Nonlocal Hamiltonian Structures and Local Lagrangian Representations

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First Nutku–Olver's "puzzle"

The ideal gas dynamic system

$$\rho_t = (\rho u)_x, \qquad u_t = \left(\frac{u^2}{2} + \rho f''(\rho) - f'(\rho)\right)_x$$

has **only one local** Hamiltonian structure of the *Dubrovin–Novikov* type

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathbf{H}_4}{\delta \rho} \\ \frac{\delta \mathbf{H}_4}{\delta u} \end{pmatrix},$$

where the Hamiltonian and other lower functionals are given by

$$\mathbf{H}_4 = \int \left[\frac{\rho u^2}{2} + \rho f'(\rho) - 2f(\rho)\right] dx, \qquad \mathbf{H}_3 = \int \rho u dx, \qquad \mathbf{H}_2 = \int \rho dx,$$

First Nutku–Olver's "puzzle"

However, for this ideal gas dynamic system

$$\rho_t = (\rho u)_x, \qquad u_t = \left(\frac{u^2}{2} + \rho f''(\rho) - f'(\rho)\right)_x$$

Ya. Nutku and P. Olver found an **absolutely new local** Hamiltonian structure determined by the *third order* purely differential operator

$$\hat{B} = \hat{R}^2 \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

where the recursion operator $\hat{R}$ is a purely differential operator of the *first* order

$$\hat{R} = D_x (W_x)^{-1}$$

and the matrix W is given by

$$W = \begin{pmatrix} u & \rho \\ f''(\rho) & u \end{pmatrix}.$$

The Problems

So, we have the set of questions:

- **How to explain the ORIGIN of this local Hamiltonian structure?**
- **How to generalize this local Hamiltonian structure to N component case?**
- **How to recognize that a given hydrodynamic type system possesses such a local Hamiltonian structure?**
- **How to construct such a local Hamiltonian structure?**
- **How many such local Hamiltonian structures a given hydrodynamic type system can possess?**
- **Are there any relationships between these local Hamiltonian structures and local Hamiltonian structures of the Dubrovin–Novikov type?**

The key idea used for opening this puzzle is hidden in the local Lagrangian representation for this hydrodynamic type system

$$S=\int\left[\frac{1}{2}\frac{\rho_x u_t-u_x\rho_t}{u_x^2-f'''(\rho)\rho_x^2}-\rho\right]dxdt.$$

Local Lagrangian representations

The Lagrangian

$$S = \int [g_k(r, r_x, r_{xx}, ...) r_t^k - h(r, r_x, r_{xx}, ...)] dxdt$$

determines the Euler–Lagrange equations

$$\hat{M}_{ik} r_t^k = \frac{\delta \mathbf{H}}{\delta r^i}$$

in the Hamiltonian form

$$r_t^i = \hat{K}^{ij} \frac{\delta \mathbf{H}}{\delta r^j},$$

where the Hamiltonian $\mathbf{H} = \int h(r, r_x, r_{xx}, ...) dx$, and the Hamiltonian operator $\hat{K}^{ij}$ is an inverse operator to the symplectic operator

$$\hat{M}_{ij} = \frac{\partial g_i}{\partial r^j_{(n)}} D_x^n - (-1)^n D_x^n \frac{\partial g_j}{\partial r^i_{(n)}}.$$

Local Lagrangian representations

Let us restrict our consideration to the case

$$S=\int[g_k(r,r_x)r_t^k-h(r,r_x,r_{xx},...)]dxdt.$$

The corresponding Euler–Lagrange equations can be written in the form

$$\hat{M}_{ik}r_t^k=\frac{\delta\mathbf{H}}{\delta r^i},$$

where the symplectic operator is given by

$$\hat{M}_{ik}=\frac{\partial g_k}{\partial r^i}-\frac{\partial g_i}{\partial r^k}-\left(\frac{\partial g_i}{\partial r_x^k}D_x+D_x\frac{\partial g_k}{\partial r_x^i}\right).$$

If we choose

$$g_k(r,r_x)=\frac{\bar{H}_k^2(\mathbf{r})}{2r_x^k},$$

then

$$\hat{M}_{ii}=\frac{\bar{H}_i}{r_x^i}D_x\frac{\bar{H}_i}{r_x^i},\qquad \hat{M}_{ik}|_{k\neq i}=\bar{H}_i\bar{H}_k\left(\frac{\beta_{ik}}{r_x^k}-\frac{\beta_{ki}}{r_x^i}\right).$$

Theorem

The Hamiltonian operator

$$\hat{K}^{ij} = \bar{\varepsilon}^{\alpha\beta} w^i_{(\alpha)} r^i_x D^{-1}_x w^j_{(\beta)} r^j_x$$

is an **inverse** *operator to the above symplectic operator* $\hat{M}_{ik}$ *iff*
1. β_{ik} *are rotation coefficients of the corresponding conjugate curvilinear coordinate net determined by the Darboux system and by the "anti-flatness" condition*

$$\partial_i \beta_{ki} + \partial_k \beta_{ik} + \sum_{m \neq i} \beta_{im} \beta_{km} = 0, \qquad i \neq k;$$

2. *affinors* $w^i_{(\alpha)}$ *determine* N *commuting hydrodynamic type systems (structural flows)*

$$r^i_{t^\alpha} = w^i_{(\alpha)} r^i_x,$$

where $w^i_{(\alpha)} = \bar{H}_{(\alpha)i} / \bar{H}_i$, *and* $\bar{H}_{(\alpha)i}$ *are solutions of the linear ODE systems*

$$\partial_i \bar{H}_{(\alpha)k} = \beta_{ik} \bar{H}_{(\alpha)i}, \qquad i \neq k, \qquad \partial_i \bar{H}_{(\alpha)i} + \sum_{m \neq i} \beta_{im} \bar{H}_{(\alpha)m} = 0;$$

3. *$\bar{\varepsilon}^{\alpha\beta}$ is a constant non-degenerate symmetric matrix such that*

$$\bar{\varepsilon}^{\alpha\beta} = \sum \bar{H}_m^{(\alpha)} \bar{H}_m^{(\beta)}, \qquad \bar{H}_i^{(\alpha)} = \bar{\varepsilon}^{\alpha\beta} \bar{H}_{(\beta)i}.$$

Corollary: The Lagrangian

$$S = \int \left[\frac{1}{2} \sum \bar{H}_k^2(\mathbf{r}) \frac{r_\tau^k}{r_x^k} - h(\mathbf{r}) \right] dxdt$$

determines an *integrable* hydrodynamic type system

$$r_\tau^i = \frac{H_i}{\bar{H}_i} r_x^i,$$

where

$$H_i = \bar{H}_i^{(\beta)} q_\beta, \qquad q_\beta = \sum \bar{H}_{(\beta)m} H_m, \qquad \partial_i q_\beta = \psi_i \bar{H}_{(\beta)i}.$$

It means that the Hamiltonian density h is determined by the linear system (here $\partial_i h = \psi_i \bar{H}_i$)

$$\partial_i \psi_k = \beta_{ki} \psi_i, \quad i \neq k, \qquad \psi_i = \partial_i H_i + \sum_{m \neq i} \beta_{im} H_m.$$

Mixed Local Hamiltonian Structures

If the rotation coefficients β_{ik} associated with a given hydrodynamic type system satisfy the flatness and anti-flatness conditions, simultaneously, then this hydrodynamic type system has two Hamiltonian structures

$$r^i_t = \{r^i, \mathbf{H}_1\}_1 = \{r^i, \mathbf{H}_2\}_2,$$

where the components of the local (Dubrovin–Novikov) Poisson structure

$$\{r^i, r^j\}_1 = \hat{A}^{ij}\delta(x - x')$$

are given by

$$\hat{A}^{ii} = \frac{1}{\bar{H}_i} D_x \frac{1}{\bar{H}_i}, \qquad \hat{A}^{ik}|_{i\neq k} = \frac{1}{\bar{H}_i \bar{H}_k}(\beta_{ki} r^i_x - \beta_{ik} r^k_x)$$

and the components of the local symplectic structure

$$\hat{M}_{jk}\{r^k, r^i\}_2 = \delta^i_j \delta(x - x')$$

are given by

$$\hat{M}_{ii} = \frac{\bar{H}_i}{r^i_x} D_x \frac{\bar{H}_i}{r^i_x}, \qquad \hat{M}_{ik}|_{i\neq k} = \bar{H}_i \bar{H}_k \left(\frac{\beta_{ik}}{r^k_x} - \frac{\beta_{ki}}{r^i_x}\right).$$

Mixed Local Hamiltonian Structures

Thus, the recursion operator of the *second* order is a product of local Hamiltonian and symplectic operators of the *first* order

$$\hat{R}^i_k = \hat{A}^{ij}\hat{M}_{jk},$$

and the above hydrodynamic type system has infinitely many local *Hamiltonian* structures of *all odd* orders

$$r^i_t = \hat{A}^{ij}\frac{\delta \mathbf{H}_1}{\delta r^j} = \hat{A}^{ij}\hat{M}_{jk}\hat{A}^{ks}\frac{\delta \mathbf{H}_0}{\delta r^s} = \hat{A}^{ij}\hat{M}_{jk}\hat{A}^{ks}\hat{M}_{sn}\hat{A}^{nm}\frac{\delta \mathbf{H}_{-1}}{\delta r^m} = \dots$$

and infinitely many local *symplectic* structures of *all odd* orders

$$\hat{M}_{ij}r^j_t = \frac{\delta \mathbf{H}_2}{\delta r^i}, \qquad \hat{M}_{ij}\hat{A}^{jk}\hat{M}_{kn}r^n_t = \frac{\delta \mathbf{H}_3}{\delta r^i}, \qquad \hat{M}_{ij}\hat{A}^{jk}\hat{M}_{kn}\hat{A}^{ns}\hat{M}_{sm}r^m_t = \frac{\delta \mathbf{H}_4}{\delta r^i}, \dots$$

Integrability

Theorem: *The nonlinear PDE system*

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \qquad i \neq j \neq k,$$

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m\neq i} \beta_{mi}\beta_{mk} = 0, \ \partial_i \beta_{ki} + \partial_k \beta_{ik} + \sum_{m\neq i} \beta_{im}\beta_{km} = 0, \ i \neq k$$

is an integrable system, which is a consequence of compatibility conditions following from the over-determined linear PDE system

$$\lambda H_i = \partial_i \left(\partial_i H_i + \sum_{n\neq i} \beta_{in} H_n \right) + \sum_{m\neq i} \beta_{mi} \left(\partial_m H_m + \sum_{n\neq m} \beta_{mn} H_n \right),$$

$$\partial_i H_k = \beta_{ik} H_i, \qquad i \neq k.$$

Example

Theorem: *Semi-Hamiltonian hydrodynamic type system cannot possess more than $N+1$ anti-flat nonlocal Hamiltonian structures.*

Example: The hydrodynamic type system

$$r_t^i = \left(\sum r^m - 2r^i\right) r_x^i, \qquad i = 1, 2, ..., N$$

has $N+1$ local Lagrangian representations

$$S_n = \int \left[\frac{1}{2} \sum \frac{(r^k)^{n-1}}{\prod\limits_{m \neq k} (r^k - r^m)} \frac{r_t^k}{r_x^k} - h_n(\mathbf{r})\right] dxdt, \qquad n = 1, 2, ..., N+1.$$

Indeed, the Lame coefficients are given by

$$\bar{H}_i^0 = \prod_{m \neq i} (r^i - r^m)^{-1/2}.$$

It means that the corresponding rotation coefficients $\tilde{\beta}_{ik}$ are mirrored to the rotation coefficients β_{ik} associated with the elliptic coordinates.

Conclusion

In this talk we were able to find answers to a set of important questions.

- **The Lagrangian formulation for nonlocal Hamiltonian structures of hydrodynamic type systems associated with the anti-flatness condition is presented.**
- **The existence of two Hamiltonian structures associated with the flatness and the anti-flatness conditions implies an infinite set of local Hamiltonian structures of odd higher orders.**
- **The description of multi-Hamiltonian structures associated with anti-flatness conditions is equivalent to the description of local multi-Hamiltonian structures of the Dubrovin–Novikov type.**
- **The existence of one local Hamiltonian structure for the Egorov hydrodynamic type system implies the existence of infinitely many local Hamiltonian structures of odd higher degrees.**
- **Non-local Hamiltonian structures associated with the anti-flatness condition are extended to an arbitrary "co-dimension".**