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"Real Monge-Ampère operators and (almost) complex, (almost) product, hyperkähler, hypersymplectic, generalized complex and other geometric structures"

Talk at I. Krasilschik Seminar

"Geometry of Differential Equations"

Moscow Independent University
and
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Russian Academy of Sciences

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Plan

1° Short introduction in Lychagin's geometric approach to Monge - Ampère equations

$$\boxed{Af_{xx} + 2Bf_{xy} + Cf_{yy} + D(f_{xx}\cdot f_{yy} - f_{xy}^2)}$$

$$+ E = 0, \quad x, y \in U \subset \mathbb{R}^2,$$

A, B, C, D, E - functions on

$x, y, \cancel{f}, f_x, f_y$

Geometrically: M^n - smooth variety, $\pi : J^1M \to M$ or

$(\pi : T^*M \to M)$

1^{st} jet bundle (cotangent bdle)

$\omega \in \Omega^k(J^1M)$ $(\omega \in \Omega^k(T^*M))$ $k \leq n$

"forms" $\longleftrightarrow$ "operators"

$\omega \longmapsto \Delta_\omega : C(M) \to \Omega^k(M)$

Monge-Ampère operators

$\Delta_\omega(f) := \sigma_f^*(\omega)$, where

$\sigma_f : M \to J^1 M$ ($\sigma_f : M \to T^*M$)
"contact" case "symplectic case"

If $f \in C^\infty(M)$, $\omega \in \Omega^k$ $\Rightarrow \sigma_f^*(\omega) \in \Omega^k(M)$
$1 \leq k \leq \underline{\underline{n}}$

Δ_ω has "MA type nonlinearity"
(= depends on minors of Hessian matrix $\| f_{x_i x_j} \|_{1 \leq i,j \leq n}$)

<u>Examples:</u> $\underbrace{n=2}$, $M = \mathbb{R}^2 \ni (x,y)$

$T^*\mathbb{R}^2 = \mathbb{R}^4 \ni (x,y,p,q)$, $J^1\mathbb{R}^2 = \mathbb{R}^5 \ni (x,y,\underline{\underline{u}},p,q)$

- $\omega \in \Omega^2(T^*\mathbb{R}^2)$

$\boxed{\omega = dp \wedge dq - dx \wedge dy}$, $\Delta_\omega = (\det \mathrm{Hess} - 1)$

$\Delta_\omega(f) = (f_{xx} f_{yy} - f_{xy}^2 - 1)$

- $\omega \in \Omega^1(J^1\mathbb{R}^1)$ $\underline{\omega = u\,dp - p\,dx}$

$$\Delta_\omega(f) = f\cdot f_{xx} - f_x = \Gamma_f^*(\omega)$$

(we omit "volume forms" $dx\wedge dy$ and dx).

- $M=\mathbb{R}^3$, $T^*M=\mathbb{R}^6$, (x,y,z,p,q,r)

  $\omega\in\Omega^3(T^*\mathbb{R}^3)$ $\quad \omega = dp\wedge dq\wedge dz - dp\wedge dy\wedge dz - dx\wedge dq\wedge dz - dx\wedge dy\wedge dr$

  $\Delta_\omega(f) = \det \mathrm{Hess}(f) - \Delta f$ — special Lagrangian MAO.

  $\boxed{\det \mathrm{Hess} f - \Delta f = 0}$

- $M=\mathbb{R}^2$, $T^*\mathbb{R}^2$, $\Omega = dp\wedge dx + dq\wedge dy$

  canonical symplectic form

  $\boxed{\Delta_\Omega \equiv 0} \Rightarrow \Omega\in\ker\Gamma_f^* \quad \forall f$

"forms" $\longrightarrow$ "operator" correspondence is not 1:1

One consider the (graded) ideal $\mathcal{I}_\Omega \subset \Omega^\cdot(T^*M)$ generated by Ω, ρ ($\rho = p\,dq$)

The quotient $\Omega^\cdot(T^*M)/\mathcal{I}_\Omega$

consists of effective forms:

$\underbrace{\Omega^{\bullet}_{\varepsilon}(T^*M)}:= \Omega^{\bullet}(T^*M)/\mathcal{J}_{\Omega}$

We denote by $\underset{\smile}{T}: \Omega^k(T^*M) \to \Omega^{k+2}(T^*M)$ $(+2)$-rising operator $T(\omega) = \underline{\Omega \wedge \omega}$,

by $\underset{\smile}{\perp}: \Omega^k(T^*M) \to \Omega^{k-2}(T^*M)$ (-2)-lowering operator of the contraction: $\perp(\omega) = \iota_{\varepsilon}(\omega)$ - the $X_{\frac{\partial}{\partial q}\wedge\frac{\partial}{\partial p}}$

Canonical bivector $\Omega^{-1} \in \Lambda^2(TM)$ substitution to ω.

The commutator $[\perp, T] = \perp \circ T - T \circ \perp$ - 0-order (scalar) operator:

$$[\perp, T](\omega) = (n-k)\,\omega, \quad \omega \in \Omega^k(T^*M)$$

<u>Observation</u>: The triple $(T, \perp, [\perp, T])$ defines a representation of sl_2 in $\Omega^{\bullet}(TM)$ ("sl_2-triple")

Skew-orthogonality condition $(k=n)$

$\Omega \wedge \omega = 0 \Leftrightarrow \omega \in \Omega^n_{\varepsilon}(T^*M)$

$\forall k \ \perp\omega = 0$

Weak Hodge

Lepage - Hodge - Lychagin theorem

Any k-form $\omega \in \Omega^k(T^*M)$ admits the following unique decomposition (finite sum)

$$\omega = \omega_0 + T\omega_1 + T^2\omega_2 + \dots$$

where ω_i are effective $(n-2i)$-forms $(0 \le i \le [n/2])$. $\perp \omega_i = 0$

Few words about the proof:

$sl_2(\mathbb{C})$ - 2×2 complex matrices; trace 0

$\langle e, f, h \rangle$ - basis

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, commutation rel's

$[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$

$\rho: sl_2(\mathbb{C}) \to gl(V_{\mathbb{C}})$,

We denote by E, F, H images of generators $\{E, F, H\} \subset gl(V)$ - sl_2-triple

The representation ρ is a real rep if $V_{\mathbb{C}} = V \otimes \mathbb{C}$ and $\rho(sl_2(\mathbb{R})) \subset gl(V)$

The representation is irreducible if V has no proper s/spaces invariant w.r.t $\rho(sl_2(\mathbb{R})) \subset gl(V)$

Thm (Weyl): Each f.d. reps of sl_2 admits a splitting in a direct sum of irreducibles

Corollary: Let (V, Ω) - $2n$ dimensional and $W = \Lambda^{\cdot}(V_{\mathbb{C}}^{*})$ - graded by deg. Then $(W[n], T)$ - satisfied Hard Lefschetz

Hard-Lefschetz: a pair (W, A)

$W = \bigoplus_{\ell \in \mathbb{Z}} W^{\ell}$ - f.d. real vector space

$A \in End(W)$, $A(W^{\ell}) \subset W^{\ell+2}$ and

$A^{\ell}: W^{-\ell} \to W^{\ell}$ - isomorphism $\forall \ell \geq 0$

$W[n]$ - shifted grading: $W^{\ell} \to W^{\ell+n}$

The pair $(\Omega^{\cdot}(T^{*}M), T)$ satisfies Hard Lefschetz (Lychagin & Lepage)

Kähler structure

$g(JX, JY) = g(X, Y)$ Figure : Erich Kähler

- Kähler manifold: $(M, J, g)-$ a complex manifold with a compatible hermitian metric g (which is called Kähler) if the following condition holds:

$$d\omega = 0.$$

- Here ω (the Kähler form) is a $(1, 1)-$ form which is determined by

$$\omega(X, Y) = g(JX, Y)$$ (1,1) hermitian

and determines g via

$$g(X, Y) = g(JX, JY) = \omega(X, JY).$$

Calabi-Yau structure

ω

Figure : Calabi and Yau

$\Omega \in \Lambda^{n,0}(M)$

complex dim

Calabi-Yau structure: $(M^n, g, J, \Omega)-$ a Calabi-Yau variety if

- $((M^n, g, J)-$ a Kähler manifold with a Kähler form ω;
- $\Omega-$non-zero constant $(n, 0)-$form (holomorphic volume form);

  covariantly

- $$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega}.$$

nowhere zero!

(g, J, Ω, ω) - CY

Calabi Conjecture:

Compact Kähler m-fd a unique Kähler g in the same classe whose Ricci form is any 2-form rep. $c_1(M)$

- Calabi Conjecture: $((M^n, g, J)-$ a Kähler manifold with a Kähler form ω;
- Let f be a smooth real function on M satisfying

$F = e^f$

$$\int_M e^f \omega^n = \int_M \omega^n.$$

normalisation cond.

- Then there exists a smooth function ϕ such that $\int_M \phi vol_g = 0$ and
  ($vol_g = \omega^n$)
  - $$R := \omega + i\partial\bar\partial\phi -$$
    a positive $(1,1)-$form (a Kähler form of some metric g').
  - $$\boxed{(\omega + i\partial\bar\partial\phi)^n = e^f\omega^n.}$$
    (solve for all f)
- This is a Monge-Ampére equation.

!! Complex MAE

Harvey-Lawson

Calibrated geometries
Kähler

$\Delta f = \det \mathrm{Hess} f$

$n=3$

f

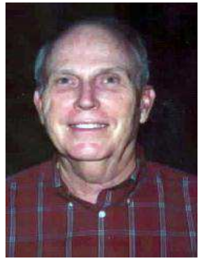
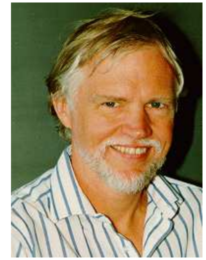

Figure : Harvey and Lawson

$\alpha = dz_1 \wedge \cdots \wedge dz_n$

- Special Lagrangian equation: $\mathbb{C}^n$ is enabled with a Kähler structure $g + i\omega$ and the canonical complex volume form α. A real submanifold L^n in $\mathbb{C}^n$ is a special lagrangian if

$$\omega|_L = 0 \text{ et } \Im(\alpha)|_L = 0. \iff$$

combination of subminors of Hess

- A graph $L = \{(x + i\frac{\partial f}{\partial x}),\ x \in \mathbb{R}^n\}$ is special lagrangian iff f is a solution of a MAE:
  - $n = 2$: $\Delta f = 0$
  - $n = 3$: $\Delta f - \mathrm{hess}(f) = 0$

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$\Delta f - \mathrm{Hess} f = 0$

HyperKähler Structure

- HyperKähler Structure, $4D$: $4D$ manifold M^4 is enabled with a HyperKähler structure $(\omega_1, \omega_2, \omega_3)$ if this triple of symplectic forms with $\omega_i \wedge \omega_j = 0$, $i \neq j$ and independent of $i = j$.
- M is a Kähler manifold (M, J_i, g), $\omega_i(X, Y) = g(J_i X, Y)$ $i = 1, 2, 3$, and $J_i = \omega_j^{-1}\omega_k$, $(i, j, k) = (1, 2, 3)$
- If $(M, \omega_i, ß = 1, 2, 3)-$ HyperKähler, taking the complex structure J_1 and two-form $\omega' = \omega_1 + i\partial\bar{\partial}\phi$, obtain a Monge-Ampère operator $\Delta(\phi)$:

$$\omega' \wedge \omega' = \Delta(\phi)\omega_1 \wedge \omega_1$$

$\Delta \equiv \Delta_{\omega'}$ in our previous notations!

- $\rho = -i\partial\bar{\partial} \log \Delta(\phi)-$ the Ricci form of ω'.

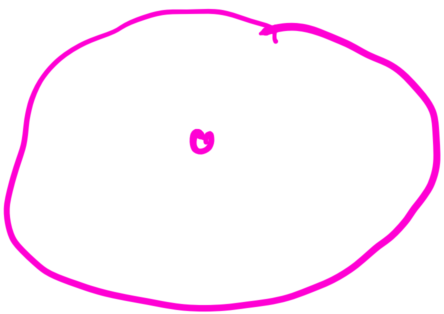
Table 1.

Constant coeff. MAO

on $M = \mathbb{R}^2$, $T^*M = \mathbb{R}^4$

(q_1, q_2) (q_1, q_2, p_1, p_2)

	$\boldsymbol{\Delta}_\omega = 0$	ω	$pf(\omega)$
Laplace	$\Delta f = 0$	$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	1
"wave"	$\Box f = 0$	$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	-1
	$\frac{\partial^2 f}{\partial q_1^2} = 0$	$\cancel{dq_1 \wedge dp_2}$	0



$\subset \mathbb{R}^2$

Geometric Structures on $T^*\mathbb{R}^2$.

$\mathbb{R}^4 = T^*\mathbb{R}^2$

Let (Ω, ω) be a Monge-Ampère structure on $X = \underline{\mathbb{R}^4}$. The field of endomorphisms $A_\omega : X \to TX \otimes T^*X$ is defined by

$$\omega(\cdot,\cdot) = \Omega(A_\omega \cdot, \cdot).$$

REMARK The tensor

$$J_\omega = \frac{A_\omega}{\sqrt{|pf(\omega)|}} \neq 0$$

gives

- an almost-complex structure on X if $pf(\omega) > 0$.

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REMARK The tensor

$$J_\omega = \frac{A_\omega}{\sqrt{|pf(\omega)|}}$$

gives

- an almost-complex structure on X if $pf(\omega) > 0$.
- an almost-product structure on X if $pf(\omega) < 0$.

THEOREM (Lychagin-R.)

Let $\omega \in \Omega^2_\varepsilon(\mathbb{R}^4)$ be an effective non-degenerate 2-form on $(\mathbb{R}^4, \Omega)$.

- The following assertions are equivalent:
- The equation $\Delta_\omega = 0$ is locally equivalent to one of two linear equations: $\Delta f = 0$ ou $\square f = 0$;
- The tensor J_ω is integrable;
- the normalized form $\frac{\omega}{\sqrt{|pf(\omega)|}}$ is closed.

J_ω

$d\left(\frac{\omega}{\sqrt{|pf(\omega)|}}\right)$

Δ_ω linearised !!!

Courant Bracket

M^n - real, $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$

T −tangent bundle of M and T^*− cotangent bundle.

$$(X+\xi, Y+\eta) = \frac{1}{2}(\xi(Y) + \eta(X)),$$

(n, n)

-natural indefinite interior product on $T \oplus T^*$.

The Courant bracket on sections of $T \oplus T^*$ is

$$[X+\xi, Y+\eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi).$$

1) skew-symmetric
2) No Jacobi! But: $Jac(X+\xi, Y+\eta, Z+\rho)$ - "exact"
"form" ⇝ Nijenhuis tensor for what??

Generalized Complex Geometry

Figure : Hitchin

DEFINITION [Hitchin]: An almost generalized complex structure is a bundle map $\mathbb{J} : T \oplus T^* \to T \oplus T^*$ with

$4n$

M^n $2n$ $2n$

$$\mathbb{J}^2 = -1,$$

and

$$(\mathbb{J}\cdot, \cdot) = -(\cdot, \mathbb{J}\cdot).$$

An almost generalized complex structure is integrable if the spaces of sections of its two eigenspaces are closed under the Courant bracket.

2D SMAE and Generalized Complex Geometry

ω

- **DEFINITION** A Monge-Ampère equation $\Delta_\omega = 0$ has a divergent type if the corresponded form can be chosen closed : $\omega' = \omega + \mu\Omega$.

$A_\omega \quad \Omega$

- **PROPOSITION** (B.Banos)
  Let $\Delta_\omega = 0$ be a Monge-Ampère divergent type equation on $\mathbb{R}^2$ with closed ω (which might be non-effective). The generalized almost-complex structure defined by

$$\mathbb{J}_\omega = \begin{pmatrix} A_\omega & \Omega^{-1} \\ -\Omega(1 + A_\omega^2 \cdot, \cdot) & -A_\omega^* \end{pmatrix}$$

is integrable.

$\mathbb{J}_\omega^2 = -1$

$N_{\mathbb{J}}$ - Jacobiator $= 0$

Hitchin pairs (after M.Crainic)

= "sympl."

A Hitchin pair is a pair of bivectors π and Π, $\Pi-$ non-degenerate, satisfying

$$\begin{cases} [\Pi,\Pi] = [\pi,\pi] \neq 0 \\ [\Pi,\pi] = 0. \end{cases} \tag{6}$$

PROPOSITION There is a 1-1 correspondence between Generalized complex structure

$$\mathbb{J} = \begin{pmatrix} A & \pi_A \\ \sigma & -A^* \end{pmatrix}$$

– G.C.S of thi

with σ non degenerate and Hitchin pairs of bivector (π, Π). In this correspondence

$$\begin{cases} \sigma = \Pi^{-1} \\ A = \pi \circ \Pi^{-1} \\ \pi_A = -(1 + A^2)\Pi \end{cases}$$

Hitchin pair of bivectors in $4D$

Π is non-degenerate $\Rightarrow$ two 2-forms ω and Ω, not necessarily closed and $\omega(\cdot,\cdot) = \Omega(A\cdot,\cdot)$.
A generalized lagrangian surface: closed under A, or equivalently, bilagrangian: $\omega|_L = \Omega|_L = 0$.
Locally, L is defined by two functions u and v satisfying a first order system:

Crainic-Hitchin a.g.c.st.

Jacobi systems

G.C. str. ?

$n=2$

$n=4$

$$
\begin{cases}
a + b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} + d\frac{\partial v}{\partial x} + e\frac{\partial v}{\partial y} + f \det J_{u,v} = 0 \\
A + B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y} + D\frac{\partial v}{\partial x} + E\frac{\partial v}{\partial y} + E \det J_{u,v} = 0
\end{cases}
$$

$$
J_{u,v} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}
$$

Such a system generalizes both MAE and Cauchy-Riemann systems and is called a Jacobi system.

integ. g.c. str. $\Rightarrow$

Invariants for effective 3-forms

- To each effective 3-form $\omega \in \Omega^3_\varepsilon(\mathbb{R}^6)$, we assign the following geometric invariants:
- the ~~Lychagin-R.~~ metric defined by

$$g_\omega(X, Y) = \frac{(\iota_X\omega) \wedge (\iota_Y\omega) \wedge \Omega}{\Omega^3},$$

- the Hitchin tensor defined by

$$g_\omega = \Omega(A_\omega \cdot, \cdot),$$

- The Hitchin pfaffian defined by

g_ω , $pf(\omega)$

$$pf(\omega) = \frac{1}{6} tr A_\omega^2.$$

non-integrable!

	$\Delta_\omega = 0$	ω	$\varepsilon(q_\omega)$	$pf(\omega)$
1	$\mathrm{hess}(f) = 1$	$dq_1 \wedge dq_2 \wedge dq_3 + \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(3,3)$	ν^4
2	$\Delta f - \mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - \nu^2 \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(0,6)$	$-\nu^4$
3	$\square f + \mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 + \nu^2 \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(4,2)$	$-\nu^4$
4	$\Delta f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(0,3)$	0
5	$\square f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(2,1)$	0
6	$\Delta_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 - dp_2 \wedge dq_1 \wedge dq_3$	$(0,1)$	0
7	$\square_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 + dp_2 \wedge dq_1 \wedge dq_3$	$(1,0)$	0
8	$\cancel{\frac{\partial^2 f}{\partial q_1^2}} = 0$	$dp_1 \wedge dq_2 \wedge dq_3$	$(0,0)$	0
9		0	$(0,0)$	0

Table : Classification of effective 3-formes in dimension 6

3D Generalized Calabi-Yau structures

$\Delta_\omega \to J_\Delta$

- **A generalized almost Calabi-Yau structure** on a 6D-manifold X is a 5-uple $(g, \Omega, A, \alpha, \beta)$ where
  - g is a (pseudo) metric on X,
  - Ω is a symplectic on X,
  - A is a smooth section $X \to TX \otimes T^*X$ such that $A^2 = \pm Id$ and such that
    $$g(U, V) = \Omega(AU, V)$$
    for all tangent vectors U, V,
  - α and β are (eventually complex) decomposable 3-forms whose associated distributions are the distributions of A eigenvectors and such that
    $$\frac{\alpha \wedge \beta}{\Omega^3} \text{ is constant.}$$

- A generalized Calabi-Yau structure $(g, \Omega, K, \alpha, \beta)$ is integrable if α and β are closed.

mod $\leftarrow (\omega, \Omega) \partial Sp$

Generalized CY and MA

Each nondegenerate Monge-Ampère structure (Ω, ω_0) defines a generalized almost Calabi-Yau structure $(q_\omega, \Omega, A_\omega, \alpha, \beta)$ with

$$d(\omega) = d\left(\frac{\omega_0}{\sqrt[4]{|\lambda(\omega_0)|}}.\right) = 0$$

The generalized Calabi-Yau structure associated with the equation

$$\boxed{\Delta(f) - \mathrm{hess}(f) = 0}$$

is the canonical Calabi-Yau structure of $\mathbb{C}^3$

$$\begin{cases} g = -\sum_{j=1}^{3} dx_j.dx_j + dy_j.dy_j \\ A = \sum_{j=1}^{3} \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau associated with the equation

$$\square(f) + \mathrm{hess}(f) = 0$$

is the pseudo Calabi-Yau structure

$$\begin{cases} g = dx_1.dx_1 - dx_2.dx_2 + dx_3.dx_3 + dy_1.dy_1 - dy_2.dy_2 + dx_3.dx_3 \\ A = \frac{\partial}{\partial x_1} \otimes dy_1 - \frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 \\ + \frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau structure associated with the equation

$$\boxed{\mathrm{hess}(f) = 1}$$

is the "real" Calabi-Yau structure

$$\begin{cases} g = \sum_{j=1}^{3} dx_j . dy_j \\ A = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3 \\ \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

SO(6)-equiv

- **THEOREM** A SMAE $\Delta_\omega = 0$ on $\mathbb{R}^3$ associated to an effective non-degenerated form ω is locally equivalent to on of three following equations:
- $\Delta_\omega \to$ $$\begin{cases} \mathrm{hess}(f) = 1 \\ \Delta f - \mathrm{hess}(f) = 0 \\ \Box f + \mathrm{hess}(f) = 0 \end{cases}$$
- iff the correspondingly defined generalized Calabi-Yau structure is integrable and locally flat.

The End

of

the 1st part

Thanks for

your time!