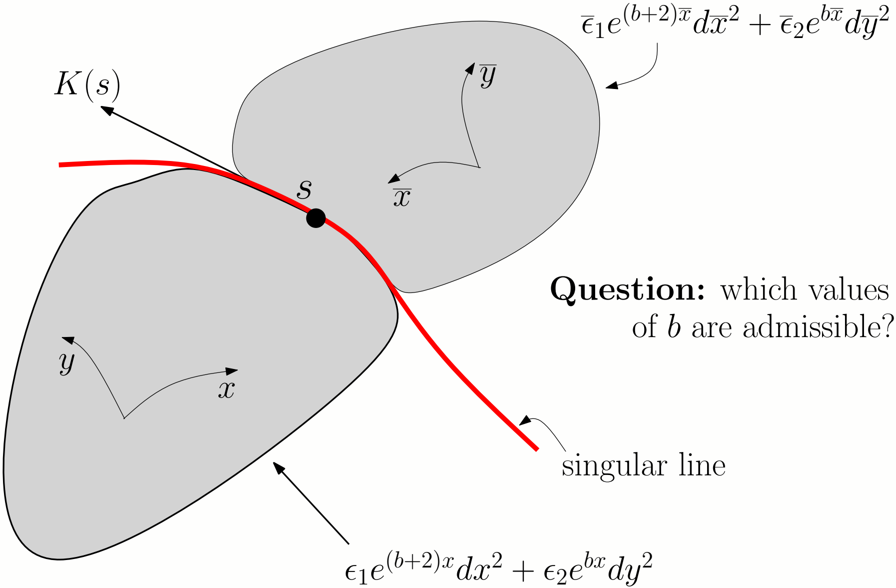
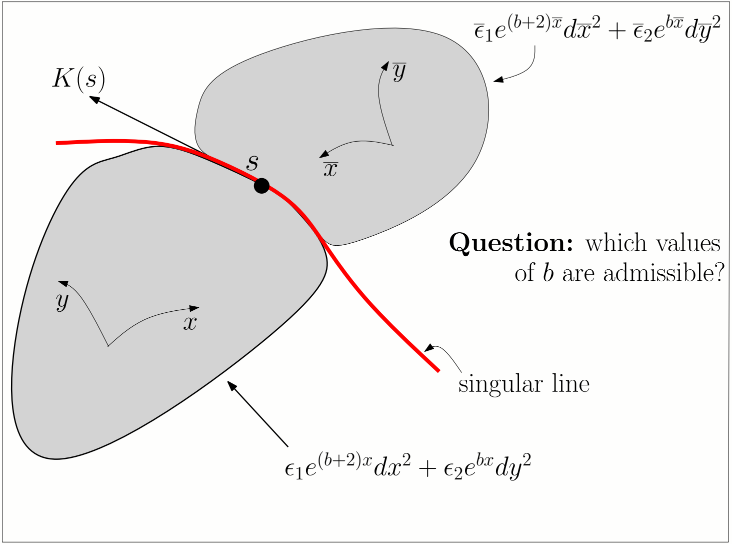


Question: which values of b are admissible?



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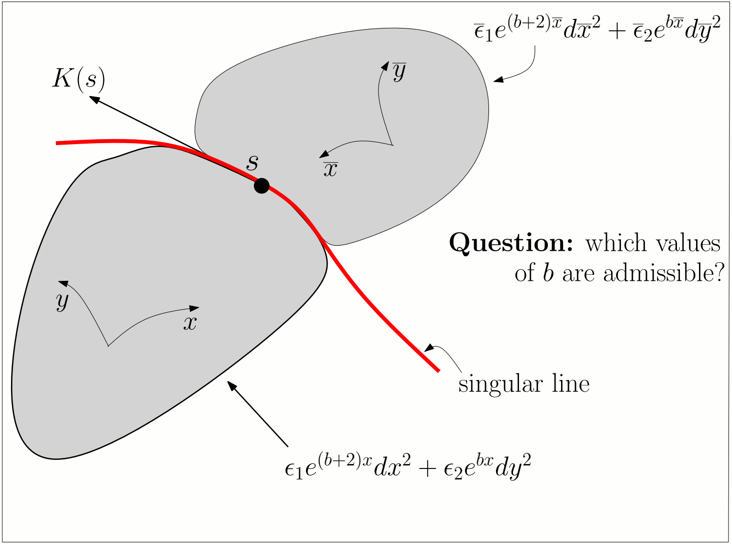
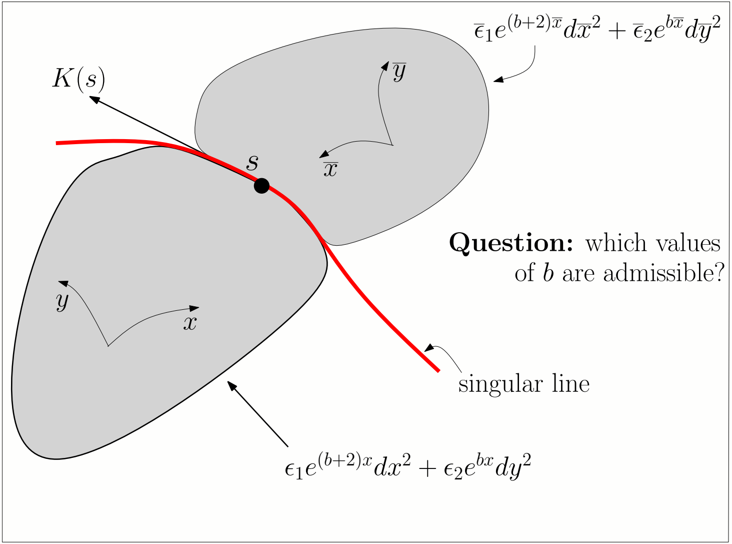
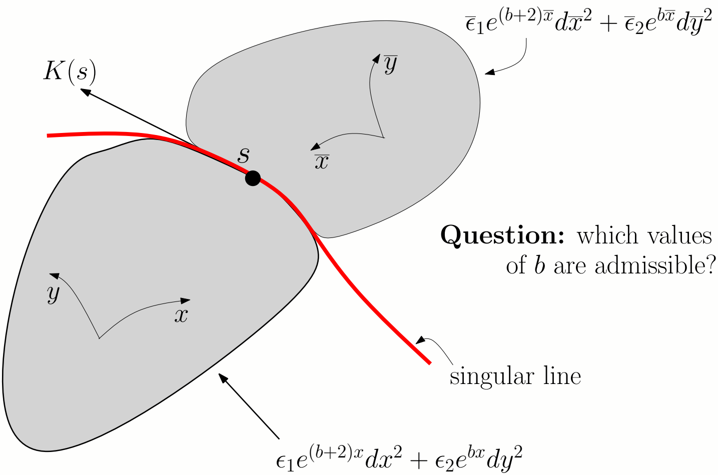


By studying the behavior of the Gaussian curvature R, $g^{-1}(dR, dR)$, $\Delta(R)$



By studying the behavior of the Gaussian curvature R, $g^{-1}(dR, dR)$, $\Delta(R)$ we realize that the only possibility is that $-2 < b < -1$.



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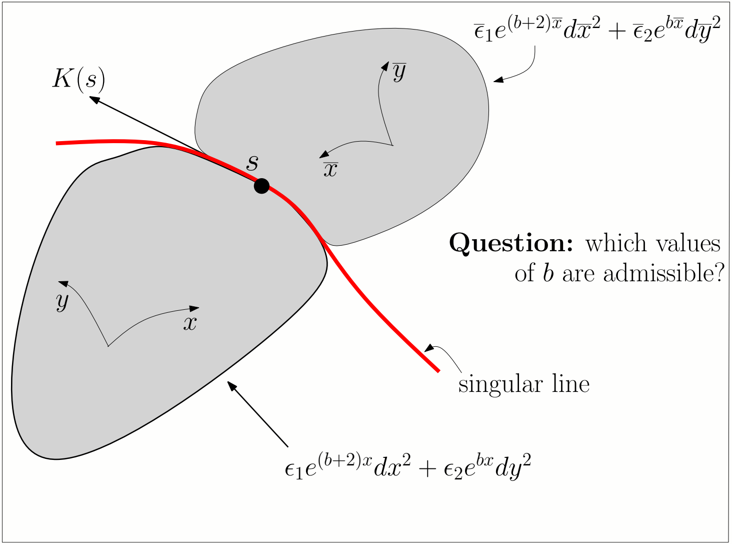

$$g = (1 + yx^h)^{-\frac{h+1}{h}} dxdy\,, \qquad K = x^{h+1}\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}\,, \quad H = -x\frac{\partial}{\partial x} + hy\frac{\partial}{\partial y}$$

Metrics with at least 2 projective vector fields: non-regular case

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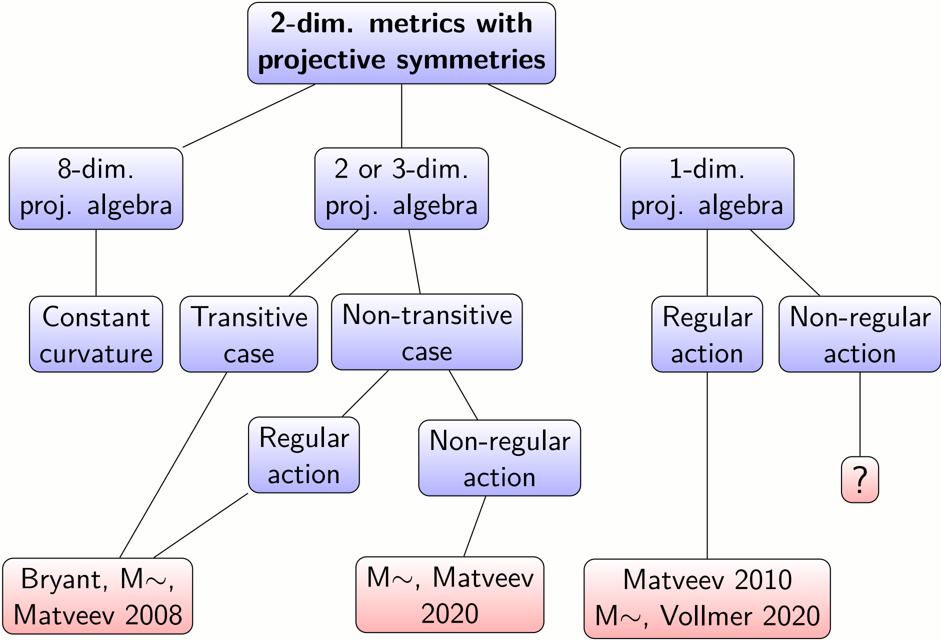
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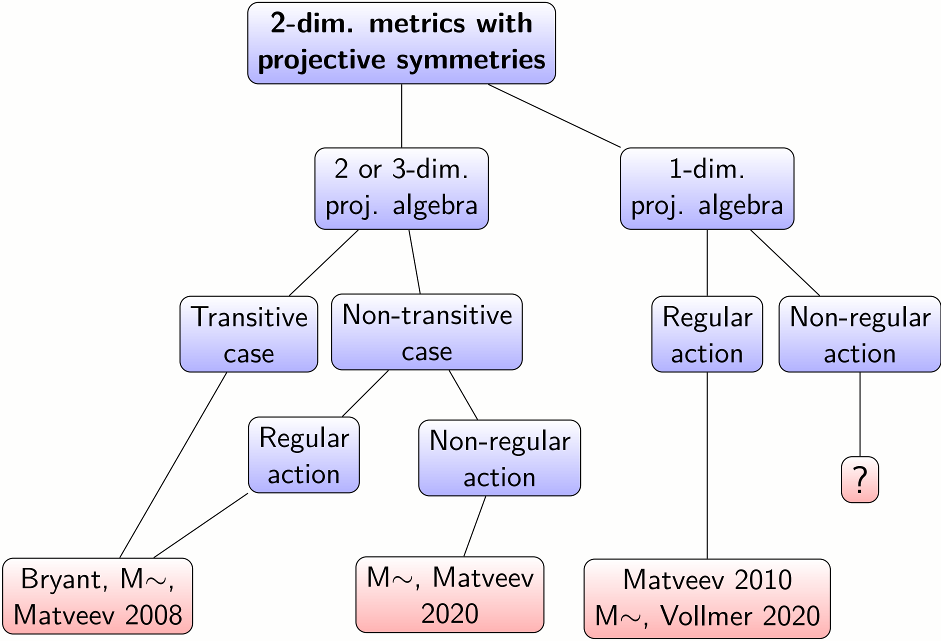
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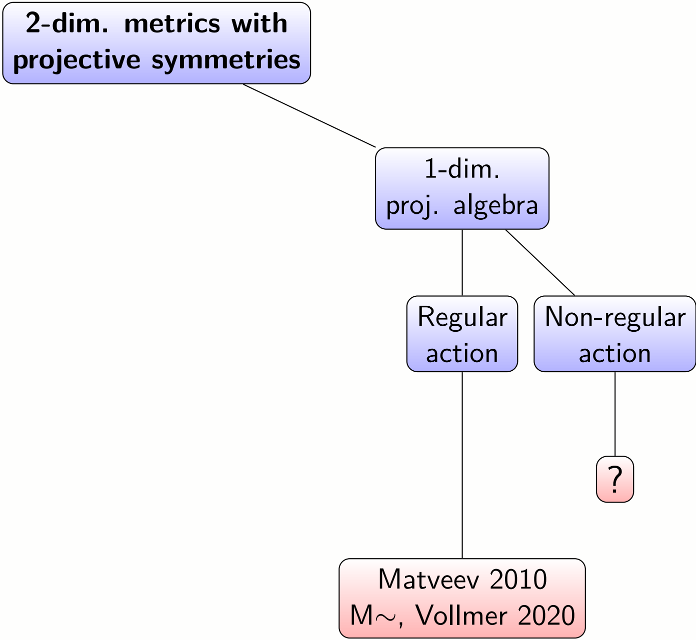
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Metrics with only one projective symmetry

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$$\nabla_a \sigma^{bc} - \frac{1}{3}(\delta_a^c \nabla_i \sigma^{ib} + \delta_a^b \nabla_i \sigma^{ic}) = 0 \,, \quad \sigma^{ij} := \det(g)^{\frac{1}{3}} g^{ij} \,. \tag{10}$$

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$$\sigma_1\,,\ \sigma_2 \text{ solutions of } (10) \ \Rightarrow\ K_1\sigma_1 + K_2\sigma_2 \ \text{ are solution of } (10)$$

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that is any metric of the form $g[K_1, K_2]$ is projectively equivalent to g_1 (and g_2).

Metrics with only one projective symmetry

We have already seen that for solving such problem it is crucial to solve the "Metrizability Equations"

$$\nabla_a \sigma^{bc} - \frac{1}{3}(\delta_a^c \nabla_i \sigma^{ib} + \delta_a^b \nabla_i \sigma^{ic}) = 0\,, \quad \sigma^{ij} := \det(g)^{\frac{1}{3}} g^{ij}\,. \tag{11}$$

$K_1\sigma_1 + K_2\sigma_2$ are solutions and from them we obtain the following metrics

$$g[K_1, K_2] = \frac{K_1 \det(g_1)^{-\frac{2}{3}} g_1 + K_2 \det(g_2)^{-\frac{2}{3}} g_2}{\det(K_1 \det(g_1)^{-\frac{2}{3}} g_1 + K_2 \det(g_2)^{-\frac{2}{3}} g_2)^2}$$

that is any metric of the form $g[K_1, K_2]$ is projectively equivalent to g_1 (and g_2).

Key observation: The flow of (the unique up to a constant) projective vector field send a metric $g[K_1, K_2]$ in one of the same form.

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Key observation: The flow of (the unique up to a constant) projective vector field send a metric $g[K_1, K_2]$ in one of the same form.
We can then distinguish non-isometric metrics by studying some invariants, for instance the length of the projective vector field along its flow.

Example

The metric

$$g = k\frac{y-x}{xy}\left(\frac{e^{-3x}}{x}dx^2 + h\frac{e^{-3y}}{y}dy^2\right)$$

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The list is much more long, see

G. Manno, A. Vollmer: Normal forms of two-dimensional metrics admitting exactly one essential projective vector field, *J. Math. Pure Appl.*, **135** (2020), 26–82.

3-dimensional case

Theorem (Kiosak, Matveev (2010))

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Idea: If one has a simple local description of a pair g_1 and g_2 of 3-dimensional projectively equivalent metrics, then we have all by using the formula

$$g[K_1, K_2] = \frac{\left(K_1 \det(g_1)^{\frac{1}{4}} g_1^{-1} + K_2 \det(g_2)^{\frac{1}{4}} g_2^{-1}\right)^{-1}}{\det\left(K_1 \det(g_1)^{\frac{1}{4}} g_1^{-1} + K_2 \det(g_2)^{\frac{1}{4}} g_2^{-1}\right)}$$

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For metrics g_1 and g_2 such that the (1,1)–tensor

$$\left|\frac{\det(g_2)}{\det(g_1)}\right|^{\frac{1}{4}} g_2^{-1} g_1$$

is diagonalizable, we have a simple local description.

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Metrics g_1 and g_2 assume either the form

$$\begin{aligned} g_1 = {} & \pm (F_1 - F_2)(F_1 - F_3)\,(dx^1)^2 \pm (F_2 - F_1)(F_2 - F_3)\,(dx^2)^2 \\ & \pm (F_3 - F_1)(F_3 - F_2)\,(dx^3)^2 \\ g_2 = {} & \pm \frac{(F_1 - F_2)(F_1 - F_3)}{F_1^2 F_2 F_3}\,(dx^1)^2 \pm \frac{(F_2 - F_1)(F_2 - F_3)}{F_1 F_2^2 F_3}\,(dx^2)^2 \\ & \pm \frac{(F_3 - F_1)(F_3 - F_2)}{F_1 F_2 F_3^2}\,(dx^3)^2 \end{aligned}$$

where $F_i = F_i(x^i)$,

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where $F_i = F_i(x^i)$, or the form

$$g_1 = \zeta(z)\,(h \pm dz^2) \qquad g_2 = \frac{\zeta(z)}{Z(z)\,\rho^2}\left(\frac{h}{\rho} \pm \frac{dz^2}{Z(z)}\right)$$

where $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$, $h_{ij} = h_{ij}(x, y)$, and

$$\zeta(z) = Z(z) - \rho\,, \quad \rho \in \mathbb{R}\,.$$

Examples

The metric, for $k_i \neq 0$,

$$g = k_1 \left(\tfrac{1}{x} - \tfrac{1}{y}\right) \left(\tfrac{1}{x} - \tfrac{1}{z}\right) e^{2x}\, dx^2 + k_2 \left(\tfrac{1}{y} - \tfrac{1}{x}\right) \left(\tfrac{1}{y} - \tfrac{1}{z}\right) e^{2y}\, dy^2$$
$$+ k_3 \left(\tfrac{1}{z} - \tfrac{1}{x}\right) \left(\tfrac{1}{z} - \tfrac{1}{y}\right) e^{2z}\, dz^2$$

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The list of course is much longer, see
G. Manno, A. Vollmer: 3-dimensional Levi-Civita metrics with projective vector fields, `https://arxiv.org/abs/2110.06785`

Complex case

What about if we replace the concept of unparametrized geodesic with that of *J–planar curve*?

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More precisely, if (M, J, ∇) is a real $2n$-dimensional smooth manifold equipped with a complex structure J and a complex connection ∇, i.e., a torsion free affine connection such that $\nabla J = 0$, a *J-planar curve* is a regular curve $\gamma : I \subseteq \mathbb{R} \to M$ such that

$$\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma + \beta(t)J(\dot\gamma)$$

for some smooth functions $\alpha, \beta \in C^\infty(I)$.

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Promising results have been obtained with Jan Schumm and Andreas Vollmer.