Necessary conditions for super-intetegrability

Case of natural systems in constant curvature spaces

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- Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{X}_H(\boldsymbol{x}), \qquad \boldsymbol{x} \in M^{2n}, \qquad t \in \mathbb{C},$$

  where M^{2n} is a complex analytic symplectic manifold, and $H : M^{2n} \to \mathbb{C}$ is an analytic Hamiltonian.
- Particular solution $\mathbb{C} \ni t \mapsto \boldsymbol{\varphi}(t) \in M^{2n}$.
- Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \boldsymbol{X}'_H(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{y},$$

- Differential Galois group $\mathcal{G}$ of VE is an algebraic subgroup of $\mathrm{Sp}(2n, \mathbb{C})$.

Morales-Ramis Theorem

General systems

Theorem

If the system $\boldsymbol{X}_H$ is integrable in the Liouville sense with first integrals which are meromorphic in a neighbourhood of the phase curve $\Gamma \subset M^{2n}$ corresponding to $\varphi(t)$, then the differential Galois group $\mathcal{G}$ of VE is virtually Abelian.

What is it the differential Galois group of an equation?

Consider linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \boldsymbol{A}\cdot\boldsymbol{y}, \qquad \boldsymbol{y} \quad \text{where} \quad \boldsymbol{A} = [a_{ij}], \quad a_{ij} \in K, \tag{L}$$

$$\left(K, \frac{\mathrm{d}}{\mathrm{d}t}\right) \quad \text{differential field, e.g.,} \quad K = \mathbb{C}(t).$$

The Picard-Vessiot extension $F \supset K$ contains n linearly independent solutions of (L), i.e.,

$$\boldsymbol{y} = (y^1, \ldots, y^n) \in F^n$$

Fundamental matrix $\boldsymbol{Y} \in \mathrm{GL}(n, F)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y} = \boldsymbol{A}\cdot\boldsymbol{Y} \tag{1}$$

What is it the differential Galois group of an equation?

Definition

The differential Galois group $\mathcal{G} := \mathrm{Gal}(F/K)$ of Picard-Vessiot $F \supset K$ is the group of differential automorphism $\sigma : F \to F$, such that $\sigma_{|K} = \mathrm{Id}_K$.

For $\sigma \in \mathcal{G}$ we have $\boldsymbol{M}_\sigma \in \mathrm{GL}(n, C)$,

$$\mathcal{G} \ni \mapsto \boldsymbol{M}_\sigma \in \mathrm{GL}(n, C), \qquad \sigma(\boldsymbol{Y}) = \boldsymbol{Y} \cdot \boldsymbol{M}_\sigma$$

Theorem

Group $\mathrm{Gal}(F/K)$ *is a linear algebraic subgroup of* $\mathrm{GL}(n, C)$.

Lie algebra of a differential Galois group.

An element $\boldsymbol{X} \in \mathfrak{g} \subset \mathbb{M}(n, C)$ can be consider as a linear vector field on C^n

$$C^n \ni \boldsymbol{x} \longmapsto \boldsymbol{X} \cdot \boldsymbol{x} \in C.$$

1. If a holomorphic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^n, \tag{N}$$

   has independent holomorphic first integrals $F_1, \ldots, F_m$, then the variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{y}, \tag{VE}$$

   have polynomial independent first integrals $f_1, \ldots, f_m$, which are
2. invariants of the differential Galois group G of VE, and this implies that
3. $f_1, \ldots, f_m$ are first integrals of each $\boldsymbol{X} \in \mathfrak{g}$.

Basic implications

Hamiltonian case

1. If a holomorphic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{X}_H(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^n, \tag{H}$$

   has independent commuting holomorphic first integrals $F_1, \ldots, F_m$, then the variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \boldsymbol{X}'_H(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{y}, \tag{VEH}$$

   have polynomial commuting and independent first integrals $f_1, \ldots, f_m$, which are
2. invariants of the differential Galois group G of VE, and this implies that
3. $f_1, \ldots, f_m$ are commuting first integrals of each $\boldsymbol{X} \in \mathfrak{g}$.

Key Lemma

Lemma

If Lie algebra $\mathfrak{g} \subset \mathrm{sp}(2n, \mathbb{C})$ *admits* n *independent and commuting first integrals then it is Abelian.*

Normal variational equations

We can reduce VEH by one degree of freedom using variational energy integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z} = \boldsymbol{A}_{\mathrm{N}} \cdot \boldsymbol{z}, \qquad \boldsymbol{z} \in \mathbb{C}^{2(n-1)} \tag{NVE}$$

The differential Galois group of (NVE) is a subgroup of $\mathrm{Sp}(2(n-1), \mathbb{C})$.

Our main theorem

Theorem (I)

Assume that a holomorphic Hamiltonian system with n degrees admits $2n-1$ first integrals which are meromorphic in a neighbourhood U of a phase curve Γ and independent in $U \setminus \Gamma$. Then the Lie algebra $\mathfrak{g}_{\mathrm{N}}$ of the differential Galois group $\mathcal{G}_{\mathrm{N}}$ of the normal variational equations along Γ is the zero algebra, i.e., $\mathcal{G}_{\mathrm{N}}$ is a finite subgroup of $\mathrm{Sp}(2n-2,\mathbb{C})$.

Proof

- $\mathcal{G}_{\mathrm{N}}$ admits $2n-2$ independent rational first integrals $f_1, \ldots, f_{2n-2}$;

Proof

- $\mathcal{G}_{\mathrm{N}}$ admits $2n-2$ independent rational first integrals $f_1, \ldots, f_{2n-2}$;
- for each $Y \in \mathfrak{g}_{\mathrm{N}} \subset \mathrm{sp}(2n-2, \mathbb{C})$, $Y(f_i) = 0$ for $i = 1, \ldots, 2n-2$, thus $Y = 0$.

Natural systems with homogeneous potentials

Hamiltonian function

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}),$$

where $V(\boldsymbol{q})$ is a homogeneous function of degree $k \in \mathbb{Z}^{\times}$.

Darboux Points and Particular Solutions

Assumption

Potential $V \in \mathbb{C}(\boldsymbol{q})$ is homogeneous and $\deg V = k \in \mathbb{Z}^{\star}$.

Definition

A proper Darboux point $\boldsymbol{d} \in \mathbb{C}^n$ of potential $V(\boldsymbol{q})$ is a solution of

$$V'(\boldsymbol{d}) = \boldsymbol{d}, \qquad \boldsymbol{d} \neq \boldsymbol{0}.$$

Particular solution

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \quad \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d}, \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

Phase curve Γ_{ε}:

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right)$$

Variational equations

$$\ddot{\boldsymbol{x}} = -\varphi(t)^{k-2} V''(\boldsymbol{d})\boldsymbol{x}.$$

If $V''(\boldsymbol{d})$ is diagonalisable, then in an appropiate base

$$\ddot{y}_i = -\lambda_i \varphi(t)^{k-2} y_i, \qquad 1 \leq i \leq n, \tag{2}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $V''(\boldsymbol{d})$. One of these eigenvalues, let us say λ_n is $k-1$.

Differential Galois group

$$\mathcal{G} \subset \mathcal{G}(\lambda_1) \times \cdots \times \mathcal{G}(\lambda_n) \subset \mathrm{Sp}(2n, \mathbb{C}), \qquad \mathcal{G}(\lambda_i) \subset \mathrm{Sp}(2, \mathbb{C}).$$

and

$$\mathcal{G}_{\mathrm{N}} \subset \mathcal{G}(\lambda_1) \times \cdots \times \mathcal{G}(\lambda_{n-1}) \subset \mathrm{Sp}(2n-2, \mathbb{C}), \qquad \mathcal{G}(\lambda_i) \subset \mathrm{Sp}(2, \mathbb{C}).$$

Hence

$$\mathfrak{g} \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,$$

and

$$\mathfrak{g}_{\mathrm{N}} \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1},$$

where $\mathfrak{g}_i$ is a Lie subalgebra of $\mathrm{sp}(2, \mathbb{C})$, for $i = 1, \ldots, n$.

Transformation to hypergeometric equations

$$\ddot{\eta} = -\lambda\varphi(t)^{k-2}\eta,$$

$$\Gamma_\varepsilon : \qquad \varepsilon = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{k}\varphi^k.$$

Differential Galois group $\mathcal{G}(k,\lambda) \subset \mathrm{Sp}(2,\mathbb{C})$.
Yoshida transformation

$$z := \frac{1}{\varepsilon k}\varphi(t)^k.$$

$$\left.\begin{aligned} &z(1-z)\eta'' + [c-(a+b+1)z]\eta' - ab\eta = 0,\\ &a+b = \frac{k-2}{2k}, \qquad ab = -\frac{\lambda_i}{2k}, \qquad c = 1-\frac{1}{k}. \end{aligned}\right\} \qquad \text{(H)}$$

Differenial Galois group $G(k,\lambda) \subset \mathrm{GL}(2,\mathbb{C})$.

Properties of $G(k,\lambda)^\circ$

Fact

The identity component $\mathcal{G}(k,\lambda)^\circ$ is isomorphic to $G(k,\lambda)^\circ$.

Lemma

If $G(k,\lambda)^\circ$ is solvable then it is Abelian.

Lemma (K)

The identity component $G(k,\lambda)^\circ$ of the differential Galois group of hypergeometric equation (H) *is Abelian if and only if (k,λ) belong to the following list*

Properties $G(k,\lambda)^\circ$

case	k	λ
1.	± 2	λ
2.	k	$p + \frac{k}{2}p(p-1)$
3.	k	$\frac{1}{2}\left(\frac{k-1}{k} + p(p+1)k\right)$
4.	3	$-\frac{1}{24} + \frac{1}{6}(1+3p)^2, \quad -\frac{1}{24} + \frac{3}{32}(1+4p)^2$ $-\frac{1}{24} + \frac{3}{50}(1+5p)^2, \quad -\frac{1}{24} + \frac{6}{25}(1+5p)^2$
5.	4	$-\frac{1}{8} + \frac{2}{9}(1+3p)^2$

Properties $G(k,\lambda)^{\circ}$

case	k	λ	
6.	5	$-\frac{9}{40}+\frac{5}{18}\left(1+3p\right)^2,$	$-\frac{9}{40}+\frac{2}{5}\left(1+5p\right)^2$
7.	-3	$\frac{25}{24}-\frac{1}{6}\left(1+3p\right)^2,$	$\frac{25}{24}-\frac{3}{32}\left(1+4p\right)^2$
		$\frac{25}{24}-\frac{3}{50}\left(1+5p\right)^2,$	$\frac{25}{24}-\frac{6}{25}\left(1+5p\right)^2$
8.	-4	$\frac{9}{8}-\frac{2}{9}\left(1+3p\right)^2$	
9.	-5	$\frac{49}{40}-\frac{5}{18}\left(1+3p\right)^2,$	$\frac{49}{40}-\frac{2}{5}(1+5p)^2$

where p is an integer and λ an arbitrary complex number.

Theorem

Assume that the Hamiltonian a natural Hamiltonian system system with a homogeneous potential $V \in \mathbb{C}(\boldsymbol{q})$ of degree $k \in \mathbb{Z}^{\star}$ satisfies the following conditions:

1. *there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$, and*
2. *matrix $V''(\boldsymbol{d})$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n = k - 1$;*
3. *the system admits $2n - 1$ functionally independent first integrals $F_1 = H, F_2, \ldots, F_{2n-1}$ which are meromorphic in a connected neighbourhood of phase curve Γ_ε.*

Our Theorem

Theorem (continuation)

Then each (k, λ_i) belongs to the list from Lemma K, and moreover

- *if $|k| > 2$, then each pair (k, λ_i) for $1 \leq i \leq n-1$, belongs to items 3–9 of the table from Lemma K;*
- *if $|k| \leq 2$, then each pair (k, λ_i), for $1 \leq i \leq n-1$ belongs to the following list*

case	k	λ
I.	-2	$1 - r^2$
II.	-1	1
III.	1	0
IV.	2	r^2

(3)

where $r \in \mathbb{Q}^{\star}$;

Separable potential

$$V = Aq_1^k + Bq_2^k$$

- Darboux points $\boldsymbol{d}_1 = (0, (\frac{1}{Bk})^{1/(k-2)})$ and $\boldsymbol{d}_2 = ((\frac{1}{Ak})^{1/(k-2)}, 0)$ for $k \neq 2$; for $\boldsymbol{d} = (1, 0)$ and $\boldsymbol{d} = (0, 1)$
- non-trivial eigenvalues $\lambda(\boldsymbol{d}_i) = 0$ for $k \neq 2$; for $k = 2$ $\lambda(\boldsymbol{d}_1) = B/A$ and $\lambda(\boldsymbol{d}_2) = A/B$
- by our theorem, if V is maximally super-integrable, then either $k = -2$, or $k = 1$ or $k = 2$ and, in this last case, $A/B = r^2$ for $r \in \mathbb{Q}^\star$.

Radial potential

$$V = \alpha r^k, \quad r = \sqrt{q_1^2 + q_2^2}$$

- infintely many Darboux points
- non-trivial eigenvalue at each of them $\lambda(\boldsymbol{d}) = 1$. Thus, by our theorem, if V is superintegrable, then $k = -1$ or $k = 2$.

Three body problem

$$V = \frac{1}{k}\left[(q_1 - q_2)^k + (q_2 - q_3)^k + (q_3 - q_1)^k\right], \qquad k \in \mathbb{Z} \setminus \{0, 1\}$$

$$F_2 = p_1 + p_2 + p_3,$$

Lemma

Assume that $k \in \mathbb{Z} \setminus \{-2, 0, 1, 2, 4\}$. Then the potential V is not integrable by meromorphic first integrals in the Liouville sense.

- $k = 4$, one additional first integral F_3;

Lemma

Assume that $k \in \mathbb{Z} \setminus \{0, 1, -2\}$. Then the potential V is not maximally superintegrable by meromorphic first integrals.

Three body problem

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- $k = 4$, one additional first integral F_3;
- $k = 2$ two additional first integrals F_3 and F_4;

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Three body problem

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Lemma

Assume that $k \in \mathbb{Z} \setminus \{-2, 0, 1, 2, 4\}$. Then the potential V is not integrable by meromorphic first integrals in the Liouville sense.

- $k = 4$, one additional first integral F_3;
- $k = 2$ two additional first integrals F_3 and F_4;
- $k = -2$ three additional first integrals F_3, F_4 and F_5;

Lemma

Assume that $k \in \mathbb{Z} \setminus \{0, 1, -2\}$. Then the potential V is not maximally superintegrable by meromorphic first integrals.

$k = -2$

Potential

$$V(q_1, q_2) = V(r\cos\varphi, r\sin\varphi) = \frac{1}{r^2}U(\varphi).$$

A Darboux point is given by

$$(c_1, c_2) = c(\cos\varphi_0, \sin\varphi_0), \qquad U'(\varphi_0) = 0, \quad U(\varphi_0) \neq 0.$$

Non-trivial eigenvalue of $V''(\boldsymbol{c})$ is

$$\lambda = 1 + \frac{U''(\varphi_0)}{k\, U(\varphi_0)}, \qquad k = -2.$$

$k = -2$

Necessary condition for the super-integrability is $\lambda = 1 - s^2$, with $s \in \mathbb{Q}^\times$ Thus, from relation

$$\lambda = 1 + \frac{U''(\varphi_0)}{k\; U(\varphi_0)}, \qquad k = -2.$$

after setting

$$U(\varphi) = \frac{1}{[f(\varphi)]^2}.$$

we obtain the relation

$$f''(\varphi_0) = -s^2 f(\varphi_0),$$

If we assume that f satisfies this relation identically then we find two independent solutions for f

$$f_1(\varphi) = \cos(s\varphi), \quad f_2(\varphi) = \sin(s\varphi)$$

and therefore

$$U_1(\varphi) = \frac{1}{\cos^2(s\varphi)}, \quad U_2(\varphi) = \frac{1}{\sin^2(s\varphi)}.$$

$k = -2$

Hamiltonian system given by

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \frac{a}{r^2\cos^2(n\varphi)} = \frac{p_r^2}{2} + \frac{1}{r^2}G, \tag{4}$$

where G is

$$G = \frac{p_\varphi^2}{2} + \frac{a}{\cos^2(n\varphi)}. \tag{5}$$

is integrable. For $n \in \mathbb{Q}^\times$ it is super-integrable

$$F_1 = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2G)^{\frac{n-2k}{2}} \frac{p_r^{2k}}{r^{n-2k}} \left[p_\varphi \cos(n\varphi) + \frac{n-2k}{2k+1} r p_r \sin(n\varphi) \right]$$

$$F_2 = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2G)^{\frac{n-2k-1}{2}} \frac{p_r^{2k}}{r^{n-2k}} \left[2G\sin(n\varphi) - \frac{n-2k}{2k+1} r p_r p_\varphi \cos(n\varphi) \right].$$

Generalisation to a non-flat metric

We consider natural systems Hamiltonian systems defined by

$$H^{(\kappa)} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathrm{S}_\kappa^2(r)}\right) + V(r,\varphi).$$

where

$$\mathrm{C}_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ 1 & \text{for} \quad \kappa = 0, \\ \cosh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0, \end{cases} \tag{6}$$

$$\mathrm{S}_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ x & \text{for} \quad \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0. \end{cases} \tag{7}$$

These functions satisfy the following identities

$$\mathrm{C}_\kappa^2(x) + \kappa\,\mathrm{S}_\kappa^2(x) = 1, \quad \mathrm{S}_\kappa'(x) = \mathrm{C}_\kappa(x), \quad \mathrm{C}_\kappa'(x) = -\kappa\,\mathrm{S}_\kappa(x). \tag{8}$$

Generalisation

We consider potentials of the form

$$V^{(\kappa)}(r,\varphi) := \frac{1}{\mathrm{S}_\kappa^2(r)}U(\varphi). \tag{9}$$

These potentials are separable. In fact

$$G := \frac{1}{2}p_\varphi^2 + U(\varphi), \tag{10}$$

is a first integral of the system and we have also

$$H = \frac{1}{2}p_r^2 + \frac{1}{\mathrm{S}_\kappa^2(r)}G. \tag{11}$$

Generalisation

Theorem

Let potential $V^{(\kappa)}$ satisfies the following assumption: there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. If $V^{(\kappa)}$ is super-integrable, then

$$\lambda := 1 - \frac{1}{2}\frac{U''(\varphi_0)}{U(\varphi_0)} = 1 - s^2,$$

for a certain non-zero rational number s.

Case $\mathbb{S}^2$

Take $\kappa = 1$, and

$$V_n^{(\kappa)}(r, \varphi) := \frac{1}{\mathsf{S}_\kappa^2(r)} U(\varphi), \tag{12}$$

where

$$U(\varphi) = \frac{a}{\cos^2(n\varphi)} + \frac{b}{\sin^2(n\varphi)}.$$

If $n \in \mathbb{Q}^\times$, then this potential is super-integrable.

Case $\mathbb{S}^2$

Assume, for simplicity that $n \in \mathbb{N}^{\times}$, and $a, b \in \mathbb{R}$, explicit forms of first integrals

$$
\begin{aligned}
I_1 = & \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} (2G)^{n-j} \left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} \right)^{2n-2j-1} p_r^{2j} \\
& \times \left[G p_\varphi \sin(2n\varphi) \frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} - \frac{2(n-j)}{2j+1} p_r (G\cos(2n\varphi) + b - a) \right],
\end{aligned}
$$

$$
\begin{aligned}
I_2 = & \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} (2G)^{n-j} \left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} \right)^{2n-2j-1} p_r^{2j} \\
& \times \left[\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} (G\cos(2n\varphi) + b - a) + \frac{n-j}{2j+1} p_r p_\varphi \sin(2n\varphi) \right].
\end{aligned}
$$