On certain classes of integrable and superintegrable homogeneous potentials

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Outline

1. Monomial potentials
2. Direct method
3. Application of the direct method to monomial potentials. Results
4. Superintegrability and classical separation
5. Separation for systems with first integral of higher order in the momenta

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Systems with homogeneous potentials

$$H = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + V(\boldsymbol{q}),$$

$V(\boldsymbol{q})$ homogeneous of degree $k \in \mathbb{Z}^\star$

$$\dot{\boldsymbol{q}} = \boldsymbol{p}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial V}{\partial \boldsymbol{q}}$$

proper Darboux point

$$\operatorname{grad} V(\boldsymbol{d}) = \boldsymbol{d}.$$

One can easily generalise to the case

$$H = \frac{1}{2}\boldsymbol{p}^T M\boldsymbol{p} + V(\boldsymbol{q}),$$

M is a symmetric non-singular constant matrix

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Monomial potentials

Potential V is equivalent to W iff $V(\boldsymbol{q}) = W(A\boldsymbol{q})$, where A is $n \times n$ matrix satisfying $AA^T = \alpha\,\mathrm{Id}_n$ for a certain $\alpha \in \mathbb{C}^\star$.

Lemma

If a polynomial potential V of degree $k > 2$ does not have any proper Darboux point, then it is either equivalent to the following one

$$V_{k,l} = \alpha(q_2 - \mathrm{i}q_1)^{k-l}(q_2 + \mathrm{i}q_1)^l, \qquad \alpha \in \mathbb{C}^\star.$$

for some $l = 2, \ldots, k-2$, or $k = 2s$ and V has factor $(q_2 \pm \mathrm{i}q_1)$ with multiplicity s.

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Real form of monomial potentials – Drach-type potentials

$$z_1 = q_1 + \mathrm{i}q_2, \qquad z_2 = q_1 - \mathrm{i}q_2,$$
$$y_1 = \frac{1}{2}(p_1 - \mathrm{i}p_2), \quad y_2 = \frac{1}{2}(p_1 + \mathrm{i}p_2).$$

In these variables the Hamiltonian function has the form

$$H = 2y_1y_2 + z_1^l z_2^{k-l}.$$

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Direct method – first remarks

$$H = 2p_1p_2 + V(q_1, q_2), \quad \deg V = k$$

Hamilton equations

$$\dot{q}_1 = 2p_2, \qquad \dot{p}_1 = -\frac{\partial V}{\partial q_1},$$
$$\dot{q}_2 = 2p_1, \qquad \dot{p}_2 = -\frac{\partial V}{\partial q_2},$$

are invariant with respect to

$$t \to -t, \qquad q_i \to q_i, \qquad p_i \to -p_i, \qquad i = 1, 2.$$

Assumption that V is homogeneous of degree $k \Longrightarrow$ Hamilton equations are quasi-homogeneous with weights 2 for coordinates and k for momenta.

Direct method – form of the first integral

We can assume that a first integral I polynomial at momenta is

- either even or odd in the momenta (p_1, p_2),
- quasi-homogeneous of a certain weight-degree M, i.e.

$$I(\lambda^2 q_1, \lambda^2 q_2, \lambda^k p_1, \lambda^k p_2) = \lambda^M I(q_1, q_2, p_1, p_2).$$

For $M = 8$

$$I = \sum_{j=0}^{8} A_j(q_1, q_2) p_1^{8-j} p_2^j + \sum_{j=0}^{6} B_j(q_1, q_2) p_1^{6-j} p_2^j + \sum_{j=0}^{4} C_j(q_1, q_2) p_1^{4-j} p_2^j$$
$$+ \sum_{j=0}^{2} D_j(q_1, q_2) p_1^{2-j} p_2^j + F(q_1, q_2),$$

It depends on $(M+2)^2/4 = 25$ unknown functions of two variables.

Direct method – form of the first integral

Weight-homogeneity implies that functions $\{A_j\}_{j=0}^{8}$ must be homogeneous functions of variables q_1 and q_2 of the same order e.g i, similarly $\{B_j\}_{j=0}^{6}$, $\{C_j\}_{j=0}^{4}$, $\{D_j\}_{j=0}^{2}$ and F are homogeneous $\deg B_j = i + k$, $\deg C_j = i + 2k$, $\deg D_j = i + 3k$ and $\deg F = i + 4k$. Then $M = 8k + 2i$, where $i = 0, 1, \dots, 8$.

New coordinates $x = q_1$ and $z = q_2/q_1$. If $P(q_1, q_2)$ is a homogeneous function of degree k, then
$P(q_1, q_2) = q_1^k P(1, q_2/q_1) = x^k P(1, z) = x^k p(z)$, where
$p(z) := P(1, z)$.

Hamilton equations

$$\dot{x} = 2p_2, \qquad \dot{p}_1 = x^{k-1}(zv' - kv), \qquad ' = \frac{\mathrm{d}}{\mathrm{d}z}$$

$$\dot{z} = \frac{2}{x}(p_1 - zp_2), \qquad \dot{p}_2 = -x^{k-1}v'.$$

Direct method – transformed form

Change of time $\mathrm{d}t \to x^{-1}\mathrm{d}t$ in transformed Hamilton equations into

$$\dot{x} = 2xy_2, \qquad \dot{y}_1 = x^k(zv' - kv),$$
$$\dot{z} = 2(y_1 - zy_2), \qquad \dot{y}_2 = -x^k v'$$

The form of the first integral

$$I = x^i \sum_{j=0}^{8} a_j(z) y_1^{8-j} y_2^j + x^{i+k} \sum_{j=0}^{6} b_j(z) y_1^{6-j} y_2^j + x^{i+2k} \sum_{j=0}^{4} c_j(z) y_1^{4-j} y_2^j$$
$$+ x^{i+3k} \sum_{j=0}^{2} d_j(z) y_1^{2-j} y_2^j + x^{i+4k} f(z), \qquad i = 0, \ldots, 8.$$

$\dot{I} = 0$ yields $R = 0$, R polynomial in the momenta (y_1, y_2)

$$R = R_9 + R_7 + R_5 + R_3 + R_1 = 0,$$

$$R_9 = 2x^i \left\{ a_0' y_1^9 + (ia_8 - za_8') y_2^9 + \sum_{j=0}^{7} (ia_j + a_{j+1}' - za_j') y_1^{8-j} y_2^{j+1} \right\}$$

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Four families

$$v = z^d, \qquad d = k - l$$

1	$H_1 = 2p_1p_2 + q_1^d q_2^d$	$I_2 = q_1p_1 - q_2p_2$
2	$H_1 = 2p_1p_2 + \dfrac{q_2^d}{\sqrt{q_1}}$	$I_2 = 2p_1(q_2p_2 - p_1q_1) + \dfrac{q_2^{d+1}}{\sqrt{q_1}}$
3	$H_1 = 2p_1p_2 + q_1q_2^d$	$I_2 = p_1^2 + \dfrac{q_2^{d+1}}{d+1}$
4	$H_1 = 2p_1p_2 + \dfrac{q_2^d}{q_1^{d+2}}$	$I_2 = (p_1q_1 - p_2q_2)^2 - \dfrac{2q_2^{d+1}}{q_1^{d+1}}$

Superintegrability for some elements of family 2

$$V = q_1^{-1/2} q_2^2, \quad I = -18 p_1 p_2^3 + q_1^{-1/2}(48 p_1^2 q_1^2 - 72 p_1 p_2 q_1 q_2 - 9 p_2^2 q_2^2) - 32 q_2^3,$$

$$V = q_1^{-1/2} q_2^3, \quad I = 100 p_1^2 p_2^4 + q_1^{-1} q_2^4(-380 p_1^2 q_1^2 + 556 p_1 p_2 q_1 q_2 + 25 p_2^2 q_2^2) + 20 q_1^{-1/2} p_1 (16 p_1^3 q_1^3 - 40 p_1^2 p_2 q_1^2 q_2 + 30 p_1 p_2^2 q_1 q_2^2 + 5 p_2^3 q_2^3) + 128 q_1^{-1/2} q_2^8,$$

$$V = q_1^{-1/2} q_2^{-5/2}, \quad I = 4 p_1^3 (p_1 q_1 - p_2 q_2) - 6 p_1^2 q_1^{-1/2} q_2^{-3/2} + q_2^{-3},$$

$$V = q_1^{-1/2} q_2^{-7/2}, \quad I = 100 p_1^2 (p_1 q_1 - p_2 q_2)^4 - 20 q_1^{-1/2} q_2^{-5/2} p_1 (11 p_1^3 q_1^3 - 35 p_1^2 p_2 q_1^2 q_2 + 45 p_1 p_2^2 q_1 q_2^2 - 5 p_2^3 q_2^3) + q_1^{-1} q_2^{-5} (201 p_1^2 q_1^2 - 606 p_1 p_2 q_1 z_2 + 25 p_2^2 q_2^2) - 128 q_1^{-1/2} q_2^{-15/2},$$

Superintegrability for some elements of family 3

$$V = q_1 q_2^{-5/3}, \quad I = 2p_1^2(p_1q_1 - p_2q_2)^2$$
$$+ q_2^{-2/3}(p_1q_1 - p_2q_2)(27p_2q_2 - 11p_1q_1) + 32q_1^2q_2^{-4/3},$$
$$V = q_1 q_2^{-4/5}, \quad I = 896p_1^6p_2(p_1q_1 - p_2q_2) + 224q_2^{-4/5}p_1^4(2p_1^2q_1^2$$
$$+ 66p_1p_2q_1q_2 - 65p_2^2q_2^2) + 16q_2^{-3/5}p_1^2(476p_1^2q_1^2 + 5215p_1p_2q_1q_2$$
$$- 5025p_2^2q_2^2) + 5q_2^{-2/5}(9072p_1^2q_1^2 + 32920p_1p_2q_1q_2 - 30625p_2^2q_2^2)$$
$$+ 102400q_1^2q_2^{-1/5},$$
$$V = q_1 q_2^{-7/5}, \quad I = 4p_1^4(p_1q_1 - p_2q_2)^2 - 4p_1^2q_2^{-2/5}(11p_1q_1 - 15p_2q_2)$$
$$\times (p_1q_1 - p_2q_2) + q_2^{-4/5}(201p_1^2q_1^2 - 810p_1p_2q_1q_2 + 625p_2^2q_2^2)$$
$$- 640q_1^2q_2^{-6/5},$$

Superintegrability for some elements of family 4

$$V = q_1^2 q_2^{-4}, \quad I = 2p_1^3(p_1 q_1 - p_2 q_2) - q_2^{-3}(4p_1^2 q_1^2 + p_1 p_2 q_1 q_2 + p_2^2 q_2^2) + q_1^3 q_2^{-6}$$

$$V = q_1^3 q_2^{-5}, \quad I = 4p_1^4 + q_2^{-8}(q_1^4 - 6p_1^2 q_1^2 q_2^4 - 4p_1 p_2 q_1 q_2^5 - 2p_2^2 q_2^6),$$

$$V = q_1^{1/2} q_2^{-5/2}, \quad I = 4p_1^3(p_1 q_1 - p_2 q_2) - 6p_1^2 q_1^{1/2} q_2^{-3/2} + q_2^{-3},$$

$$V = q_1^{-1/3} q_2^{-5/3}, \quad I = 4p_1^2(p_1 q_1 - p_2 q_2)^2 + 2p_1(2p_2 q_2 - 3p_1 q_1) q_1^{-1/3} q_2^{-2/3} + q_1^{-2/3} q_2^{-4/3},$$

$$V = q_1^{-2/3} q_2^{-4/3}, \quad I = 2p_1(p_1 q_1 - p_2 q_2)^2 + q_1^{-2/3} q_2^{-1/3}(p_2 q_2 - 2p_1 q_1)$$

Other integrable potentials

$$V = q_1^2 q_2^5, \quad I = 16p_1^3(p_1q_1 - p_2q_2) + z_2^6(4p_1^2q_1^2 - 8p_1p_2q_1q_2 + p_2^2q_2^2) - q_1^3 q_2^{12},$$

$$V = q_1^2 q_2^{-7/4}, \quad I = p_1^6 + 8q_2^{-3/4}p_1^3(4p_2q_2 - p_1q_1) + 8q_2^{-3/2}(5p_1^2q_1^2 - 4p_1p_2q_1q_2 + 8p_2^2q_2^2) - 48q_1^3q_2^{-9/4},$$

$$I_1 = 2p_1^3(p_1q_1 - p_2q_2) - q_2^{-3/4}(13p_1^2q_1^2 - 80p_1p_2q_1q_2 + 64p_2^2q_2^2) + 64q_1^3q_2^{-3/2},$$

$$V = q_1^2 q_2^{-10/7}, \quad I = 4p_1^6 + 14q_2^{-3/7}p_1^3(7p_2q_2 - 4p_1q_1) + 343q_2^{-6/7}(p_1q_1 - 7p_2q_2)(p_1q_1 - p_2q_2) - 2058q_1^3q_2^{-9/7},$$

$$V = q_1^{-2/3} q_2^{-7/3}, \quad I = 8p_1(p_1q_1 - p_2q_2)^3 - q_1^{-2/3}q_2^{-4/3}(13p_1^2q_1^2 - 44p_1p_2q_1q_2 + 4p_2^2q_2^2) + 16q_1^{-1/3}q_2^{-8/3},$$

$$I_1 = 4p_1^2(p_1q_1 - p_2q_2)^4 - 4q_1^{-2/3}q_2^{-4/3}p_1(p_1q_1 - p_2q_2)(2p_1^2q_1^2 + p_2^2q_2^2 - 6p_1p_2q_1q_2) + q_1^{-4/3}q_2^{-8/3}(10p_1^2q_1^2 - 16p_1p_2q_1q_2 + p_2^2q_2^2) - 3q_1^{-1}q_2^{-4},$$

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Problems

1. How to find superintegrable cases for families 1-4 of potentials?
2. How to find superintegrable cases for potentials, that do not belong to families 1-4 of potentials?

Problem 1

Potentials from families 1-4 possess additional first integral that is quadratic in the momenta $\Longrightarrow$ classical theory of separation of variables.

Family 1

$$H_1 = 2p_1p_2 + q_1^d q_2^d, \quad I_2 = q_1p_1 - q_2p_2$$

Canonical transformation of coordinates

$$q_1 = r(\cos\varphi - \mathrm{i}\sin\varphi) = re^{-\mathrm{i}\varphi}, \qquad q_2 = r(\cos\varphi + \mathrm{i}\sin\varphi) = re^{\mathrm{i}\varphi},$$

and conjugated momenta

$$p_1 = \frac{e^{\mathrm{i}\varphi}}{2}\left(p_r + \frac{\mathrm{i}}{r}p_\varphi\right), \qquad p_2 = \frac{e^{-\mathrm{i}\varphi}}{2}\left(p_r - \frac{\mathrm{i}}{r}p_\varphi\right).$$

Hamiltonian H_1 and first integral I_2 in new variables read

$$H_1 = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} + V(r), \quad V(r) = r^{2d},$$
$$J_2 = -\mathrm{i}\,I_2 = p_\varphi.$$

In variables $x_1 = r\cos\varphi$, and $x_2 = r\sin\varphi$ and momenta y_1, y_2

$$H_1 = \frac{1}{2}(y_1^2 + y_2^2) + r^{2d}, \quad r^2 = x_1^2 + x_2^2, \quad J_2 = y_1x_2 - y_2x_1.$$

Superintegrable radial potentials

Theorem

The radial potential is super-integrable iff $d = -1/2$ or $d = 1$.

- for $d = -1/2$

$$H_3 = (p_1 - p_2)(p_1 q_1 - p_2 q_2) - \frac{q_1 + q_2}{2\sqrt{q_1 q_2}},$$

- for $d = 1$

$$H_3 = 2p_2^2 + q_1^2,$$

- connection with the Bertrand theorem.

Family 2

$$H_1 = 2p_1p_2 + \frac{q_2^d}{\sqrt{q_1}}, \quad I_2 = 2p_1(q_2p_2 - p_1q_1) + \frac{q_2^{d+1}}{\sqrt{q_1}}$$

Relation

$$I_2 = -2p_1^2q_1 + q_2H_1,$$

change of independent variable $t \to \tau$, $\mathrm{d}\tau/\mathrm{d}t = 1/\sqrt{q_1}$,

$$q_1' = 2p_2\sqrt{q_1}, \qquad p_1' = \frac{1}{2}\frac{q_2^d}{q_1},$$

$$q_2' = 2p_1\sqrt{q_1}, \qquad p_2' = -dq_2^{d-1},$$

$$I_2 = -\frac{1}{2}q_2'^2 + q_2H_1.$$

It gives

$$\int \frac{\mathrm{d}q_2}{\sqrt{2(q_2H_1 - H_2)}} = \tau + \beta_1,$$

Family 2

$$H_1^{-1}\sqrt{2(q_2H_1 - I_2)} = \tau + \beta_1, \quad \overset{\beta_1=0}{\Longrightarrow} \quad q_2 = \frac{2I_2 + H_1^2\tau^2}{2H_1}$$

substitution into H_1

$$H_1 = \frac{1}{2q_1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau}\frac{\mathrm{d}q_2}{\mathrm{d}\tau} + \frac{q_2^d}{\sqrt{q_1}}$$

$$\frac{H_1\tau}{2q_1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + \frac{(2I_2 + H_1^2\tau^2)^d}{(2H_1)^d\sqrt{q_1}} = H_1$$

change of dependent variable

$$q_1 = \left(\frac{I_2}{H_1}\right)^{2d}\frac{v^2}{H_1^2}.$$

$$\frac{\tau}{v}\frac{\mathrm{d}v}{\mathrm{d}\tau} + \frac{(2I_2 + H_1^2\tau^2)^d}{(2I_2)^d v} = 1$$

Family 2

change of independent variable

$$\tau \longmapsto x = -\frac{H_1^2\tau^2}{2I_2}$$

$$2x\frac{\mathrm{d}v}{\mathrm{d}x} - v + (1-x)^d = 0$$

after differentiation

$$x(1-x)\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{1-(1-2d)x}{2}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{d}{2}v = 0$$

Gauss hypergeometric differential equation

$$x(1-x)w'' + [c-(a+b+1)x]w' - abw = 0,$$

with parameters

$$a = -\frac{1}{2}, \qquad b = -d, \qquad c = \frac{1}{2}.$$

Construction of first integral for Family 2

Its general solution

$$v(x) = \beta_2\sqrt{x} + \beta_3\ {}_2F_1\left(-\frac{1}{2}, -d, \frac{1}{2}, x\right)$$

substitution into non-homogeneous equation gives $\beta_3 = 1$

$$q_1 = \frac{1}{H_1^2}\left[\beta_2\sqrt{2(H_1q_2 - I_2)} + \left(\frac{I_2}{H_1}\right)^d\ {}_2F_1\left(-\frac{1}{2}, -d, \frac{1}{2}, 1 - \frac{H_1}{I_2}q_2\right)\right]^2$$

After substitution $\tau = \tau(q_2)$ one can construct the following first integral

$$H_3 = \sqrt{2}\beta_2 = \frac{\sqrt{q_1}\,H_1 - \left(\frac{I_2}{H_1}\right)^d\ {}_2F_1\left(-\frac{1}{2}, -d, \frac{1}{2}, 1 - \frac{H_1}{I_2}q_2\right)}{\sqrt{q_2H_1 - I_2}}.$$

Problem

When this first integral is an algebraic function?

Superintegrability for Family 2

Theorem

Hamiltonian system given by

$$H = 2p_1p_2 + \frac{q_2^d}{\sqrt{q_1}}$$

is super-integrable with algebraic additional first integral iff $d = p$, or $d = -(2p-1)/2$, for a positive integer p.

Family 3

$$H_1 = 2p_1p_2 + q_1q_2^d, \quad I_2 = p_1^2 + \kappa q_2^{d+1}, \quad \kappa = \frac{1}{d+1}$$

First quadrature

$$\beta_1 + t = -\int^{q_2} \frac{\mathrm{d}x}{2\sqrt{I_2 - \kappa\, x^{d+1}}}$$

we substitute solution $q_2(t, \beta_1)$ into the Hamiltonian

$$H_1 = \frac{1}{2}\frac{\mathrm{d}q_1}{\mathrm{d}t}\,\frac{\mathrm{d}q_2}{\mathrm{d}t} + q_1q_2^d$$

and solve the obtained quadrature with respect to q_1.

Lie algebra of first integrals

Let $I_1 = H_1$ and I_2 are the action variables, and ω_1, ω_2, are the corresponding angle variables. Let us note that ω_2 is a first integral.

$$\rho = \alpha \left(p_1 q_1 - \frac{2p_2 q_2}{d+1} \right)$$

$$\{\rho, I_1\} = (\alpha + \beta) I_1, \quad \{\rho, I_2\} = 2\alpha I_2, \quad \{I_1, I_2\} = 0$$

$$\rho = -(\alpha + \beta) I_1 \omega_1 - 2\alpha I_2 \omega_2 + F(I_1, I_2)$$

Additional integral of motion H_3 is a function on the action variables I_1, I_2 and one angle variable ω_2.

Second angle variable

$$\omega_2 = \frac{\rho}{2\alpha I_2} + \frac{d-1}{d+1}\,\frac{I_1}{2I_2}\int^{q_2}\frac{\mathrm{d}x}{2\sqrt{I_2-\kappa\,x^{d+1}}}$$

ω_2 is the multi-valued function on the whole phase space.

$$H_3 = 4\alpha(d+1)I_2\omega_2 = 2(d+1)\rho + (d-1)\alpha I_1 \int^{q_2}\frac{\mathrm{d}x}{\sqrt{I_2-\kappa\,x^{d+1}}}\,.$$

Substitution

$$y = \frac{\kappa}{I_2}x^{d+1},$$

gives

$$\int\frac{\mathrm{d}x}{\sqrt{I_2-\kappa\,x^{d+1}}} = \frac{1}{(d+1)\sqrt{I_2}}\left(\frac{I_2}{\kappa}\right)^{\frac{1}{d+1}}\int\frac{y^{-\frac{d}{d+1}}\mathrm{d}y}{\sqrt{1-y}}.$$

Construction of H_3 for Family 3

substitution

$$\int \frac{y^{-\frac{d}{d+1}}\mathrm{d}y}{\sqrt{1-y}} = (d+1)y^{\frac{1}{d+1}}v(y),$$

gives that $v = v(y)$ satisfies

$$y(1-y)v'' + \left(\frac{d+2}{d+1} - \frac{3d+5}{2d+2}y\right)v' - \frac{1}{2d+2}v = 0.$$

Hypergeometric equation with parameters

$$a = \frac{1}{2}, \quad b = \frac{1}{d+1}, \quad c = 1 + \frac{1}{d+1}.$$

Final form of H_3

$$H_3 = 2(d+1)\rho + (d-1)\alpha\frac{I_1 q_2}{\sqrt{I_2}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{d+1}, 1+\frac{1}{d+1}, \frac{\kappa q_2^{d+1}}{I_2}\right).$$

Superintegrability for Family 3

Theorem

Hamiltonian system given by

$$H = 2p_1p_2 + q_1q_2^d$$

is super-integrable with algebraic additional first integral iff d takes the form

$$d = \frac{1-p}{p}, \quad or \quad d = \frac{1+2p}{1-2p}$$

for $p \in \mathbb{N}$.

Polynomial first integrals

Lemma

If

$$d = \frac{1+2p}{1-2p} \quad \textit{for a certain} \quad p \in \mathbb{N}, \tag{1}$$

then

$$\widetilde{H}_3 := H_3 I_2^p, \tag{2}$$

is a first integral polynomial in the momenta of degree $2p+1$.

Family 4

$$H_1 = 2p_1p_2 + q_1^{-d-2}q_2^d, \quad I_2 = (p_1q_1 - p_2q_2)^2 - 2q_1^{-d-1}q_2^{d+1}$$

Observation

deg $V = -2 \Longrightarrow$ separation in polar coordinates

$$q_1 = r(\cos\varphi - \mathrm{i}\sin\varphi) = re^{-\mathrm{i}\varphi}, \qquad q_2 = r(\cos\varphi + \mathrm{i}\sin\varphi) = re^{\mathrm{i}\varphi},$$

and for the corresponding momenta

$$p_1 = \frac{e^{\mathrm{i}\varphi}}{2}\left(p_r + \frac{\mathrm{i}}{r}p_\varphi\right), \qquad p_2 = \frac{e^{-\mathrm{i}\varphi}}{2}\left(p_r - \frac{\mathrm{i}}{r}p_\varphi\right).$$

Family 4

$$H_1 = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} + \frac{e^{2\mathrm{i}(d+1)\varphi}}{r^2}, \quad J_2 = \frac{1}{2}p_\varphi^2 + e^{2\mathrm{i}(d+1)\varphi}.$$

one relation

$$H_1 = \frac{1}{2}p_r^2 + \frac{1}{r^2}J_2.$$

a new independent variable τ, $\mathrm{d}\tau/\mathrm{d}t = 1/r^2$

$$p_r = \frac{r'}{r^2}, \qquad p_\varphi = \varphi',$$

separation relations

$$H_1 = \frac{r'^2}{2r^4} + \frac{1}{r^2}J_2, \quad \text{and} \quad J_2 = \frac{\varphi'^2}{2} + e^{2\mathrm{i}(d+1)\varphi}.$$

Two quadratures for Family 4

$$\int \frac{\mathrm{d}r}{r\sqrt{2(H_1 r^2 - J_2)}} = \tau + C_1, \qquad \int \frac{\mathrm{d}\varphi}{\sqrt{2\left(J_2 - e^{2\mathrm{i}(d+1)\varphi}\right)}} = \tau + C_2.$$

after integration

$$\frac{1}{\sqrt{2J_2}} \arctan \sqrt{\frac{H_1 r^2 - J_2}{J_2}} = \tau + C_1,$$

$$\frac{\mathrm{i}}{(d+1)\sqrt{2J_2}} \operatorname{arctanh}\left[\sqrt{\frac{J_2 - e^{2\mathrm{i}(d+1)\varphi}}{J_2}}\right] = \tau + C_2$$

First integral

$$I = (d+1)\sqrt{2J_2}(C_2 - C_1) = \mathrm{i}\operatorname{arctanh}\left[\sqrt{\frac{J_2 - e^{2\mathrm{i}(d+1)\varphi}}{J_2}}\right]$$

$$-(d+1)\arctan\sqrt{\frac{H_1 r^2 - J_2}{J_2}}.$$

Construction of additional first integral

$$\arctan z = \frac{\mathrm{i}}{2} \ln \left(\frac{1 - \mathrm{i}z}{1 + \mathrm{i}z} \right), \qquad \operatorname{arctanh} z = \frac{1}{2} \ln \left(\frac{1 + z}{1 - z} \right).$$

Form of first integral

$$H_3 = 2^{d+2} H_1^{d+1} \exp(\mathrm{i}\, I) = \frac{e^{-2\mathrm{i}(d+1)\varphi}}{r^{2(d+1)}} \left(\sqrt{2J_2} + p_\varphi \right)^2 \left(\sqrt{2J_2} + \mathrm{i}\, r p_r \right)^{2(d+1)}$$

or in original variables

$$H_3 = \frac{1}{q_2^{2(d+1)}} (p_2 q_2 - p_1 q_1 + \sqrt{J_2})^2 \left(p_1 q_1 + p_2 q_2 + \sqrt{J_2} \right)^{2(d+1)},$$

Theorem

Hamiltonian system given by

$$H_1 = 2 p_1 p_2 + q_1^{-d-2} q_2^d$$

is super-integrable with algebraic additional first integral iff d is a rational number.

Outline

1 Monomial potentials

2 Direct method

3 Application of the direct method to monomial potentials. Results

4 Superintegrability and classical separation

5 **Separation for systems with first integral of higher order in the momenta**

Separation for systems with first integral of higher order in the momenta

$$H_1 = 2p_1p_2 + q_1^3 q_2^{-9/5},$$
$$H_2 = 4p_1^4 - 10\left(3p_1^2q_1^2 - 30p_1p_2q_1q_2 + 25p_2^2q_2^2\right)q_2^{-4/5} + 225q_1^4q_2^{-8/5}$$

bi-Hamiltonian approach

$$X_{H_1} = P\mathrm{d}H_1 = P'\mathrm{d}H_2.$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

Looking for compatible Poisson bi-vectors P and P'

$$[\![P, P']\!] = 0, \quad [\![P', P']\!] = 0$$

$$P' = \mathcal{L}_Z P, \quad Z = \sum Z^k \partial_k.$$

Separation – control matrix

$$Z^1 = -4q_2p_1^2\,, \quad Z^2 = 0\,, \quad Z^3 = 25q_2^{1/5}(3p_1q_1 + 5p_2q_2)\,,$$
$$Z^4 = -5q_2^{-6/5}\left((2p_1p_2q_2^{9/5} + q_1^3)\sqrt{10} - 3p_1q_2^{2/5}q_1^2 - 15p_2q_2^{7/5}q_1\right)$$

control matrix F

$$P'\mathrm{d}H_i = P\sum_{j=1}^{2} F_{ij}\,\mathrm{d}H_j, \qquad i = 1, 2.$$

is non-degenerate matrix

$$F = \begin{pmatrix} 10q_2^{1/5}(-15q_1 + q_2^{2/5}\sqrt{10}p_1) & \frac{1}{2} \\ -500q_2^{1/5}(2q_2p_1^2 + 15q_2^{1/5}q_1^2 - 3q_2^{3/5}\sqrt{10}p_1q_1 + 5q_2^{8/5}\sqrt{10}p_2) & 0 \end{pmatrix}$$

Separation variables from recursion operator

Eigenvalues or the recursion operator $\mathcal{N} = P' \cdot P^{-1} = \mathcal{L}_Z P \cdot P^{-1}$

$$A(\lambda) = (\lambda - u_1)(\lambda - u_2) = \lambda^2 - 10q_2^{1/5}(q_2^{2/5}\sqrt{10}p_1 - 15q_1)\lambda$$
$$+ 250q_2^{2/5}\left(15q_1^2 + \sqrt{10}q_2^{2/5}(5p_2q_2 - 3q_1p_1) + 2p_1^2q_2^{4/5}\right)$$

are variables of separation. Corresponding momenta

$$v_{1,2} = -\frac{q_1}{40\sqrt{10}q_2^{3/5}} - \frac{p_1}{200\, q_2^{1/5}} \pm \frac{u_1 - u_2}{200\, q_2^{4/5}}, \qquad \{u_j, v_k\} = \delta_{ij}\,.$$

Separation relations

$$-2\,(\sigma v_k)^3\, u_k + H_2 + 2u_k H_1 = 0\,, \qquad k = 1, 2, \quad \sigma = 10^{3/2}\,.$$

$$\beta_1 - t = H_2 \int^{v_1} \frac{\mathrm{d}v}{(\sigma^3 v^3 + H_1)^2} + H_2 \int^{v_2} \frac{\mathrm{d}v}{(\sigma^3 v^3 + H_1)^2}\,,$$
$$\beta_2 = \int^{v_1} \frac{\mathrm{d}v}{2(\sigma^3 v^3 - H_1)} + \int^{v_2} \frac{\mathrm{d}v}{2(\sigma^3 v^3 - H_1)}$$